Electronic Journal of Differential Equations, Vol. 2010(2010), No. 10, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXPLICIT ESTIMATES FOR SOME MIXED INTEGRAL INEQUALITIES

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Abstract. The main objective of this paper is to establish explicit estimates for some mixed integral inequalities, which can be used as tools in the study of qualitative behavior of solutions to certain integral equations. Discrete analogues and some applications of one of our results are also given.

1. Introduction

Inequalities with explicit estimates are among the most powerful and widely used analytic tools in the study of various dynamic systems. They enable us to obtain valuable information about solutions of equations without the need to know in advance the solutions explicitly. Extensive surveys of such inequalities may be found in the monographs [5, 6, 7, 10] and the references cited therein. It is easy to see that some inequalities with explicit estimates available in the literature are not directly applicable to study the qualitative properties of solutions of many dynamical systems. For instance, the integral equation

$$u(t,x) = f(t,x) + \int_0^t \int_0^s \int_0^\alpha G(s-\tau,x,y)F(\tau,y,u(\tau,y))\,dy\,d\tau\,ds, \tag{1.1}$$

for $t \in [0,T], x \in [0,\alpha]$, arising in the study of partial differential equations of the form

$$u_{tt}(t,x) - au_{xxt}(t,x) = F(t,x,u(t,x)), \quad t \in [0,T], x \in [0,\alpha], \tag{1.2}$$

with the initial conditions

$$u(0,x) = \phi(x), u_t(0,x) = \psi(x), \quad x \in [0,\alpha], \tag{1.3}$$

and the boundary conditions

$$u(t,0) = u(T,\alpha) = 0, \quad t \in [0,T], \tag{1.4}$$

where $G(t,x,y)$ is the Green's function for the equation $w_t(t,x) = aw_{xx}(t,x)$ with the zero Dirichlet boundary data, a is a positive constant and $T > 0$, $\alpha > 0$ are finite but can be arbitrarily large constants. For more details, see [1, 9]. It seems that, in the study of certain basic results, equation (1.1) can be dealt with a more satisfactory manner than dealing directly with (1.2)-(1.4). Indeed, one need a new

2000 *Mathematics Subject Classification.* 34K10, 35K10, 39A10.
Key words and phrases. Integral inequalities; qualitative behavior; approximate solutions; dependency of solutions on parameters; closeness of solutions.

Submitted September 21, 2009. Published January 16, 2010.

insight to handle the equations like (1.1) for which the available inequalities in the literature lose their direct applicability.

From the considerations such as above, it is desirable to find explicit estimates on certain new inequalities which will be equally important to achieve a diversity of desired goals. Motivated by the desire to widen the scope of the inequalities with explicit estimates, in the present paper we offer new explicit estimates on some basic integral inequalities which can be used as tools for handling the equations like (1.1). Discrete analogues of the main results and some applications to illustrate the usefulness of one of our results are also given. We hope that our results here will reveal as a model for future investigations.

2. Statement of Results

Let $\mathbb{R}$ be the set of real numbers, $N_0 = \{0, 1, 2, \dots\}$, $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_1 = [1, \infty)$, $B = \prod_{i=1}^{m} [c_i, d_i] \subset \mathbb{R}^m$, $(c_i < d_i)$, $E = \mathbb{R}_+ \times B$ and $'$ denotes the derivative. For any function u defined on B we denote by $\int_B u(y)\,dy$ the m-fold integral $\int_{c_1}^{d_1} \dots \int_{c_m}^{d_m} u(y_1, \dots, y_m)\,dy_m \dots dy_1$. Let $N_i[\alpha_i, \beta_i] = \{\alpha_i, \alpha_i + 1, \dots, \beta_i\}$ $(\alpha_i < \beta_i)$, $\alpha_i, \beta_i \in N_0$ for $i = 1, 2, \dots, m$, $\Omega = \prod_{i=1}^{m} N_i[\alpha_i, \beta_i] \subset \mathbb{R}^m$, $H = N_0 \times \Omega$ and for any function z defined on N_0 we define the operator Δ by $\Delta z(n) = z(n+1) - z(n)$. For any function Ω we define the m-fold sum over Ω by

$$\sum_{\Omega} w(y) = \sum_{y_1=\alpha_1}^{\beta_1} \cdots \sum_{y_m=\alpha_m}^{\beta_m} w(y_1, \dots, y_m).$$

Clearly $\sum_\Omega w(y) = \sum_\Omega w(x)$ for $x, y \in \Omega$. We denote by $C(S_1, S_2)$ and $D(S_1, S_2)$ respectively the class of continuous and discrete functions from the set S_1 to the set S_2. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively and assume that all integrals, sums and products involved exist and are finite.

Our main results are given in the following theorem.

