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GENERALIZED SOLUTIONS TO THE GKDV EQUATION

MAURILIO MARCIO MELO

Abstract. In this article we study the Cauchy problem in $\mathcal{G}_2((0,T)\times\mathbb{R})$ (the algebra of generalized functions, in the sense of Colombeau) for the generalized Korteweg-de Vries equation, with initial condition $\varphi\in\mathcal{G}_2(\mathbb{R})$, which contains $H^s(\mathbb{R})$, for $s\in\mathbb{R}$.

1. Introduction

In [4] and [5], Biagioni and Oberguggenberger have shown that the nonlinear theory of the generalized functions, introduced by Colombeau [9], can be used to deal with the Cauchy problem for the nonlinear evolution equations. Following this approach, in this article we study the Cauchy problem

$$u_t + u_{xxx} + a(u)u_x = 0, \quad u(0)=\varphi, \tag{1.1}$$

where $a(u)=u^3$.

Equations of the form (1.1) are known as generalized Korteweg-de Vries (gKdV), as opposed to the ordinary KdV, when $a(u)=u$, and modified KdV (mKdV), when $a(u)=u^2$. The KdV equation was derived by Korteweg-de Vries as a model for long waves propagating in a channel. Subsequently, the mKdV equation has been showed relevant in a number of different physical systems. In fact, a large class of hyperbolic models has been reduced to these equations. Another reason to study them is their relation with inverse scattering theory.

The space chosen to deal with this problem is the Colombeau algebra $\mathcal{G}_2(\Omega)$, $\Omega=(0,T)\times\mathbb{R}$ which we will describe in section 2. The KdV and mKdV equations were studied in the same context in [5] and [8], respectively. The authors obtained results of existence and uniqueness of solutions in $\mathcal{G}_2(\Omega)$ to the Cauchy problem for these equations and initial condition in $\mathcal{G}_2(\mathbb{R})$.

The KdV and mKdV equations have an infinite number of conserved quantities, see [21]. But, in general, if $a(u)\neq u,u^2$, this fact is not true. This property was used in the proof of the existence of solutions to (1.1) for the cases $a(u)=u,u^2$, see [5] and [8].

In [17], results of existence and uniqueness of solutions were established in $\mathcal{G}_2((0,T)\times\mathbb{R})$ to the Cauchy problem for the equation

$$u_t - 30u^2u_x + 20u_xu_{xx} + 10uu_{xxx} - u_{xxxxx} = 0, \tag{1.2}$$

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which belongs to the Lax hierarchy [16] for the KdV equation.

In [19], the Cauchy problem was studied for the equation

$$u_t = \sum_{\alpha_1} \partial^{\alpha_1} G_m(u), \tag{1.3}$$

where $\alpha_1 \in \mathbf{N}^n$ and $G_m(u)$ is the gradient in u of the functional $F_m(u)$, where $F_m(u)$ is constant along solutions to the KdV equation in dimension n

$$u_t = u\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} + \sum_{i,j=1}^{n} \frac{\partial^3 u}{\partial x_i \partial^2 x_j}.$$

We observe that the function $a(u) = u^3$, satisfies the one-sided growth condition

$$\limsup_{|u|\to\infty} |u|^{-4} a(u) \le 0, \tag{1.4}$$

which allows global solutions to problem (1.1) in H^s, $s > 3/2$, according to [14]. If $a(u) = u^4$ the problem (1.1), in Sobolev spaces, is considered as a critical case by various reasons: there are no known global results in the usual Sobolev space; for some initial condition in H^1, Martel and Merle have shown in [20], the finite time blow up; finally, the 4^{th} power is the only one for which the problem (1.1) has no continuous dependence on the initial condition when the time for existence of solutions depend on L^2-norm of φ, see [6].

In [2], [7] and [18] other evolution equations such as Benjamin-Ono (BO), Smith (S), Cubic Schrödinger, Lax hierarchy, are studied in Colombeau's algebras.

The paper is organized as follows: In Section 2, we introduce notation and some definitions.

In section 3, we study problem (1.1) in the case $a(u) = u^3$. In Lemmas 3.1 and 3.2, we establish estimates in which we use the results obtained by Kato in [14, Lemmas 4.2 and 4.3]. These estimates will be used in the proof of Theorem 3.3, of existence and uniqueness of solutions to the Cauchy problem (1.1) with $a(u) = u^3$.

