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BEHAVIOR AT INFINITY OF ψ-EVANESCENT SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In this article we present some necessary and sufficient conditions for the existence of ψ-evanescent solution of the nonhomogeneous linear differential equation $x' = A(t)x + f(t)$, which is related to the notion of ψ-ordinary dichotomy for the equation $x' = A(t)x$. We associate that with the condition of ψ-ordinary dichotomy for the homogeneous linear differential equation $x' = A(t)x$.

1. Introduction

The existence of ψ-bounded and ψ-stable solutions on $\mathbb{R}_+$ for systems of ordinary differential equations has been studied by many authors; such as Akinyele [1], Avramescu [2], Boi [4, 5], Constantin [6], Diamandescu [9, 10, 11]. Also, in [5, 9, 10, 11] the authors prove several sufficient conditions of the ψ-evanescence at ∞, $-\infty$ for the solutions of linear differential equations.

The purpose of this paper is to provide a condition for the existence of ψ- evanescent solution of the equations $x' = A(t)x + f(t)$, which is concerned with the notion of ψ-ordinary dichotomy for the equation $x' = A(t)x$. We shall deal with the existence of ψ-evanescent solution of nonhomogeneous equations, which have been studied in recent works, such as [5, 9, 11].

Denote by $\mathbb{R}^d$ the d-dimensional Euclidean space. Elements in this space are denoted by $x = (x_1, x_2, \dots, x_d)^T$ and their norm by $\|x\| = \max\{|x_1|, |x_2|, \dots, |x_d|\}$. For real $d\times d$ matrices A, we define the norm $|A| = \sup_{\|x\|\leqslant 1} \|Ax\|$. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_- = (-\infty, 0]$, $J = \mathbb{R}_-$,$J = \mathbb{R}_+$ or $J = \mathbb{R}$. Let $\psi_i : J \to (0, \infty)$, $i = 1, 2, \dots, d$ be continuous functions and let

$$\psi = \operatorname{diag}\{\psi_1, \psi_2, \dots, \psi_d\}.$$

Definition 1.1. A function $f : J \to \mathbb{R}^d$ is said to be

- ψ-bounded on J if ψf is bounded on J.
- ψ-integrable on J if f is measurable and ψf is Lebesgue integrable on J.

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In $\mathbb{R}^d$, consider the following equations on J.

$$x' = A(t)x + f(t), \tag{1.1}$$

$$x' = A(t)x. \tag{1.2}$$

where $A(t)$ is a continuous $d \times d$ matrix function and $f(t)$ is a continuous function for $t \in J$.

By a solution of (1.1), we mean a continuous function satisfying (1.1) for almost t in J. Let $Y(t)$ be the fundamental matrix of (1.2) with $Y(0) = I_d$, the identity $d \times d$ matrix. A $d \times d$ matrix P is said to be a projection matrix if $P^2 = P$. If P is a projection, then so is $I_d - P$. Two projections $P, I_d - P$ are called supplementary.

Definition 1.2. Equation (1.2) is said to have a ψ-ordinary dichotomy on J if there exist positive constants K, L and two supplementary projections P_1, P_2 such that

$$|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leqslant K \quad \text{for } s \leqslant t; s,t \in J, \tag{1.3}$$

$$|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)| \leqslant L \quad \text{for } t \leqslant s; s,t \in J. \tag{1.4}$$

Also we say that (1.2) has a ψ-ordinary dichotomy on J with two supplementary projections P_1, P_2.

Remark 1.3. It is easily verified that if (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ and on $\mathbb{R}_-$ with two supplementary projections P_1, P_2 then (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}$ with two supplementary projections P_1, P_2. Note that for $\psi = I_d$, we obtain the notion of ordinary dichotomy (see [7, 8])

Theorem 1.4 ([4, 9]). *(a) Equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_+$ for every ψ-integrable function f on $\mathbb{R}_+$ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$.*

(b) Suppose that (1.2) *has a ψ-ordinary dichotomy and* $\lim_{t\to\infty}|\psi(t)Y(t)P_1| = 0$. *Let f be a ψ-integrable function on $\mathbb{R}_+$. Then every ψ-bounded solution x(t) of* (1.1) *on $\mathbb{R}_+$ is such that* $\lim_{t\to\infty}\|\psi(t)x(t)\| = 0$.