Theorem 2.1. *Let $u, p, q, f \in C(E, \mathbb{R}_+)$ and $c \geq 0$ be a constant.*

(A1) Let $L \in C(E \times \mathbb{R}_+, \mathbb{R}_+)$ be such that

$$0 \leq L(t, x, u) - L(t, x, v) \leq M(t, x, v)(u - v), \tag{2.1}$$

for $u \geq v \geq 0$, where $M \in C(E \times \mathbb{R}_+, \mathbb{R}_+)$. If

$$u(t, x) \leq p(t, x) + q(t, x) \int_0^t \int_0^s \int_B L(\tau, y, u(\tau, y))\,dy\,d\tau\,ds, \tag{2.2}$$

for $(t, x) \in E$, then for $(t, x) \in E$,

$$\begin{aligned} u(t, x) \leq p(t, x) &+ q(t, x) \Big(\int_0^t \int_0^s \int_B L(\tau, y, p(\tau, y))\,dy\,d\tau\,ds \Big) \\ &\times \exp\Big(\int_0^t \int_0^s \int_B M(\tau, y, p(\tau, y)) q(\tau, y)\,dy\,d\tau\,ds \Big). \end{aligned} \tag{2.3}$$

(A2) Let $g \in C(\mathbb{R}_+, \mathbb{R}_+)$ be a nondecreasing function, $g(u) > 0$ on $(0, \infty)$. If

$$u(t, x) \leq c + \int_0^t \int_0^s \int_B f(\tau, y) g(u(\tau, y))\,dy\,d\tau\,ds, \tag{2.4}$$

for $(t,x) \in E$, *then for* $0 \le t \le t_1$; $t, t_1 \in \mathbb{R}_+$, $x \in B$,

$$u(t,x) \le W^{-1}\Big[W(c) + \int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds\Big], \tag{2.5}$$

where

$$W(r) = \int_{r_0}^r \frac{d\sigma}{g(\sigma)}, \quad r > 0, \tag{2.6}$$

$r_0 > 0$ *is arbitrary and* W^{-1} *is the inverse of* W *and* $t_1 \in \mathbb{R}_+$ *is chosen so that*

$$W(c) + \int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds \in \mathrm{Dom}(W^{-1}),$$

for all $t \in \mathbb{R}_+$ *lying in* $0 \le t \le t_1$ *and* $x \in B$.

(A3) If

$$u^2(t,x) \le c + \int_0^t \int_0^s \int_B f(\tau,y)u(\tau,y)\,dy\,d\tau\,ds, \tag{2.7}$$

for $(t,x) \in E$, *then*

$$u(t,x) \le \sqrt{c} + \frac{1}{2}\int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds, \tag{2.8}$$

for $(t,x) \in E$.

(A4) Let $g(u)$ *be as in part (A2). If*

$$u^2(t,x) \le c + \int_0^t \int_0^s \int_B f(\tau,y)u(\tau,y)g(u(\tau,y))\,dy\,d\tau\,ds, \tag{2.9}$$

for $(t,x) \in E$, *then for* $0 \le t \le t_2$; $t, t_2 \in \mathbb{R}_+$, $x \in B$,

$$u(t,x) \le W^{-1}\Big[W(\sqrt{c}) + \frac{1}{2}\int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds\Big], \tag{2.10}$$

where W, W^{-1} *are as in part (A2) and* $t_2 \in \mathbb{R}_+$ *is chosen so that*

$$W(\sqrt{c}) + \frac{1}{2}\int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds \in \mathrm{Dom}(W^{-1}),$$

for all $t \in \mathbb{R}_+$ *lying in* $0 \le t \le t_2$ *and* $x \in B$.

(A5) Suppose that $u \in C(E, \mathbb{R}_1)$, $c \ge 1$. *If*

$$u(t,x) \le c + \int_0^t \int_0^s \int_B f(\tau,y)u(\tau,y)\log u(\tau,y)\,dy\,d\tau\,ds, \tag{2.11}$$

for $(t,x) \in E$, *then*

$$u(t,x) \le c^{\exp\left(\int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds\right)}, \tag{2.12}$$

for $(t,x) \in E$.

(A6) Let $u \in C(E, \mathbb{R}_1), c \ge 1$ *and* $g(u)$ *be as in (A2). If*

$$u(t,x) \le c + \int_0^t \int_0^s \int_B f(\tau,y)u(\tau,y)g(\log u(\tau,y))\,dy\,d\tau\,ds, \tag{2.13}$$

for $(t,x) \in E$, *then for* $0 \le t \le t_3$; $t, t_3 \in \mathbb{R}_+, x \in B$,

$$u(t,x) \le \exp\Big(W^{-1}\Big[W(\log c) + \int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds\Big]\Big), \tag{2.14}$$

where W, W^{-1} are as in part (A2) and $t_3 \in \mathbb{R}_+$ be chosen so that

$$W(\log c) + \int_0^t \int_0^s \int_B f(\tau, y)\, dy\, d\tau\, ds \in \operatorname{Dom}(W^{-1}),$$

for all $t \in \mathbb{R}_+$ lying in the interval $0 \le t \le t_3$ and $x \in B$.