In Section 4, we give a sketch of the proof establishing that the solution to (1.1) in $\mathcal{G}_2((0,T)\times\mathbb{R})$, given in the Theorem 3.3, is related to the solution obtained by Kato [14]. We also state the result of existence and uniqueness of solutions in $\mathcal{G}_2((0,T)\times\mathbb{R})$ for (1.1) with $a(u)$ satisfying condition (1.4). Finally, in Remark 4.4, we show what a soliton described by gKdV equation leads to an example of a nonzero solution to equation of (1.1) in the Colombeau algebra $\mathcal{G}((0,T)\times\mathbb{R})$, whose restriction to $t = 0$ is zero in $\mathcal{G}(\mathbb{R})$. This shows that we do not have uniqueness of solutions to the problem (1.1) with initial condition in $\mathcal{G}(\mathbb{R})$.

2. Notation and some basic definitions

Let Ω be an open subset of $\mathbb{R}^n$, $s \in \mathbb{R}$. We denote by $H^s(\Omega)$ the usual Sobolev space L^2-type; i.e., $H^s = J^{-s}L^2$, with norm $\|u\|_s = \|J^s u\|_0 = (J^s u, J^s u)^{1/2}$, where $J = (1-\Delta)^{1/2}$; Δ is the Laplacian, $\|\cdot\|_0$ is the norm in L^2 and $(\cdot,\cdot)$ its inner product; $H^\infty(\Omega) = \cap_{k\in\mathbb{Z}} H^k(\Omega)$, $H^{-\infty}(\Omega) = \cup_{k\in\mathbb{Z}} H^k(\Omega)$, $[J^s, M_u] = J^s M_u - M_u J^s$ is the commutator operator, where M_u is the multiplication by u operator and $\mathcal{D}'(\Omega)$ is the distribution space.

Next we exhibit the algebra where we study the Cauchy problem for the gKdV equation, the space $\mathcal{G}_2(\Omega)$ (algebra of the generalized functions type Colombeau modeled in the space L^2).

Let $I = (0,1)$ and $\Omega \subset \mathbb{R}^n$ be an open set. We set

$$\begin{aligned}
\mathcal{E}_2[\Omega] &= (H^\infty(\Omega))^I = \{\widehat{u} : I \to H^\infty(\Omega), \varepsilon \in I \to \widehat{u}_\varepsilon \in H^\infty(\Omega)\},\\
\mathcal{E}_{M,2}[\Omega] &= \big\{\widehat{u} \in \mathcal{E}_2[\Omega] : \forall \alpha \in N^n,\ \exists N > 0 \text{ and } C > 0,\\
&\qquad \text{such that } \|\partial^\alpha \widehat{u}_\varepsilon\|_0 \le C\varepsilon^{-N},\ \text{for small } \varepsilon\big\},\\
\mathcal{N}_2[\Omega] &= \big\{\widehat{u} \in \mathcal{E}_{M,2}[\Omega] : \forall \alpha \in N^n \text{ and } M > 0, \exists C > 0,\\
&\qquad \text{such that } \|\partial^\alpha \widehat{u}_\varepsilon\|_0 \le C\varepsilon^{M},\ \text{for small } \varepsilon\big\}.
\end{aligned}$$

We observe that $\mathcal{E}_{M,2}[\Omega]$ is an algebra with partial derivatives and $\mathcal{N}_2[\Omega]$ is an ideal of $\mathcal{E}_{M,2}[\Omega]$ which is invariant under derivatives.

The Colombeau's algebra modelled in L^2 is defined as the quotient space

$$\mathcal{G}_2[\Omega] = \frac{\mathcal{E}_{M,2}[\Omega]}{\mathcal{N}_2[\Omega]}.$$

Its elements $u, v, \dots$ are called generalized functions in Ω. The multiplication and derivatives in $\mathcal{G}_2[\Omega]$ are defined on the representatives.

If in the definition of $\mathcal{G}_2$ we consider the space H^∞ replaced by $W^{\infty,\infty} = \cap_{k\in\mathbb{Z}} W^{k,\infty}$, we obtain $\mathcal{G}$ (algebra of generalized functions defined by Colombeau, see [1]).

Remark 2.1 ([5])**.** There is an embedding of $H^{-\infty}(\mathbb{R}^n)$ into $\mathcal{G}_2(\mathbb{R}^n)$ obtained in the following way: we fix $\rho \in S(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \rho(x)dx = 1, \quad \int_{\mathbb{R}^n} x^\alpha \rho(x)dx = 0, \quad \forall \alpha \in \mathbf{N}^n,\ |\alpha| \ge 1.$$

Let $\iota : w \to (w * \rho_\varepsilon)_\varepsilon$, where $\rho_\varepsilon(x) = \frac{1}{\varepsilon^n}\rho(x/\varepsilon)$. This defines a linear injection of $H^{-\infty}(\mathbb{R}^n)$ into $\mathcal{E}_{M,2}[\mathbb{R}^n]$, which induces an embedding $H^{-\infty}(\mathbb{R}^n)$ into $\mathcal{G}_2(\mathbb{R}^n)$; so we can see $H^\infty(\mathbb{R}^n)$ as a subalgebra of $\mathcal{G}_2(\mathbb{R}^n)$.