2. Preliminaries

Lemma 2.1. *Equation* (1.2) *has a ψ-ordinary dichotomy on J with two supplementary projections P_1, P_2 if and only if two following conditions are satisfied for all $\xi \in \mathbb{R}^d$:*

$$\|\psi(t)Y(t)P_1\xi\| \leqslant K\|\psi(s)Y(s)\xi\| \quad \textit{for } s \leqslant t; s,t \in J \tag{2.1}$$

$$\|\psi(t)Y(t)P_2\xi\| \leqslant L\|\psi(s)Y(s)\xi\| \quad \textit{for } t \leqslant s; s,t \in J \tag{2.2}$$

Proof. If (1.2) has a ψ-ordinary dichotomy on J then

$$\|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)y\| \leqslant K\|y\| \quad \text{for } s \leqslant t; s,t \in J \tag{2.3}$$

$$\|\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)y\| \leqslant L\|y\| \quad \text{for } t \leqslant s; s,t \in J \tag{2.4}$$

for any vector $y \in \mathbb{R}^d$. Choose $y = \psi(s)Y(s)\xi$, we obtain (2.1), (2.2). Conversely, if (2.1) (2.2) are true, for any vector $y \in \mathbb{R}^d$, putting $\xi = Y^{-1}(s)\psi^{-1}(s)y$ we get (2.3), (2.4). This implies that (1.2) has a ψ-ordinary dichotomy on J. The proof is complete. $\square$

Remark 2.2. If (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections P_1, P_2 then there exist positive constants K_P, L_P such that

$$\|\psi(t)Y(t)P_1\xi\| \leqslant K_P\|\xi\|, \quad \|\psi(t)Y(t)\xi\| \geqslant L_P\|P_2\xi\|$$

for all $\xi \in \mathbb{R}^d$, all $t \geqslant 0$.

Indeed, let $s = 0$, we deduce from (2.1) that $\|\psi(t)Y(t)P_1\xi\| \leqslant K\|\psi(0)\xi\| \leqslant K_P\|\xi\|$ for all $t \geqslant 0$, where $K_P = K|\psi(0)|$. Let $t = 0$, we deduce from (2.2) that $\|\psi(0)P_2\xi\| \leqslant L\|\psi(s)Y(s)\xi\|$, for all $s \geqslant 0$. Then $\|\psi(s)Y(s)\xi\| \geqslant L_P\|P_2\xi\|$, for all $s \geqslant 0$, where $L_P = [L|\psi^{-1}(0)|]^{-1}$.

Now, let $X_1 = \{u \in \mathbb{R}^d | u = x(0), x(t)$ is a ψ-bounded solution of (1.2) on $\mathbb{R}_+\}$ and let $X_0 = \{u \in \mathbb{R}^d | u = x(0), x(t)$ is a solution of (1.2) on $\mathbb{R}_+$ such that $\psi(t)x(t) \to 0$, as $t \to \infty$ $\}$.

Lemma 2.3. *If* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$ and Q_1, Q_2 are two supplementary projections, then* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections Q_1, Q_2 if and only if*

$$X_0 \subset Q_1\mathbb{R}^d \subset X_1 \tag{2.5}$$

Proof. The "only if" part. Suppose that (1.2) has a ψ-dichotomy with two supplementary projections Q_1, Q_2, we show that (2.5) holds. First, we prove $Q_1\mathbb{R}^d \subset X_1$. For any $u \in Q_1\mathbb{R}^d$, there exists $v \in \mathbb{R}^d$ such that $u = Q_1 v$. Let $y(t)$ be a solution of (1.2) such that $y(0) = u$. It follows from Remark 2.2 that

$$\|\psi(t)y(t)\| = \|\psi(t)Y(t)u\| = \|\psi(t)Y(t)Q_1 v\| \leqslant K_Q\|v\| \quad \text{for } t \geqslant 0,$$

where K_Q is a positive constant. This implies that $u \in X_1$. Hence $Q_1\mathbb{R}^d \subset X_1$. We prove $X_0 \subset Q_1\mathbb{R}^d$. For $u \in X_0$, let $x(t)$ be a solutions of (1.2) such that $x(0) = u$. It implies that

$$\|\psi(t)x(t)\| \to 0, \text{ as } t \to \infty \tag{2.6}$$

From Remark 2.2, we have

$$\|\psi(t)x(t)\| = \|\psi(t)Y(t)u\| \geqslant L_Q\|Q_2 u\|, \qquad \text{for } t \geqslant 0 \tag{2.7}$$

where L_Q is a positive constant. The relations (2.6) and (2.7) imply $Q_2 u = 0$, then $u \in Q_1\mathbb{R}^d$. Thus (2.5) holds.