The discrete analogues of the inequalities in Theorem 2.1 are given as follows.

Theorem 2.2. *Let $u, p, q, f \in D(H, \mathbb{R}_+)$ and $c \ge 0$ is a constant.*

(B1) Let $L \in D(H \times \mathbb{R}_+, \mathbb{R}_+)$ be such that

$$0 \le L(n, x, u) - L(n, x, v) \le M(n, x, v)(u - v), \tag{2.15}$$

for $u \ge v \ge 0$, where $M \in D(H \times \mathbb{R}_+, \mathbb{R}_+)$. If

$$u(n, x) \le p(n, x) + q(n, x) \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} L(\tau, y, u(\tau, y)), \tag{2.16}$$

for $(n, x) \in H$, then for $(n, x) \in H$,

$$\begin{aligned} u(n, x) \le p(n, x) + q(n, x) \Big(\sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} L(\tau, y, p(\tau, y)) \Big) \\ \times \prod_{s=0}^{n-1} \Big[1 + \sum_{\tau=0}^{s-1} \sum_{\Omega} M(\tau, y, p(\tau, y)) q(\tau, y) \Big]. \end{aligned} \tag{2.17}$$

(B2) Let $g(u)$ be as in Theorem 2.1 part (A2). If

$$u(n, x) \le c + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y) g(u(\tau, y)), \tag{2.18}$$

for $(n, x) \in H$, then for $0 \le n \le n_1$; $n, n_1 \in N_0, x \in \Omega$,

$$u(n, x) \le W^{-1} \Big[W(c) + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y) \Big], \tag{2.19}$$

where W, W^{-1} are as in Theorem 2.1 part (A2) and $n_1 \in N_0$ be chosen so that

$$W(c) + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y) \in \operatorname{Dom}(W^{-1}),$$

for all $n \in N_0$ lying in $0 \le n \le n_1$ and $x \in \Omega$.

(B3) If

$$u^2(n, x) \le c + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y) u(\tau, y), \tag{2.20}$$

for $(n, x) \in H$, then

$$u(n, x) \le \sqrt{c} + \frac{1}{2} \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y), \tag{2.21}$$

for $(n, x) \in H$.

(B4) Let $g(u)$ be as in Theorem 2.1 part (A2). If

$$u^2(n, x) \le c + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} f(\tau, y) u(\tau, y) g(u(\tau, y)), \tag{2.22}$$

for $(n,x)\in H$, then for $0\le n\le n_2$; $n,n_2\in N_0$, $x\in\Omega$,

$$u(n,x)\le W^{-1}[W(\sqrt{c})+\frac{1}{2}\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)], \tag{2.23}$$

where W, W^{-1} are as in Theorem 2.1 part (A2) and $n_2\in N_0$ be chosen so that

$$W(\sqrt{c})+\frac{1}{2}\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)\in \mathrm{Dom}(W^{-1}),$$

for all $n\in N_0$ lying in $0\le n\le n_2$ and $x\in\Omega$.

(B5) Suppose that $u\in D(H,\mathbb{R}_1)$, $c\ge 1$. If

$$u(n,x)\le c+\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)u(\tau,y)\log u(\tau,y), \tag{2.24}$$

for $(n,x)\in H$, then

$$u(n,x)\le c^{\prod_{s=0}^{n-1}[1+\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)]}, \tag{2.25}$$

for $(n,x)\in H$.

(B6) Let $u\in D(H,\mathbb{R}_1)$, $c\ge 1$ and $g(u)$ be as in Theorem 2.1 part (A2). If

$$u(n,x)\le c+\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)u(\tau,y)g(\log u(\tau,y)), \tag{2.26}$$

for $(n,x)\in H$, then for $0\le n\le n_3$; $n,n_3\in N_0$, $x\in\Omega$,

$$u(n,x)\le \exp(W^{-1}[W(\log c)+\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)]), \tag{2.27}$$

where W, W^{-1} are as in Theorem 2.1 part (A2) and $n_3\in N_0$ is chosen so that

$$W(\log c)+\sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1}\sum_{\Omega}f(\tau,y)\in \mathrm{Dom}(W^{-1}),$$

for all $n\in N_0$ lying in $0\le n\le n_3$ and $x\in\Omega$.

Remark 2.3. We note that the inequalities given in Theorems 2.1 and 2.2 can be considered as new variants of the similar inequalities given in [5, 6, 7] and they can be used as tools in certain situations in which the earlier inequalities are not directly applicable.

3. Proofs of Theorems 2.1 and 2.2

The proofs resemble one another, we give the details for (A1)–(A4) and (B5)–(B6) only. The proofs of (A5), (A6) and (B1)–(B4) can be completed by following the proofs of the above noted inequalities and closely looking at the similar results given in [5, 6]. To prove (A1)–(A4), it is sufficient to assume that $c>0$, since the standard limiting argument can be used to treat the remaining case, see [5, p. 108].