Definition 2.2. For $u \in \mathcal{G}_2((0,T)\times\mathbb{R})$, the restriction of u to $\{0\}\times\mathbb{R}$ is the class of $\widehat{u}_\varepsilon(0,\cdot)$ in $\mathcal{G}_2(\mathbb{R})$, where $\widehat{u}_\varepsilon$ is a representative of u. We denote this class by $u|_{\{t=0\}}$ or $u(0)$.

Definition 2.3. We say that $u \in \mathcal{G}_2(\mathbb{R}^n)$ is *associated with the distribution* $w \in H^{-\infty}(\mathbb{R}^n)$ if there is a representative $\widehat{u}$ of u such that $\widehat{u}_\varepsilon(\cdot) \to w$ in $\mathcal{D}'(\mathbb{R}^n)$ as $\varepsilon \to 0$. We denote it by $u \approx w$. We say that $u,\ v \in \mathcal{G}_2(\mathbb{R}^n)$ are associated if $u - v \approx 0$.

Definition 2.4. We say that $u \in \mathcal{G}_2(\Omega)$ is of $r - (\log)^{1/j}$-type, $2 \le r \le \infty$, $j \ge 1$, if it has a representative $\widehat{u} \in \mathcal{E}_{M,2}[\Omega]$ such that

$$\|\widehat{u}_\varepsilon\|_{L^r} \le C(|\log\varepsilon|^{1/j}), \quad \text{for small } \varepsilon,$$

and r-bounded-type if

$$\|\widehat{u}_\varepsilon\|_{L^r} \le C, \quad \text{for small } \varepsilon.$$

Remark 2.5. We also observe a nonlinear property of generalized functions: if $F \in O_M(\mathbb{R}^l)$; i.e., F is a smooth function and, together with all its derivatives, grows at most like some power of $|x|$ as $|x| \to \infty$, then we can define $F(u_1, u_2, \dots, u_l) \in \mathcal{G}_2(\Omega)$ for $u_i \in \mathcal{G}_2(\Omega)$, $i = 1, \dots, l$, (see [1]).

Definition 2.6. Let $P(u, \partial^\alpha u)$ be a polynomial in u and its derivatives. We say that u is a solution to the problem

$$\begin{gathered} u_t = P(u, \partial^\alpha u) \quad \text{in } \mathcal{G}_2((0,T)\times\mathbb{R}), \\ u|_{\{t=0\}} = g \quad \text{in } \mathcal{G}_2(\mathbb{R}), \end{gathered}$$

if for every representative $\widehat{u} \in \mathcal{E}_{M,2}[(0,T)\times\mathbb{R}]$ of u and $\widehat{g} \in \mathcal{E}_{M,2}[\mathbb{R}]$ of g, there are $\widehat{N} \in \mathcal{N}_2[(0,T)\times\mathbb{R}]$ and $\widehat{\eta} \in \mathcal{N}_2[\mathbb{R}]$ such that

$$\begin{gathered} \widehat{u}_t = P(\widehat{u}, \partial^\alpha \widehat{u}) + \widehat{N} \quad \text{in } (0,T)\times\mathbb{R}, \\ \widehat{u}|_{\{t=0\}} = \widehat{g} + \widehat{\eta} \quad \text{in } \mathbb{R} \end{gathered}$$

We observe that the time interval in this definition is the same for all representatives. Also, we observe that the problem for representatives is the classical problem. For some properties of generalized functions see [1, 4, 9, 10, 12, 13, 22].

3. Generalized solutions

Next, in the proof of Lemma 3.1, we use the conservation laws for the equation of (1.1) obtained by Kato in [14, Th 4.2 and eq. 27]. We note that T in this result is independent of the initial condition.

Lemma 3.1. *If $u_\varepsilon \in \mathcal{C}((0,T); H^s(\mathbb{R}))$, with $s > 3/2$, is the solution of the problem*

$$u_t + u_{xxx} + u^3 u_x = 0, \quad u(0) = \varphi_\varepsilon \in H^s(\mathbb{R}), \tag{3.1}$$

given by [14, Theorem 4.1]*, then*

$$\|u_\varepsilon(t,\cdot)\|_0 = \|\varphi_\varepsilon\|_0, \tag{3.2}$$

$$\|\partial_x u_\varepsilon(t,\cdot)\|_0 \le C m_1^7(\varepsilon), \quad \textit{for small } \varepsilon, \tag{3.3}$$

$$\|\partial_x^2 u_\varepsilon(t,\cdot)\|_0 \le C(m_2(\varepsilon))^{49} \exp(cT(m_0(\varepsilon))^{5/2}), \quad \textit{for small } \varepsilon, \tag{3.4}$$

where $m_k(\varepsilon) = \max\{1, \|\varphi_\varepsilon)\|_k\}$, $k \in \mathbf{N}$ and $C = C(T)$.