We prove the "if" part. Suppose that (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections P_1, P_2. Let Q_1, Q_2 be two supplementary projections such that (2.5) holds. We will prove that (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections Q_1, Q_2. Let $\widetilde{Q}_1, \widetilde{Q}_2$ be two supplementary projections such that $\widetilde{Q}_1\mathbb{R}^d = X_0$. Applying (2.5) to P_1, P_2 we get $\widetilde{Q}_1\mathbb{R}^d = X_0 \subset P_1\mathbb{R}^d \subset X_1$. The set $X_0' = (P_1 - \widetilde{Q}_1)\mathbb{R}^d$ is a subset of $P_1\mathbb{R}^d$, supplementary to X_0. We will show that there exists a positive constant number N such that

$$\|\psi(t)Y(t)u\| \geqslant N\|u\|, \quad \text{for all } u \in X_0',\ t \geqslant 0 \tag{2.8}$$

In fact, otherwise there exists a sequence of unit vectors $\{v_n\} \subset X_0', n = 1, 2, \dots$ and a sequence of numbers $t_n \geqslant 0$ such that $\|\psi(t_n)Y(t_n)v_n\| \to 0$. By the compactness of the unit sphere in X_0', we may assume that $v_n \to v \in X_0'$ as $n \to \infty$, where v is a unit vector. By Remark 2.2 and $(v - v_n) \in X_0' \subset P_1\mathbb{R}^d$, we obtain

$$\|\psi(t_n)Y(t_n)(v - v_n)\| = \|\psi(t_n)Y(t_n)P_1(v - v_n)\| \leqslant K_P\|v - v_n\|$$

Letting $n \to \infty$, we obtain $\|\psi(t_n)Y(t_n)(v - v_n)\| \to 0$. Then $\|\psi(t_n)Y(t_n)v_n\| + \|\psi(t_n)Y(t_n)(v - v_n)\| \to 0$ as $t_n \to \infty$. Then $\|\psi(t_n)Y(t_n)v\| \to 0$ as $t_n \to \infty$. Hence $v \in X_0$. On the other hand, $v \in X_0'$, we have $v = 0$, which is a contradiction to the unit of v. Thus (2.8) holds.

From (2.8) and (2.1) we obtain

$$\begin{aligned} N\|(P_1 - \widetilde{Q}_1)u\| &\leqslant \|\psi(t)Y(t)(P_1 - \widetilde{Q}_1)u\| \\ &\leqslant \|\psi(t)Y(t)P_1u\| + \|\psi(t)Y(t)\widetilde{Q}_1u\| \\ &\leqslant K\|\psi(s)Y(s)u\| + \|\psi(t)Y(t)\widetilde{Q}_1u\| \end{aligned} \tag{2.9}$$

for $u \in \mathbb{R}^d$, $0 \leqslant s \leqslant t$. Let $t \to \infty$, we get $\|\psi(t)Y(t)\widetilde{Q}_1u\| \to 0$. From (2.9), we have

$$N\|(P_1 - \widetilde{Q}_1)u\| \leqslant \|\psi(s)Y(s)u\| \quad \text{for } s \geqslant 0 \tag{2.10}$$

From Remark 2.2 and (2.10), we have

$$\|\psi(t)Y(t)(P_1 - \widetilde{Q}_1)u\| \leqslant K_P\|(P_1 - \widetilde{Q}_1)u\| \leqslant K_PN^{-1}\|\psi(s)Y(s)u\| \quad \text{for } t, s \geqslant 0 \tag{2.11}$$

Consequently,

$$\begin{aligned} \|\psi(t)Y(t)\widetilde{Q}_1u\| &\leqslant \|\psi(t)Y(t)P_1u\| + \|\psi(t)Y(t)(P_1 - \widetilde{Q}_1)u\| \\ &\leqslant (K + K_PN^{-1})\|\psi(s)Y(s)u\| \quad \text{for } 0 \leqslant s \leqslant t \end{aligned} \tag{2.12}$$

From $\widetilde{Q}_2 = P_2 + P_1 - \widetilde{Q}_1$ and (2.11), we obtain

$$\begin{aligned} \|\psi(t)Y(t)\widetilde{Q}_2u\| &\leqslant \|\psi(t)Y(t)P_2u\| + \|\psi(t)Y(t)(P_1 - \widetilde{Q}_1)u\| \\ &\leqslant (L + K_PN^{-1})\|\psi(s)Y(s)u\| \quad \text{for } 0 \leqslant t \leqslant s \end{aligned} \tag{2.13}$$