Prof of (A1). Setting

$$e(\tau)=\int_B L(\tau,y,u(\tau,y))\,dy, \tag{3.1}$$

the inequality (2.2) can be restated as

$$u(t,x) \le p(t,x) + q(t,x)\int_0^t \int_0^s e(\tau)\,d\tau\,ds. \tag{3.2}$$

Define

$$z(t) = \int_0^t \int_0^s e(\tau)\,d\tau\,ds, \tag{3.3}$$

then, it is easy to see that $z(0)=0, z'(0)=0$ and

$$u(t,x) \le p(t,x) + q(t,x)z(t). \tag{3.4}$$

From (3.3), (3.1), (3.4) and (2.1), we observe that

$$\begin{aligned} z''(t) = e(t) &= \int_B L(t,y,u(t,y))\,dy \\ &\le \int_B \{L(t,y,p(t,y)+q(t,y)z(t)) - L(t,y,p(t,y))\}\,dy + \int_B L(t,y,p(t,y))\,dy \\ &\le z(t)\int_B M(t,y,p(t,y))q(t,y)\,dy + \int_B L(t,y,p(t,y))\,dy. \end{aligned} \tag{3.5}$$

From (3.5) and using the fact that $z(t)$ is nondecreasing in $t \in \mathbb{R}_+$, it is easy to see that

$$\begin{aligned} z(t) \le &\int_0^t \int_0^s \int_B L(\tau,y,p(\tau,y))\,dy\,d\tau\,ds \\ &+ \int_0^t z(s)\Big\{\int_0^s \int_B M(\tau,y,p(\tau,y))q(\tau,y)\,dy\,d\tau\Big\}ds. \end{aligned} \tag{3.6}$$

Clearly, the first term on the right hand side in (3.6) is nonnegative and nondecreasing in $t \in \mathbb{R}_+$. Now a suitable application of the inequality in [5, Theorem 1.3.1] to (3.6) yields

$$\begin{aligned} z(t) \le &\Big(\int_0^t \int_0^s \int_B L(\tau,y,p(\tau,y))\,dy\,d\tau\,ds\Big) \\ &\times \exp\Big(\int_0^t \int_0^s \int_B M(\tau,y,p(\tau,y))q(\tau,y)\,dy\,d\tau\,ds\Big). \end{aligned} \tag{3.7}$$

Using (3.7) in (3.4), we get the required inequality in (2.3).

Proof of (A2). Setting

$$\bar{e}(\tau) = \int_B f(\tau,y)g(u(\tau,y))\,dy, \tag{3.8}$$

the inequality (2.4) can be restated as

$$u(t,x) \le c + \int_0^t \int_0^s \bar{e}(\tau)\,d\tau\,ds. \tag{3.9}$$

Defining $z(t)$ by the right hand side of (3.9) and following the proof part (A1) given above, we get

$$z''(t) = \bar{e}(t) = \int_B f(t,y)g(u(t,y))\,dy \le g(z(t))\int_B f(t,y)\,dy. \tag{3.10}$$

From (3.10) and using the fact that $z(t)$ is nondecreasing in $t \in \mathbb{R}_+$, it is easy to see that

$$z(t) \leq c + \int_0^t g(z(s))\Big\{\int_0^s \int_B f(\tau, y)\,dy\,d\tau\Big\}ds. \tag{3.11}$$

Now a suitable application of the inequality in [5, Theorem 2.3.1] to (3.11) yields

$$z(t) \leq W^{-1}[W(c) + \int_0^t \int_0^s \int_B f(\tau, y)\,dy\,d\tau\,ds]. \tag{3.12}$$

Using (3.12) in $u(t, x) \leq z(t)$, we get the required inequality in (2.5).

Proof of (A3). Setting

$$E(\tau) = \int_B f(\tau, y)u(\tau, y)\,dy, \tag{3.13}$$

the inequality (2.7) can be restated as

$$u^2(t, x) \leq c + \int_0^t \int_0^s E(\tau)\,d\tau\,ds. \tag{3.14}$$

Define $z(t)$ by the right hand side of (3.14), then $z(0) = c, z'(0) = 0$ and $u(t, x) \leq \sqrt{z(t)}$. Following the proof of part (A1), we get

$$z''(t) = E(t) = \int_B f(t, y)u(t, y)\,dy \leq \sqrt{z(t)} \int_B f(t, y)\,dy. \tag{3.15}$$

By taking $t = \tau$ in (3.15) and integrating it over τ from 0 to t and using the fact that $z(t)$ is nondecreasing in $t \in \mathbb{R}_+$, we get

$$z'(t) \leq \sqrt{z(t)} \int_0^t \int_B f(\tau, y)\,dy\,d\tau. \tag{3.16}$$

The inequality (3.16) implies (see [5, p. 233])

$$\sqrt{z(t)} \leq \sqrt{c} + \frac{1}{2} \int_0^t \int_0^s \int_B f(\tau, y)\,dy\,d\tau\,ds. \tag{3.17}$$

The required inequality in (2.8) follows by using (3.17) in $u(t, x) \leq \sqrt{z(t)}$.