Proof. From [14, eq.18] we have (3.2). From [14, eq.19 and 27] we have respectively,

$$\|\partial_x u_\varepsilon(t,\cdot)\|_0^2 - (a_2(u_\varepsilon(t.\cdot),1) = \|\varphi_\varepsilon'\|_0^2 - (a_2(\varphi_\varepsilon),1) \tag{3.5}$$

and

$$\begin{aligned} &\frac{d}{dt}[\|\partial_x^2 u_\varepsilon(t,\cdot)\|_0^2 - \frac{5}{3}(u_\varepsilon(t,\cdot)^3 \partial_x u_\varepsilon(t,\cdot), \partial_x u_\varepsilon(t,\cdot))] \\ &= \frac{1}{2}((\partial_x u_\varepsilon(t,\cdot))^5, 1) + 5(u_\varepsilon(t,\cdot))^5 (\partial_x u_\varepsilon(t,\cdot))^3, 1), \end{aligned} \tag{3.6}$$

where $a_2(\lambda) = \lambda^5/10$. For the rest of this article, we omit the subscript ε and $(t,\cdot)$ in our notation. Then, from (3.5) we have

$$\|\partial_x u\|_0^2 = \frac{1}{10}\int u^5 dx + \|\varphi'\|_0^2 - \frac{1}{10}\int \varphi^5 dx.$$

Thus,

$$\begin{aligned}
\|\partial_x u\|_0^2 &\le \frac{1}{10}\|u\|_{L^\infty}^3\|u\|_0^2 + \|\varphi'\|_0^2 + \frac{1}{10}\|\varphi\|_{L^\infty}\|\varphi\|_0^4 \\
&\le c(\|\partial_x u\|_0^{\frac{3}{2}}\|u\|_0^{\frac{7}{2}} + \|\varphi'\|_0^2 + \|\varphi'\|_0^{\frac{1}{2}}\|\varphi\|_0^{1/2}\|\varphi\|_0^4 \\
&\le c(\|\partial_x u\|_0^{\frac{3}{2}}\|u\|_0^{\frac{7}{2}} + \|\varphi'\|_0^2 + \|\varphi'\|_0 + \|\varphi\|_1^5) \\
&\le c(\delta\|\partial_x u\|_0^2 + c(\delta)\|\varphi\|_0^{14} + \|\varphi'\|_0^2 + \|\varphi'\|_0 + \|\varphi\|_1^5),
\end{aligned}$$

since by Gagliardo-Nirenberg $\|u\|_{L^\infty} \le \|\partial_x u\|_0^{1/2}\|u\|_0^{1/2}$ see [11]; and by Young's inequalities $ab \le \delta a^p + c(\delta)b^q$. Taking $\delta = 1/(2c)$ we obtain (3.3). The right-hand side of (3.6) can be estimated by

$$\begin{aligned}
&c(\|\partial_x u\|_{L^\infty}^3\|\partial_x u\|_0^2 + \|u\|_{L^\infty}^5\|\partial_x u\|_{L^\infty}\|\partial_x u\|_0^2) \\
&\le c(\|\partial_x u\|_0^{\frac{7}{2}}\|\partial_x^2 u\|_0^{\frac{3}{2}} + \|u\|_0^{5/2}\|\partial_x u\|_0^5\|\partial_x^2 u\|_0^{1/2}) \\
&\le c(\|\partial_x u\|_0^{14} + \|\partial_x^2 u\|_0^2 + \|u\|_0^{\frac{5}{2}}(\|\partial_x u\|_0^{\frac{20}{3}} + \|\partial_x^2 u\|_0^2)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{d}{dt}[\|\partial_x^2 u\|_0^2 - \frac{5}{3}(u^3\partial_x u, \partial_x u)] \\
&\le c(\|\partial_x u\|_0^{14} + \|\partial_x^2 u\|_0^2 + \|u\|_0^{5/2}(\|\partial_x u\|_0^{20/3} + \|\partial_x^2 u\|_0^2)) \\
&\le c(\|\partial_x u\|_0^{14} + \|u\|_0^5 + \|\partial_x u\|_0^{40/3} + (1 + \|u\|_0^{5/2})\|\partial_x^2 u\|_0^2)).
\end{aligned}$$