From $\widetilde{Q}_1\mathbb{R}^d = X_0 \subset \mathbb{Q}_1\mathbb{R}^d \subset X_1$, we obtain $Q_2\widetilde{Q}_1\mathbb{R}^d \subset Q_2Q_1\mathbb{R}^d = 0$ then $Q_1\widetilde{Q}_1 = (I_d - Q_2)\widetilde{Q}_1 = \widetilde{Q}_1$. Thus

$$Q_1\widetilde{Q}_2 = Q_1(I_d - \widetilde{Q}_1) = Q_1 - \widetilde{Q}_1 \tag{2.14}$$

By the definition of X_1, there exists $N' > 0$ such that

$$\|\psi(t)Y(t)u\| \leqslant N'\|u\|, \text{ for } t \geqslant 0 \tag{2.15}$$

It follows from Lemma 2.1, (2.12), (2.13) that (2.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections $\widetilde{Q}_1, \widetilde{Q}_2$. By Remark 2.2 we have

$$\|\psi(s)Y(s)u\| \geqslant \widetilde{L}_Q\|\widetilde{Q}_2u\| \quad \text{for } s \geqslant 0\,.$$

Combining this inequality, (2.14) and (2.15) we obtain

$$\begin{aligned} \|\psi(t)Y(t)(Q_1 - \widetilde{Q}_1)u\| &\leqslant N'\|(Q_1 - \widetilde{Q}_1)u\| \\ &\leqslant N'\|Q_1\widetilde{Q}_2u\| \leqslant N'|Q_1|\|\widetilde{Q}_2u\| \\ &\leqslant K_2\|\psi(s)Y(s)u\|, \quad \text{for } t, s \geqslant 0 \end{aligned} \tag{2.16}$$

where K_2 is a positive constant. From (2.12), (2.16), we have

$$\begin{aligned} \|\psi(t)Y(t)Q_1u\| &\leqslant \|\psi(t)Y(t)\widetilde{Q}_1u\| + \|\psi(t)Y(t)(Q_1 - \widetilde{Q}_1)u\| \\ &\leqslant (K + K_PN^{-1} + K_2)\|\psi(s)Y(s)u\|, \quad \text{for } 0 \leqslant s \leqslant t \end{aligned} \tag{2.17}$$

From $Q_2 = \widetilde{Q}_2 + \widetilde{Q}_1 - Q_1$, (2.13) and (2.16), we obtain

$$\begin{aligned} \|\psi(t)Y(t)Q_2u\| &\leqslant \|\psi(t)Y(t)\widetilde{Q}_2u\| + \|\psi(t)Y(t)(\widetilde{Q}_1 - Q_1)u\| \\ &\leqslant (L + K_P N^{-1} + K_2)\|\psi(s)Y(s)u\|, \quad \text{for } 0 \leqslant t \leqslant s \end{aligned} \tag{2.18}$$

Lemma 2.1 and (2.17), (2.18) follow that (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections Q_1, Q_2. The proof is complete. □

Let $\widetilde{X}_1 = \{u \in \mathbb{R}^d | u = x(0), x(t)$ is a ψ-bounded solution of (1.2) on $\mathbb{R}_-$ $\}$, and let $\widetilde{X}_0 = \{u \in \mathbb{R}^d | u = x(0), x(t)$ is a solution of (1.2) on $\mathbb{R}_-$ such that $\psi(t)x(t) \to 0$, as $t \to -\infty$ $\}$. From Theorem 1.4 and Lemma 2.3, we obtain the following results on half-line $\mathbb{R}_-$.

Lemma 2.4. *(a) Equation* (1.1) *has at least one ψ-bounded solution on $\mathbb{R}_-$ for every ψ-integrable function f on $\mathbb{R}_-$ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_-$.*

(b) If (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_-$ and $\widetilde{Q}_1, \widetilde{Q}_2$ are two supplementary projections, then* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_-$ with two supplementary projections $\widetilde{Q}_1, \widetilde{Q}_2$ if and only if*

$$\widetilde{X}_0 \subset \widetilde{Q}_2\mathbb{R}^d \subset \widetilde{X}_1 \tag{2.19}$$

Proof. The proof of this Lemma is similar to that of Theorem 1.4 and Lemma 2.3 with the corresponding replacement ($t \geqslant s \geqslant 0$ with $0 \geqslant s \geqslant t$, P_1 with $-P_2, P_2$ with $-P_1$, ∞ with $-\infty$, $-\infty$ with ∞ ...). □

Definition 2.5. A function $x(t)$ is said to be

- ψ-evanescent at ∞ if $\lim_{t\to\infty} \|\psi(t)x(t)\| = 0$.
- ψ-evanescent at $-\infty$ if $\lim_{t\to-\infty} \|\psi(t)x(t)\| = 0$.
- ψ-evanescent at $\pm\infty$ if $\lim_{t\to\pm\infty} \|\psi(t)x(t)\| = 0$.