Proof of (A4). Setting

$$\bar{E}(\tau) = \int_B f(\tau, y)u(\tau, y)g(u(\tau, y))\,dy, \tag{3.18}$$

the inequality (2.9) can be restated as

$$u^2(t, x) \leq c + \int_0^t \int_0^s \bar{E}(\tau)\,d\tau\,ds. \tag{3.19}$$

Defining $z(t)$ by the right hand side of (3.19) and following the proof of part (A1), we get

$$z''(t) = \bar{E}(t) \leq \sqrt{z(t)}g(\sqrt{z(t)}) \int_B f(t, y)\,dy, \tag{3.20}$$

from which, it is easy to observe that

$$\frac{z'(t)}{\sqrt{z(t)}} \leq g(\sqrt{z(t)}) \int_0^t \int_B f(t, y)\,dy. \tag{3.21}$$

From (3.21), we get

$$\sqrt{z(t)} \leq \sqrt{c} + \frac{1}{2}\int_0^t g(\sqrt{z(s)})\Big\{\int_0^s \int_B f(\tau, y)\,dy\,d\tau\Big\}ds. \tag{3.22}$$

Now a suitable application of the inequality in [5, Theorem 2.3.1] to (3.22) yields

$$\sqrt{z(t)} \leq W^{-1}\Big[W(\sqrt{c}) + \frac{1}{2}\int_0^t \int_0^s \int_B f(\tau, y)\,dy\,d\tau\,ds\Big]. \tag{3.23}$$

Using (3.23) in $u(t,x) \leq \sqrt{z(t)}$, we get (2.10).

Proof of (B5). Setting

$$r(\tau) = \sum_{\Omega} f(\tau, y)u(\tau, y)\log u(\tau, y), \tag{3.24}$$

the inequality (2.24) can be restated as

$$u(n, x) \leq c + \sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1} r(\tau). \tag{3.25}$$

Define

$$z(n) = c + \sum_{s=0}^{n-1}\sum_{\tau=0}^{s-1} r(\tau), \tag{3.26}$$

then $z(0) = c, \Delta z(0) = 0$ and

$$u(n, x) \leq z(n). \tag{3.27}$$

From (3.26), (3.24), (3.27), we observe that

$$\Delta^2 z(n) = r(n) = \sum_{\Omega} f(n, y)u(n, y)\log u(n, y) \leq z(n)\log z(n)\sum_{\Omega} f(n, y). \tag{3.28}$$

From the above inequality and using the fact that $z(n)$ is nondecreasing in $n \in N_0$, we get

$$\Delta z(n) \leq \sum_{\tau=0}^{n-1} z(\tau)\log z(\tau)\sum_{\Omega} f(\tau, y) \leq z(n)\{\log z(n)\sum_{\tau=0}^{n-1}\sum_{\Omega} f(\tau, y)\}. \tag{3.29}$$

Now a suitable application of the inequality in [6, Theorem 1.2.1] to (3.29) yields

$$\begin{aligned} z(n) &\leq c\prod_{s=0}^{n-1}[1 + \log z(s)\sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau, y)] \\ &\leq c\exp(\sum_{s=0}^{n-1}\log z(s)\sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau, y)). \end{aligned} \tag{3.30}$$

From (3.30), we observe that

$$\log z(n) \leq \log c + \sum_{s=0}^{n-1}\log z(s)\sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau, y). \tag{3.31}$$

Now an application of the inequality in [6, Theorem 1.2.2] to (3.31) yields

$$\log z(n) \leq (\log c)\prod_{s=0}^{n-1}[1 + \sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau, y)] = \log c^{\prod_{s=0}^{n-1}[1+\sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau,y)]}. \tag{3.32}$$

From (3.32), we observe that

$$z(n) \le c^{\prod_{s=0}^{n-1}\left[1+\sum_{\tau=0}^{s-1}\sum_{\Omega} f(\tau,y)\right]}. \tag{3.33}$$

Using (3.33) in (3.27), we get the required inequality in (2.25).

Proof of (B6). The proof can be completed by setting

$$\bar{r}(\tau) = \sum_{\Omega} f(\tau,y)u(\tau,y)g(\log u(\tau,y)), \tag{3.34}$$

and following the proof of (B5) and closely looking at the proof of the inequality in [6, Theorem 3.5.3]. Here, we omit the details.