Integrating this inequality from 0 to $t \le T$, and using (3.2) and (3.3), we obtain

$$\begin{aligned}
&\|\partial_x^2 u\|_0^2 - \frac{5}{3}(u^3\partial_x u, \partial_x u) \\
&\le \|\varphi''\|_0^2 - \frac{5}{3}(\varphi^3\varphi', \varphi') + cT(m_1(\varepsilon))^{98} + c(1 + \|\varphi\|_0^{5/2})\int_0^t \|\partial_x^2 u\|_0^2.
\end{aligned}$$

This implies that

$$\|\partial_x^2 u\|_0^2 \le c[\|\varphi\|_0^{\frac{3}{2}}\|\partial_x u\|_0^{\frac{7}{2}} + \|\varphi''\|_0^2 + \|\varphi\|_1^5 + T(m_1(\varepsilon))^{98} + (m_0(\varepsilon))^{5/2}\int_0^t \|\partial_x^2 u\|_0^2],$$

or

$$\|\partial_x^2 u\|_0^2 \le c[\|\varphi\|_0^3 + \|\partial_x u\|_0^7 + \|\varphi''\|_0^2 + \|\varphi\|_1^5 + T(m_1(\varepsilon))^{98} + (m_0(\varepsilon))^{5/2}\int_0^t \|\partial_x^2 u\|_0^2],$$

or, using (3.3), we have the inequality

$$\|\partial_x^2 u\|_0^2 \le C(T)(m_2(\varepsilon))^{98} + c(m_0(\varepsilon))^{5/2}\int_0^t \|\partial_x^2 u\|_0^2].$$

Then, by Gronwall's lemma, we have (3.4). □

Next, in the proof of Lemma 3.2, we follow the idea in [3] and use the inequalities, obtained by Kato and Ponce in [15], valid for $s > 0$ and $1 < p < \infty$:

$$\|[J^s, M_f]g\|_{L^p} \le c(\|\partial_x f\|_{L^\infty}\|J^{s-1}g\|_{L^p} + \|J^s f\|_{L^p}\|g\|_{L^\infty}), \tag{3.7}$$

and as a consequence,

$$\|J^s(fg)\|_{L^p} \le c(\|f\|_{L^\infty}\|g\|_{L^p} + \|f\|_{L^p}\|g\|_{L^\infty}). \tag{3.8}$$

Lemma 3.2. *If $u_\varepsilon \in \mathcal{C}((0,T);H^s(\mathbb{R}))$ with $s \geq 2$, is the solution of (3.1), we have*

$$\|u_\varepsilon(t,\cdot)\|_s \leq \|\varphi_\varepsilon\|_s \exp[cT(m_2(\varepsilon))^{35} \exp cT(m_0(\varepsilon))^{5/2}]. \tag{3.9}$$

Proof. Applying J^s to equation of (3.1), multiplying by $J^s u$ and integrating in $\mathbb{R}$, give

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_s^2 &= -(J^s(u^3\partial_x u), J^s u) \\
&= -([J^s, M_{u^3}]\partial_x u + u^3 J^s \partial_x u, J^s u) \\
&\leq \|[J^s, M_{u^3}]\partial_x u\|_0 \|J^s u\|_0 + \frac{3}{2}|(u^2\partial_x u J^s u, J^s u)|.
\end{aligned}$$

Using (3.7) with $f = u^3$, $g = \partial_x u$ and $p = 2$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_s^2 &\leq c(\|\partial_x u^3\|_{L^\infty}\|J^{s-1}\partial_x u\|_0 + \|J^s u^3\|_0\|\partial_x u\|_{L^\infty})\|J^s u\|_0 \\
&\quad + \frac{3}{2}\|u\|_{L^\infty}^2 \|\partial_x u\|_{L^\infty}\|J^s u\|_0^2.
\end{aligned}$$

Using (3.8) with $f = u$, $g = u^2$ and $p = 2$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_s^2 &\leq c[\|3u^2\partial_x u\|_{L^\infty}\|J^s u\|_0 \\
&\quad + c(\|u\|_{L^\infty}\|u^2\|_0 + \|u\|_0\|u^2\|_{L^\infty})\|\partial_x u\|_{L^\infty}]\|J^s u\|_0 \\
&\quad + \frac{3}{2}\|u\|_{L^\infty}^2\|\partial_x u\|_{L^\infty}\|J^s u\|_0^2 \\
&\leq c[\|u\|_{L^\infty}^2\|\partial_x u\|_{L^\infty}\|J^s u\|_0 + (\|u\|_{L^\infty}^2\|u\|_0 \\
&\quad + \|u\|_0\|u\|_{L^\infty}^2)\|\partial_x u\|_{L^\infty}]\|J^s u\|_0 \\
&\quad + \frac{3}{2}\|u\|_{L^\infty}^2\|\partial_x u\|_{L^\infty}\|J^s u\|_0^2 \\
&\leq c\|u\|_{L^\infty}^2\|\partial_x u\|_{L^\infty}\|u\|_s^2 \leq c\|u\|_0\|\partial_x u\|_0^{3/2}\|\partial_x^2 u\|_0^{1/2}\|u\|_s^2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u(t)\|_s^2 &\leq c\|u\|_0\|\partial_x u\|_0^{3/2}\|\partial_x^2 u\|_0^{\frac{1}{2}}\|u\|_s^2 \\
&\leq c\|u\|_0(\|\partial_x u\|_0^3 + \|\partial_x^2 u\|_0)\|u\|_s^2.
\end{aligned}$$