Note that for $\psi = I_d$, we obtain the notion of evanescent solution of (1.1) at $\pm\infty$ (see [3])

Lemma 2.6. *If* (1.1) *has at least one solution on $\mathbb{R}$, ψ-evanescent at ∞ for every ψ-integrable function f on $\mathbb{R}$ then every solution of* (1.2) *is the sum of two solution of* (1.2)*, one of which is ψ-bounded on $\mathbb{R}_-$, and the other is defined on $\mathbb{R}_+$, ψ-evanescent at ∞.*

Proof. Set

$$h(t) = \begin{cases} 0 & \text{for } |t| \geqslant 1 \\ 1 & \text{for } t = 0 \\ \text{linear} & \text{for } t \in [-1, 0],\ t \in [0, 1] \end{cases}$$

Fix a solution $x(t)$ of (1.2). Then $h(t)x(t)$ is a ψ-integrable function on $\mathbb{R}$. Set $y(t) = x(t)\int_0^t h(s)ds$, we have

$$y'(t) = A(t)x(t)\int_0^t h(s)ds + h(t)x(t) = A(t)y(t) + h(t)x(t).$$

By hypothesis, the equation

$$y'(t) = A(t)y(t) + h(t)x(t)$$

has a solution $\widetilde{y}(t)$ on $\mathbb{R}$, ψ-evanescent at ∞. Set $x_1(t) = \widetilde{y}(t) - y(t) + \frac{1}{2}x(t)$ and $x_2(t) = -\widetilde{y}(t) + y(t) + \frac{1}{2}x(t)$. It follows from $\int_{-1}^0 h(t)dt = \int_0^1 h(t)dt = \frac{1}{2}$ that

$x_1(t) = \widetilde{y}(t)$ for $t \geqslant 1$; $x_2(t) = -\widetilde{y}(t)$ for $t \leqslant -1$. Then x_2 is the solution of (1.2), ψ-bounded on $\mathbb{R}_-$, x_1 is the solution of (1.2) on $\mathbb{R}_+$, ψ-evanescent at ∞. The solution $x(t)$ is the sum of two solutions $x_1(t)$ and $x_2(t)$ of (1.2), these solutions satisfy the conditions of Lemma. The proof is complete. □

3. The main results

Theorem 3.1. *Suppose that f is a ψ-integrable function on $\mathbb{R}_+$. Then* (1.1) *has at least one solution on $\mathbb{R}_+$, ψ-evanescent at ∞ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$.*

Proof. First, we prove the "if" part. By Lemma 2.3, we can consider (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections P_1, P_2 such that $P_1\mathbb{R}^d = X_o$. Let

$$g(t) = \int_0^t Y(t)P_1Y^{-1}(s)f(s)ds - \int_t^\infty Y(t)P_2Y^{-1}(s)f(s)ds.$$

It is easy to see that $g(x)$ is a solution of (1.1) on $\mathbb{R}_+$. We shall prove that $g(x)$ is ψ-evanescent at ∞ on $\mathbb{R}_+$. Since f is ψ-integrable on $\mathbb{R}_+$, it follows that for a given $\varepsilon > 0$, there exists $T > 0$ such that

$$(K+L)\int_T^\infty \|\psi(s)f(s)\|ds < \varepsilon/2.$$

By $P_1\mathbb{R}^d = X_o$, there exists $t_1 > T$ such that, for $t \geqslant t_1$,

$$|\psi(t)Y(t)P_1|\int_0^T \|Y^{-1}(s)f(s)\|ds < \varepsilon/2.$$

Then for $t \geqslant t_1$, we have

$$\begin{aligned}
\|\psi(t)g(t)\| &\leqslant \int_0^T |\psi(t)Y(t)P_1|.\|Y^{-1}(s)f(s)\|ds \\
&\quad + \int_T^t |\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)|.\|\psi(s)f(s)\|ds \\
&\quad + \int_t^\infty |\psi(t)Y(t)P_2Y^{-1}(s)\psi^{-1}(s)|.\|\psi(s)f(s)\|ds \\
&\leqslant |\psi(t)Y(t)P_1|\int_0^T \|Y^{-1}(s)f(s)\|ds + (K+L)\int_T^\infty \|\psi(s)f(s)\|ds \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon
\end{aligned}$$

This shows that $g(x)$ is ψ-evanescent at ∞. The "only if" part evidently holds, by Theorem 1.4(a). □

Similarly, we have the following Theorem.