4. Some applications

Inspired by equation (1.1), we consider the general integral equation

$$u(t,x) = h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u(\tau,y))\,dy\,d\tau\,ds, \tag{4.1}$$

where h, F are given functions and u is the unknown function to be found. We assume that $h \in C(E,R)$, $F \in C(E^2 \times R, R)$. The problem of existence of solutions for (4.1) can be dealt with the method employed in [8], see also [1, 2, 3, 4, 10] for similar results. In this section, by using a particular case of the inequality in Theorem 2.1 part (A1), we offer the conditions for the error evaluation of approximate solutions of equation (4.1) by establishing some new estimates on solutions of approximate problems. We also study the dependency of solutions of equations of the form (4.1) on parameters.

We use the following special form of the inequality in Theorem 2.1 part (A1) when $L(t,x,u) = f(t,x)u$ in the proofs of our results.

Corollary 4.1. *Let $u, p, q, f \in C(E,R)$. If*

$$u(t,x) \le p(t,x) + q(t,x)\int_0^t \int_0^s \int_B f(\tau,y)u(\tau,y)\,dy\,d\tau\,ds, \tag{4.2}$$

for $(t,x) \in E$, then for $(t,x) \in E$,

$$\begin{aligned} u(t,x) &\le p(t,x) + q(t,x)\Big(\int_0^t \int_0^s \int_B f(\tau,y)p(\tau,y)\,dy\,d\tau\,ds\Big) \\ &\quad \times \exp\Big(\int_0^t \int_0^s \int_B f(\tau,y)q(\tau,y)\,dy\,d\tau\,ds\Big). \end{aligned} \tag{4.3}$$

We call the function $u \in C(E,R)$ an ε-approximate solution to (4.1) if there exists a constant $\varepsilon \ge 0$ such that

$$\Big|u(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u(\tau,y))\,dy\,d\tau\,ds\Big\}\Big| \le \varepsilon, \tag{4.4}$$

for $(t,x) \in E$.

The following theorem deals with the estimate on the difference between the two approximate solutions of (4.1).

Theorem 4.2. *Let $u_i(t,x)(i = 1,2)$ be respectively ε_i-approximate solutions of (4.1) on E. Suppose that the function F in equation (4.1) satisfies the condition*

$$|F(t,x,\tau,y,u) - F(t,x,\tau,y,v)| \le q(t,x)f(\tau,y)|u-v|, \tag{4.5}$$

where $q, f \in C(E, \mathbb{R}_+)$. *Then*

$$
\begin{aligned}
|u_1(t,x) - u_2(t,x)| \le (\varepsilon_1 + \varepsilon_2)\Big[1 + q(t,x)\Big(\int_0^t \int_0^s \int_B f(\tau,y)\,dy\,d\tau\,ds\Big) \\
\times \exp\Big(\int_0^t \int_0^s \int_B f(\tau,y) q(\tau,y)\,dy\,d\tau\,ds\Big)\Big],
\end{aligned}
\tag{4.6}
$$

for $(t,x) \in E$.

Proof. Since $u_i(t,x)(i = 1,2)$ for $(t,x) \in E$ are respectively ε_i-approximate solutions of (4.1), we have

$$
\Big|u_i(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u_i(\tau,y))\,dy\,d\tau\,ds\Big\}\Big| \le \varepsilon_i. \tag{4.7}
$$

From (4.7) and using the elementary inequalities,

$$
|v - z| \le |v| + |z|, \quad |v| - |z| \le |v - z|, \tag{4.8}
$$

we observe that

$$
\begin{aligned}
\varepsilon_1 + \varepsilon_2 &\ge \Big|u_1(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u_1(\tau,y))\,dy\,d\tau\,ds\Big\}\Big| \\
&\quad + \Big|u_2(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u_2(\tau,y))\,dy\,d\tau\,ds\Big\}\Big| \\
&\ge \Big|\Big\{u_1(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u_1(\tau,y))\,dy\,d\tau\,ds\Big\}\Big\} \\
&\quad - \Big\{u_2(t,x) - \Big\{h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u_2(\tau,y))\,dy\,d\tau\,ds\Big\}\Big\}\Big| \\
&\ge |u_1(t,x) - u_2(t,x)| - \Big|\int_0^t \int_0^s \int_B \Big\{F(t,x,\tau,y,u_1(\tau,y)) \\
&\quad - F(t,x,\tau,y,u_2(\tau,y))\Big\}\,dy\,d\tau\,ds\Big|.
\end{aligned}
\tag{4.9}
$$

Let $w(t,x) = |u_1(t,x) - u_2(t,x)|$, $(t,x) \in E$. From (4.9) and using (4.5), we observe that

$$
\begin{aligned}
&w(t,x) \\
&\le (\varepsilon_1 + \varepsilon_2) + \int_0^t \int_0^s \int_B |F(t,x,\tau,y,u_1(\tau,y)) - F(t,x,\tau,y,u_2(\tau,y))|\,dy\,d\tau\,ds \\
&\le (\varepsilon_1 + \varepsilon_2) + q(t,x) \int_0^t \int_0^s \int_B f(\tau,y) w(\tau,y)\,dy\,d\tau\,ds.
\end{aligned}
\tag{4.10}
$$

Now an application of Corollary 4.1 yields (4.6). □

Remark 4.3. When $u_1(t,x)$ is a solution of (4.1), we have $\varepsilon_1 = 0$ and from (4.6), we see that $u_2(t,x) \to u_1(t,x)$ as $\varepsilon_2 \to 0$. Furthermore, if we put $\varepsilon_1 = \varepsilon_2 = 0$ in (4.6), then the uniqueness of solutions of (4.1) is established.