Integrating from 0 to $t \leq T$ and using Gronwall's lemma we have

$$\|u(t)\|_s^2 \leq \|\varphi\|_s^2 \exp[cT\|\varphi\|_0 \sup_{t\in[0,T]}(\|\partial_x u(t)\|_0^3 + \|\partial_x^2 u(t)\|_0)].$$

Then, from (3.3) and (3.4) we have (3.9). $\square$

Theorem 3.3. *If $\varphi \in \mathcal{G}_2(\mathbb{R})$ is such that $\varphi, \varphi', \varphi''$ are 2-$(\log(\log))^{1/5}$-type; i.e, φ have a representative $\widehat{\varphi}$ such that*

$$\|D^\alpha \widehat{\varphi}_\varepsilon\|_0 \leq C\Big(\log\big(\log(1/\varepsilon)\big)\Big)^{1/5}, \quad \text{for small } \varepsilon,\ \alpha = 0,1,2, \tag{3.10}$$

then, for all $T > 0$, there is a unique solution $u \in \mathcal{G}_2((0,T)\times\mathbb{R})$ for the problem

$$u_t + u_{xxx} + u^3 u_x = 0, \quad u(0) = \varphi. \tag{3.11}$$

Proof. For ε small enough, we have $(\log(\log(1/\varepsilon)))^{1/5} < \big(\log(1/\varepsilon)\big)^{1/70}$, thus condition (3.10) on $\widehat{\varphi}$ ensues

$$\|\widehat{\varphi}_\varepsilon\|_2 \le C\Big(\log(1/\varepsilon)\Big)^{1/70}, \tag{3.12}$$

$$\|\widehat{\varphi}_\varepsilon\|_0 \le C(\log(\log(1/\varepsilon)))^{1/5}, \quad \text{for small } \varepsilon. \tag{3.13}$$

For each $\varepsilon > 0$, let $\widehat{u}_\varepsilon$ be the solution of (3.1) with $u(0) = \widehat{\varphi}_\varepsilon$, given by [14, corollary 4.7]. From (3.9), (3.12) and (3.13) we have

$$\|u_\varepsilon(t,\cdot)\|_s \le \|\varphi_\varepsilon\|_s \exp[cT(\log(1/\varepsilon))^{1/2} \exp cT(\log(\log(1/\varepsilon)))^{1/2}].$$

Since

$$\Big(\log\big(\log(1/\varepsilon)\big)\Big)^{1/2} \ge 2cT, \quad \text{for } \varepsilon \text{ small enough},$$

we get

$$\|u(t)\|_s \le \|\varphi\|_s \varepsilon^{-N}, \quad N > 0,$$

which proves that $(\widehat{u}_\varepsilon)_\varepsilon \in \mathcal{E}_{M,2}[(0,T)\times\mathbb{R}]$. Thus class $u \in \mathcal{G}_2((0,T)\times\mathbb{R})$, whose representative is $(\widehat{u}_\varepsilon)_\varepsilon$ is, by construction, solution to problem (3.11).

For the uniqueness, let u and v in $\mathcal{G}_2((0,T)\times\mathbb{R})$ be two solutions of (3.11) with respective representatives $\widehat{u}_\varepsilon$ and $\widehat{v}_\varepsilon$, then, according to Definition 2.6, there exists $\widehat{N} \in \mathcal{N}_2[(0,T)\times\mathbb{R}]$ and $\widehat{n} \in \mathcal{N}_2[\mathbb{R}]$ such that, if $w = u - v$, we have (we omit ε and hat in our notation)

$$w_t + u^3 w_x + (u^3 - v^3)v_x + w_{xxx} = N, \quad w(0) = n. \tag{3.14}$$

By changing representatives, we may assume that $n = 0$. By [12, proposition 3.4(ii)], see also [13], it is sufficient show that

$$\|w(t)\|_0^2 \le C\varepsilon^q, \quad \text{for all } q. \tag{3.15}$$

Multiplying (3.14) by w and integrating over $\mathbb{R}$, we obtain

$$\begin{aligned}\frac{d}{dt}\|w(t)\|_0^2 &\le c(\|u(t)\|_{L^\infty}^2 + \|v(t)\|_{L^\infty}^2)(\|\partial_x u(t)\|_{L^\infty} \\ &\quad + \|\partial_x v(t)\|_{L^\infty})\|w(t)\|_0^2 + \|N\|_0\|w\|_0.\end{aligned}$$