Theorem 3.2. *Suppose that f is a ψ-integrable function on $\mathbb{R}_-$. Then* (1.1) *has at least one solution on $\mathbb{R}_-$, ψ-evanescent at $-\infty$ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_-$.*

Theorem 3.3. *Suppose that* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_+$ and f is a ψ-integrable function on $\mathbb{R}_+$. Then following statements are equivalent*

(a) *every ψ-bounded solution of* (1.2) *on $\mathbb{R}_+$ is ψ- evanescent at ∞.*
(b) *every ψ-bounded solution of* (1.1) *on $\mathbb{R}_+$ is ψ-evanescent at ∞.*

Proof. By Lemma 2.3, we consider (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections P_1, P_2 such that $P_1\mathbb{R}^d = X_o$. Let S_1 be the set of all ψ-bounded solutions of (1.1) on $\mathbb{R}_+$ and let S_2 be the set of all ψ-bounded solutions of (1.2) on $\mathbb{R}_+$. We establish a mapping h from S_2 to S_1:

$$(hx)(t) = x(t) + g(t),$$

where $g(t)$ as in the proof of Theorem 3.1. We obtain

$$\lim_{t\to\infty} \|\psi(t)(hx)(t) - \psi(t)x(t)\| = \lim_{t\to\infty} \|\psi(t)g(t)\| = 0$$

Thus $h(x)$ is ψ-bounded on $\mathbb{R}_+$. Hence $h(x)$ belongs to S_1. It is easily to verify that h is one-to-one mapping between S_2 and S_1.

Suppose that statement (a) is satisfied. Let z be arbitrary ψ-bounded solution of (1.1) on $\mathbb{R}_+$. The foregoing follow that there exists ψ-bounded solution x of (1.2) on $\mathbb{R}_+$ such that $h(x) = z$ and

$$\lim_{t\to\infty} \|\psi(t)z(t) - \psi(t)x(t)\| = 0$$

By hypothesis, x is ψ-evanescent at ∞. Thus z is ψ-evanescent at ∞. Suppose that statement (b) is satisfied, the proof is similarly. The proof is complete. $\square$

Note that the above Theorem is a supplement to Theorem 1.4(b). Similarly, we have the following Theorem.

Theorem 3.4. *Suppose that* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}_-$ and f is a ψ-integrable function on $\mathbb{R}_-$. Then following statements are equivalent*

(a) *every ψ-bounded solution of* (1.2) *on $\mathbb{R}_-$ is ψ- evanescent at $-\infty$.*
(b) *every ψ-bounded solution of* (1.1) *on $\mathbb{R}_-$ is ψ-evanescent at $-\infty$.*

Corollary 3.5. *Suppose that* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}$ and f is a ψ-integrable function on $\mathbb{R}$. Then following statements are equivalent*

(a) *every ψ-bounded solution of* (1.2) *on $\mathbb{R}_+$ is ψ- evanescent at ∞ and every ψ-bounded solution of* (1.2) *on $\mathbb{R}_-$ is ψ- evanescent at $-\infty$.*
(b) *every ψ-bounded solution of* (1.1) *on $\mathbb{R}$ is ψ-evanescent at $\pm\infty$.*

Note that the above corollary is a supplement to [11, Theorem 3.3].

Theorem 3.6. *Suppose that* (1.2) *has no non-trivial solution on $\mathbb{R}$, ψ-evanescent at ∞. Then* (1.1) *has a unique solution on $\mathbb{R}$, ψ-evanescent at ∞ for every ψ-integrable function f on $\mathbb{R}$ if and only if* (1.2) *has a ψ-ordinary dichotomy on $\mathbb{R}$.*

Proof. First, we prove the "if" part. By Lemma 2.3, we can consider (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ with two supplementary projections P_1, P_2 such that $P_1\mathbb{R}^d = X_o$. Let

$$x(t) = \int_{-\infty}^{t} Y(t)P_1Y^{-1}(s)f(s)ds - \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s)$$