Consider the equation (4.1) with

$$
v(t,x) = \bar{h}(t,x) + \int_0^t \int_0^s \int_B \bar{F}(t,x,\tau,y,v(\tau,y))\,dy\,d\tau\,ds, \tag{4.11}
$$

where $\bar{h} \in C(E, R)$, $\bar{F} \in C(E^2 \times R, R)$.

In the next theorem we provide conditions concerning the closeness of solutions of (4.1) and (4.11).

Theorem 4.4. *Suppose that F in equation (4.1) satisfies (4.5) and there exist constants $\delta_i \geq 0$ $(i = 1, 2)$ such that*

$$|h(t,x) - \bar{h}(t,x)| \leq \delta_1, \tag{4.12}$$

$$\int_0^t \int_0^s \int_B \big|F(t,x,\tau,y,z) - \bar{F}(t,x,\tau,y,z)\big|\, dy\, d\tau\, ds \leq \delta_2, \tag{4.13}$$

where F, h and $\bar{F}, \bar{h}$ are given as in (4.1) and (4.11). Let $u(t,x)$ and $v(t,x)$, for $(t,x) \in E$, be solutions of (4.1) and (4.11), respectively. Then for $(t,x) \in E$,

$$\begin{aligned} |u(t,x) - v(t,x)| \leq (\delta_1 + \delta_2)\Big[1 + q(t,x) \int_0^t \int_0^s \int_B f(\tau,y)\, dy\, d\tau\, ds \\ \times \exp\Big(\int_0^t \int_0^s \int_B f(\tau,y) q(\tau,y)\, dy\, d\tau\, ds\Big)\Big]. \end{aligned} \tag{4.14}$$

Proof. Let $r(t,x) = |u(t,x) - v(t,x)|$, $(t,x) \in E$. Using the facts that $u(t,x), v(t,x)$ are solutions of equations (4.1), (4.11) and the hypotheses, we have

$$\begin{aligned} r(t,x) &\leq \big|h(t,x) - \bar{h}(t,x)\big| \\ &\quad + \int_0^t \int_0^s \int_B |F(t,x,\tau,y,u(\tau,y)) - F(t,x,\tau,y,v(\tau,y))|\, dy\, d\tau\, ds \\ &\quad + \int_0^t \int_0^s \int_B |F(t,x,\tau,y,v(\tau,y)) - \bar{F}(t,x,\tau,y,v(\tau,y))|\, dy\, d\tau\, ds \\ &\leq (\delta_1 + \delta_2) + q(t,x) \int_0^t \int_0^s \int_B f(\tau,y) r(\tau,y)\, dy\, d\tau\, ds. \end{aligned} \tag{4.15}$$

Now an application of Corollary 4.1 yields (4.14). □

Remark 4.5. The result given in Theorem 4.4 relates the solutions of (4.1) and (4.11) in the sense that if F is close to $\bar{F}$ and h is close to $\bar{h}$, then the solutions of (4.1) and of (4.11) are also close to each other.

A slight variation of Theorem 4.4 is embodied in the following theorem.

Theorem 4.6. *Suppose that F and $\bar{F}$ in (4.1) and (4.11) satisfy the condition*

$$\big|F(t,x,\tau,y,u) - \bar{F}(t,x,\tau,y,v)\big| \leq \bar{q}(t,x)\bar{f}(\tau,y)|u - v|, \tag{4.16}$$

where $\bar{q}, \bar{f} \in C(E, \mathbb{R}_+)$ and (4.12) holds. Let $u(t,x)$ and $v(t,x)$ be solutions of (4.1) and (4.11), respectively, on E. Then for $(t,x) \in E$,

$$\begin{aligned} |u(t,x) - v(t,x)| \leq \delta_1 \Big[1 + \bar{q}(t,x)\Big(\int_0^t \int_0^s \int_B \bar{f}(\tau,y)\, dy\, d\tau\, ds\Big) \\ \times \exp\Big(\int_0^t \int_0^s \int_B \bar{f}(\tau,y)\bar{q}(\tau,y)\, dy\, d\tau\, ds\Big)\Big]. \end{aligned} \tag{4.17}$$

The proof of the above theorem is similar to that of Theorem 4.4, with suitable modifications, and hence we omit the details.
We next consider the equations

$$u(t,x) = h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u(\tau,y),\mu)\,dy\,d\tau\,ds, \tag{4.18}$$

$$u(t,x) = h(t,x) + \int_0^t \int_0^s \int_B F(t,x,\tau,y,u(\tau,y),\mu_0)\,dy\,d\tau\,ds, \tag{4.19}$$

for $(t,x) \in E$, where $h \in C(E,R)$, $F \in C(E \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and μ, μ_0 are parameters. The following theorem shows the dependency of solutions of (4.18) and (4.19) on parameters.