Gronwall's lemma implies that for $0 \le t \le T$,

$$\begin{aligned}\|w(t)\|_0^2 &\le \|N\|_0\|w\|_0 \exp[T \sup_{0\le t\le T}(\|u(t)\|_{L^\infty}^2 \\ &\quad + \|v(t)\|_{L^\infty}^2)\big(\|\partial_x u(t)\|_{L^\infty} + \|\partial_x v(t)\|_{L^\infty}\big)].\end{aligned}$$

Sobolev's embbeding implies

$$\begin{aligned}\|w(t)\|_0^2 &\le \|N\|_0\|w\|_0 \exp[T \sup_{0\le t\le T}(\|u(t)\|_s^2 \\ &\quad + \|v(t)\|_s^2)(\|\partial_x u(t)\|_{s-1} + \|\partial_x v(t)\|_{s-1})],\end{aligned}$$

or

$$\|w(t)\|_0^2 \le \|N\|_0\|w\|_0 \exp[cT \sup_{0\le t\le T}(\|u(t)\|_s^3 + \|v(t)\|_s^3)].$$

Thus, from (3.3), (3.4) and (3.9) and since $N \in \mathcal{N}_2[(0,T)\times\mathbb{R}]$, we obtain (3.15). $\square$

4. Other results

Remark 4.1. We observe that, following the same technique and using the result by Kato in [14, Corollary 4.7, Lemma 3.1 and Lemma A.6] it is possible to show that a similar result holds for problem (1.1): If $\varphi \in \mathcal{G}_2(\mathbb{R})$ and its derivatives are 2-bounded-type; i.e, φ has a representative $\widehat{\varphi}$ such that

$$\|\widehat{\varphi}_\varepsilon\|_k \le C, \quad \text{for small } \varepsilon$$

for all $k \in \mathbf{N}$ and $a(u)$ satisfies (1.4), then for all $T > 0$, there is a unique solution $u \in \mathcal{G}_2((0,T) \times \mathbb{R})$ of (1.1) which is also of 2-bounded-type. More precisely,

$$\sup_{t\in[0,T]} \|\widehat{u}_\varepsilon(t)\|_k \le \widetilde{a}(\|\widehat{\varphi}_\varepsilon\|_{k-1})\|\widehat{\varphi}_\varepsilon\|_k,$$

where $\widetilde{a}$ is a monotone increasing function depending only on a. We also observe that $a(u) = u^r$, $r < 4$, satisfies (1.4).

The following result shows that the generalized solution to the Cauchy problem (3.11), is associated with the respective classical solution $v \in C([0,T]; H^s(\mathbb{R}))$ given in [14, Corollary 4.7].

Proposition 4.2. *If $\varphi \in H^s(\mathbf{R})$, $s \ge 2$, then the solution of problem* (3.11), *with initial data $\iota(\varphi) \in \mathcal{G}_2(\mathbb{R})$ is associated with the respective classical solution $v \in C([0,T]; H^s(\mathbf{R}))$ given in* [14, Corollary 4.7].

Sketch of proof. It is based on the fact that

$$\|\rho_\varepsilon * \varphi\|_s = \|J^s(\rho_\varepsilon * \varphi)\|_0 = \|\rho_\varepsilon * J^s\varphi\|_0 \le \|\rho_\varepsilon\|_{L^1}\|\varphi\|_s$$

is bounded independently of ε, thus we have a unique generalized solution to (3.11). The continuous dependence theorem given by [14, Theorem 4.6] gives the association result. □