Then the function $x(t)$ is a ψ-bounded solution of (1.1) on $\mathbb{R}$. We shall prove that $x(t)$ is ψ-evanescent at ∞. We have, for $t > 0$,

$$\psi(t)x(t) = \psi(t)Y(t)P_1 \int_{-\infty}^{0} P_1Y^{-1}(s)f(s)ds + \psi(t)g(t),$$

where $g(t)$ as in the proof of Theorem 3.1. Since

$$\|P_1Y^{-1}(s)f(s)\| \leqslant |Y^{-1}(0)|.|\psi^{-1}(0)|.|\psi(0)Y(0)P_1Y^{-1}(s)\psi^{-1}(s)|.\|\psi(s)f(s)\|$$

and f is ψ-integrable on $\mathbb{R}$, we have that $P_1Y^{-1}(s)f(s)$ is integrable on $\mathbb{R}_-$. Let $a = \int_{-\infty}^0 P_1Y^{-1}(s)f(s)ds$. It follows from $P_1\mathbb{R}^d = X_0$ that

$$\lim_{t\to\infty} \|\psi(t)Y(t)P_1a\| = 0.$$

On the other hand, as in the proof of Theorem 3.1, we have

$$\lim_{t\to\infty} \|\psi(t)g(t)\| = 0.$$

Consequently $x(t)$ is defined on $\mathbb{R}$, ψ-evanescent at ∞. The uniqueness of solution $x(t)$ result from (1.2) has no non-trivial on $\mathbb{R}$, ψ-evanescent solution at ∞. Indeed, suppose that y is a solution on $\mathbb{R}$ of (1.1), ψ-evanescent at ∞ then $x-y$ is a solution solution on $\mathbb{R}$ of (1.2), ψ-evanescent at ∞. We conclude $x = y$ since $x - y$ is the trivial solution of (1.2).

Now, we prove the "only if" part. Suppose that (1.1) has a unique ψ-bounded solution on $\mathbb{R}$ for every ψ- integrable function f on $\mathbb{R}$. For each $u \in \mathbb{R}^d$, denote by $x = x(t)$ the solution of (1.2), $x(0) = u$. By Lemma 2.6, we get $x = x_1 + x_2$, where x_2 is a ψ-bounded solution of (1.2) on $\mathbb{R}_-$, x_1 is a solutions of (1.2) on $\mathbb{R}_+$ and ψ-evanescent at ∞. Thus $x_1(0) \in X_0$ and $x_2(0) \in \widetilde{X}_1$. It follows from $u = x_1(0) + x_2(0)$ that

$$\mathbb{R}^d = X_0 + \widetilde{X}_1. \tag{3.1}$$

For any $v \in X_0 \cap \widetilde{X}_1$, denote by $x(t)$ the solution of (1.2) such that $x(0) = v$. Thus $x(t)$ is a solution on $\mathbb{R}$ of (1.2), ψ-evanescent at ∞. By hypothesis, (1.2) has no non-trivial solution on $\mathbb{R}$, ψ-evanescent at ∞, then $x(t)$ is the trivial solution. This implies $v = 0$. Consequently

$$X_0 \cap \widetilde{X}_1 = 0 \tag{3.2}$$

The relations (3.1) and (3.2) imply that $\mathbb{R}^d$ is the direct sum of X_0 and $\widetilde{X}_1$. Every ψ-integrable function f on $\mathbb{R}_+$, or on $\mathbb{R}_-$ is the restriction of a ψ-integrable function f on $\mathbb{R}$, it follows that (1.2) satisfies Theorem 1.4(a) and Lemma 2.4(a). Hence (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ and has a ψ-ordinary dichotomy on $\mathbb{R}_-$. Let P_1, P_2 be two projections such that $\operatorname{Im} P_1 = X_0$, $\operatorname{Im} P_2 = \widetilde{X}_1$. Lemmas 2.3 and 2.4(b) follow that (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}_+$ and has a ψ-ordinary dichotomy on $\mathbb{R}_-$ with two supplementary projections P_1, P_2. Remark 1.3 follows that (1.2) has a ψ-ordinary dichotomy on $\mathbb{R}$ with two supplementary projections P_1, P_2. The proof is complete. $\square$

Similarly, we have the following Theorem.