Theorem 4.7. *Suppose that F in* (4.18), (4.19) *satisfies the conditions*

$$|F(t,x,\tau,y,u,\mu) - F(t,x,\tau,y,\bar{u},\mu)| \le q_0(t,x) f_0(\tau,y)|u - \bar{u}|, \tag{4.20}$$

$$|F(t,x,\tau,y,u,\mu) - F(t,x,\tau,y,u,\mu_0)| \le N|\mu - \mu_0|, \tag{4.21}$$

where $q_0, f_0 \in C(E, \mathbb{R}_+)$ and $N \ge 0$ is a constant. Let $u_1(t,x)$ and $u_2(t,x)$ be the solutions of (4.18) *and* (4.19) *respectively. Then for $(t,x) \in E$,*

$$\begin{aligned} |u_1(t,x) - u_2(t,x)| \le N|\mu - \mu_0| \Big[1 + q_0(t,x) \Big(\int_0^t \int_0^s \int_B f_0(\tau,y)\,dy\,d\tau\,ds \Big) \\ \times \exp\Big(\int_0^t \int_0^s \int_B f_0(\tau,y) q_0(\tau,y)\,dy\,d\tau\,ds \Big) \Big]. \end{aligned} \tag{4.22}$$

Proof. Let $z(t,x) = |u_1(t,x) - u_2(t,x)|$, $(t,x) \in E$. Using the facts that $u_1(t,x)$ and $u_2(t,x)$ are respectively the solutions of (4.18) and (4.19), and the hypotheses, we have

$$\begin{aligned} z(t,x) &\le \int_0^t \int_0^s \int_B |F(t,x,\tau,y,u_1(\tau,y),\mu) - F(t,x,\tau,y,u_2(\tau,y),\mu)|\,dy\,d\tau\,ds \\ &\quad + \int_0^t \int_0^s \int_B |F(t,x,\tau,y,u_2(\tau,y),\mu) - F(t,x,\tau,y,u_2(\tau,y),\mu_0)|\,dy\,d\tau\,ds \\ &\le N|\mu - \mu_0| + q_0(t,x) \int_0^t \int_0^s \int_B f_0(\tau,y) z(\tau,y)\,dy\,d\tau\,ds. \end{aligned} \tag{4.23}$$

Now an application of Corollary 4.1 yields (4.22), which shows the dependency of solutions of (4.18) and (4.9) on parameters. □

Remark 4.8. We can use the special version of the inequality in Theorem 2.2 part (B1) (as in Corollary 4.1) to establish results similar to those given above for the solutions of sum-difference equation of the form

$$u(n,x) = h(n,x) + \sum_{s=0}^{n-1} \sum_{\tau=0}^{s-1} \sum_{\Omega} F(n,x,\tau,y,u(\tau,y)), \tag{4.24}$$

where $h \in D(H,R)$, $F \in D(H^2 \times R, R)$. We also note that, many generalizations, extensions, variants and applications of the inequalities given in this paper are possible. However, we shall not pursue the detailed treatment here.

References

[1] G. Andrews; *On existence of solution to the equation* $u_{tt} = u_{xxt} + \sigma(u_x)_x$, J. Differential Equations 35(1980), 200-231.
[2] C. Corduneanu; *Integral Equations and Applications*, Cambridge University Press, Cambridge, 1991.
[3] R. B. Guenther and J. W. Lee; *Partial Differential Equations*, Prentice-Hall, Englewood, Cliffs, N. J., 1998.
[4] V. P. Mikhailov; *Partial Differential Equations*, Mir Publishers, Moscow, 1978.
[5] B. G. Pachpatte; *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.
[6] B. G. Pachpatte; *Inequalities for Finite Difference Equations*, Marcel Dekker, Inc., New York, 2002.
[7] B. G. Pachpatte; *Integral and Finite Difference Inequalities and Applications*, North-Holland Mathematics Studies, Vol. 205, Elsevier Science B. V., Amsterdam, 2006.
[8] B. G. Pachpatte; *On mixed Volterra-Fredholm type integral equations*, Indian J. Pure and Appl. Math. 17(1986), 488-496.
[9] B. G. Pachpatte; *On mixed problem for a class of nonlinear partial differential equations*, J. Natural and Phys. Sci. 13(1999), 121-132.
[10] W. Walter; *Differential and Integral Inequalities*, Springer-Verlag, Berlin, New York, 1970.

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