Remark 4.3. The δ-Dirac distribution is in $\mathcal{G}_2(\mathbb{R})$, (see Remark 2.1). If we replace the embedding of $H^{-\infty}(\mathbf{R})$ into $\mathcal{G}_2(\mathbb{R})$ in Remark 2.1 by $\iota(w)$, whose representative is given by $\widehat{w}_\varepsilon = \widehat{\iota(w)}_\varepsilon = w * \rho_{h(\varepsilon)}$, we obtain that the net $\widehat{\delta}_\varepsilon = \delta * \rho_{h(\varepsilon)} = \rho_{h(\varepsilon)}$, with $\widehat{\delta}_\varepsilon(x) = \rho_{h(\varepsilon)}(x) = \frac{1}{h(\varepsilon)}\rho(\frac{x}{h(\varepsilon)})$, is a representative of the generalized function $\iota(\delta)$. It is possible to choose $h(\varepsilon)$, such that condition (3.10) holds; i.e., $\|\widehat{\delta}_\varepsilon\|_2 \le C(\log(\log\frac{1}{\varepsilon}))^{1/5}$. Indeed, since $\|\widehat{\delta}_\varepsilon\|_2^2 = \|\widehat{\delta}_\varepsilon\|_0^2 + \|\widehat{\delta}''_\varepsilon\|_0^2$, we obtain

$$\|\widehat{\delta}_\varepsilon\|_2^2 = \int_{\mathbb{R}} \frac{1}{h^2(\varepsilon)}\rho^2(\frac{x}{h(\varepsilon)})dx + \int_{\mathbb{R}} \frac{1}{h^6(\varepsilon)}[\rho''(\frac{x}{h(\varepsilon)})]^2 dx = \frac{c_1}{h(\varepsilon)} + \frac{c_2}{h^5(\varepsilon)}.$$

By the Implicit Function Theorem, we can choose $h(\varepsilon)$ such that $\frac{c_1}{h(\varepsilon)} + \frac{c_2}{h^5(\varepsilon)} = C(\log(\log(1/\varepsilon)))^{2/5}$. Therefore, problem (3.11) with initial condition $\varphi = \iota(\delta)$ has a unique solution $u \in \mathcal{G}_2((0,T) \times \mathbb{R})$.

Remark 4.4. We observe that (3.11) has a solitary wave solution, see [23]:

$$u(t,x) = [10c \sec h^2(\frac{3}{2}\sqrt{c}(x + x_0 - ct))]^{1/3}.$$

Taking $c = x_0 = \frac{1}{\varepsilon}$, as in the proof given in [5], we have that the generalized function u with a representative given by

$$\widehat{u}_\varepsilon(t,x) = [\frac{10}{\varepsilon} \sec h^2(\frac{3}{2}\frac{1}{\sqrt{\varepsilon}}(x + \frac{1}{\varepsilon} - \frac{1}{\varepsilon}t))]^{1/3},$$

is a nonzero solution of the equation of (3.11) which belongs to algebra $\mathcal{G}((0,T)\times\mathbb{R})$, as defined in [1]. The restriction of u to $t=0$ vanishes in $\mathcal{G}(\mathbb{R})$, yielding the non-uniqueness of solutions in $\mathcal{G}((0,T)\times\mathbb{R})$ to problem (3.11). Indeed, introducing the notation $\xi=\frac{3}{2}\frac{1}{\sqrt{\varepsilon}}(x+\frac{1}{\varepsilon}-\frac{1}{\varepsilon}t)$, we can check that each derivative of $\widehat{u}_\varepsilon$ is of the form $\sum a_{mn}\varepsilon^{-j}\sec h^{m+\frac{2}{3}}(\xi)\tan h^n(\xi)$, where $m\geq 0$, $n\geq 1$ and $j\geq 1$. Then for each r, the absolute value of the r-derivative of $\widehat{u}_\varepsilon(0,\cdot)$ is bounded by

$$c\varepsilon^{-j}|\sec h^{2/3}(\frac{3}{2}\frac{1}{\sqrt{\varepsilon}}(x+\frac{1}{\varepsilon})|,$$

where $j=j(r)\in\mathbf{N}$ and $c=c(r)>0$. Since

$$|\sec h^{2/3}(\frac{3}{2}\frac{1}{\sqrt{\varepsilon}}(x+\frac{1}{\varepsilon})|\leq|\sec h^{2/3}(\frac{1}{\sqrt{\varepsilon}})|\leq 2^{2/3}\exp(-\frac{2}{3}\frac{1}{\sqrt{\varepsilon}}),$$

for $x+\frac{1}{\varepsilon}\geq\frac{2}{3}$, then all derivatives of $\widehat{u}_\varepsilon(0,\cdot)$ are bounded from above by any positive power of ε, thus $u(0,\cdot)$ is zero in $\mathcal{G}(\mathbb{R})$. On the other hand, $\widehat{u}_\varepsilon(1,0)=\sqrt[3]{10/\varepsilon}\to\infty$ as $\varepsilon\to 0$. Therefore u is not equal to zero in $\mathcal{G}((0,T)\times\mathbb{R})$, if $T>1$.

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Maurílio Márcio Melo
Instituto de Matemática e Estatística, Universidade Federalde Goiás, Caixa Postal 131, 74001-970 – Goiânia, Brazil
E-mail address: melo@mat.ufg.br