Theorem 3.7. *Suppose that* (1.2) *has no non-trivial solution on* $\mathbb{R}$*,* ψ*-evanescent at* $-\infty$*. Then* (1.1) *has a unique solution on* $\mathbb{R}$*,* ψ*-evanescent at* $-\infty$ *for every* ψ*-integrable function* f *on* $\mathbb{R}$ *if and only if* (1.2) *has a* ψ*-ordinary dichotomy on* $\mathbb{R}$*.*

Now, consider the equations

$$x'(t) = [A(t) + B(t)]x(t), \tag{3.3}$$

$$x'(t) = [A(t) + B(t)]x(t) + f(t) \tag{3.4}$$

where $B(t)$ is a $d \times d$ continuous matrix function on $\mathbb{R}_+$ and f is a ψ-integrable function on $\mathbb{R}_+$. We have the following result.

Theorem 3.8. *Suppose that* (1.2) *has a* ψ*-ordinary dichotomy on* $\mathbb{R}_+$*. If* $\delta = \sup_{t\geqslant 0} |\psi(t)B(t)\psi^{-1}(t)|$ *is sufficiently small, then following statements are equivalent*

(a) *every ψ-bounded solution of* (3.3) *on* $\mathbb{R}_+$ *is* ψ*- evanescent at* $+\infty$*.*
(b) *every ψ-bounded solution of* (3.4) *on* $\mathbb{R}_+$ *is* ψ*-evanescent at* $+\infty$*.*

Proof. By [4, Theorem 3.7], equation (3.3) has a ψ-ordinary dichotomy on $\mathbb{R}_+$. By Theorem 3.3, we have the conclusion. $\square$

With similar proof, we can conclude that $J = \mathbb{R}_-$.

Theorem 3.9. *Suppose that* (1.2) *has a* ψ*-ordinary dichotomy on* $\mathbb{R}_-$ *and* $\delta = \sup_{t\leqslant 0} |\psi(t)B(t)\psi^{-1}(t)|$ *is sufficiently small. Then the following statements are equivalent*

(a) *every ψ-bounded solution of* (3.3) *on* $\mathbb{R}_-$ *is* ψ*- evanescent at* $-\infty$*.*
(b) *every ψ-bounded solution of* (3.4) *on* $\mathbb{R}_-$ *is* ψ*-evanescent at* $-\infty$*.*

References

[1] O. Akinyele; *On partial stability and boundedness of degree k*, Atti. Acad. Naz. LinceiRend. Cl. Sei. Fis. Mat. Natur., Vol. (8), No 65 (1978), pp. 259-264.
[2] C. Avramescu; *Asupra comportãrii asimptotice a solutiilor unor ecuatii funcionable*, Analele Universitãtii din Timisoara, Seria Stiinte Matamatice-Fizice, Vol. VI, 1968, pp. 41-55.
[3] C. Avramescu; *Evanescent solutions for linear ordinary differential equations*, Electronic Journal Qualytative Theory of Differential Equations, Vol 2002, No 9, pp. 1-11.
[4] Pham Ngoc Boi; *On the ψ- dichotomy for homogeneous linear differential equations*. Electronic Journal of Differential Equations, Vol. 2006 (2006), No. 40, pp. 1-12.
[5] Pham Ngoc Boi; *Existence of ψ-bounded solution on $\mathbb{R}$ for nonhomogeneous linear differential equations*. Electronic Journal of Differential Equations, Vol. 2007 (2007). No. 52, pp. 1-10.
[6] A. Constantin; *Asymptotic properties of solution of differential equation*, Analele Universitãtii din Timisoara, Seria Stiinte Matamatice, Vol. XXX, fasc. 2-3,1992, pp. 183-225.
[7] W. A. Coppel; *Dichotomies in Stability Theory* Springer-Verlag Berlin Heidelberg New York, 1978.
[8] J. L. Daletskii; M. G. Krein; *Stability of solutions of differential equations in Banach spaces*, American Mathematical Society Providence, Rhode Island 1974.
[9] A. Diamandescu; *Existence of ψ-bounded solutions for a system of differential equations* , Electronic Journal of Differential Equations, Vol. 2004 (2004), No. 63, pp. 1-6.
[10] A. Dimandescu; *A note on the existence of ψ-bounded solutions for a system of differential equations on* $\mathbb{R}$, Electronic Journal of Differential Equations, Vol. 2008 (2008), No. 128, pp. 1-11.
[11] A. Dimandescu; *ψ-bounded solutions for linear differential systems with Lebesgue ψ-integrable funtions on* $\mathbb{R}$ as right-hand sides, Electronic Journal of Differential Equations, Vol. 2009 (2009), No. 05, pp. 1-12.

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