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EFFECT OF HYPERVISCOSITY ON THE NAVIER-STOKES TURBULENCE

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Abstract. In this article, we modified the Navier-Stokes equations by adding a higher order artificial viscosity term to the conventional system. We first show that the solution of the regularized system converges strongly to the solution of the conventional system as the regularization parameter approaches zero, for each dimension $d \leq 4$. Then we show that the use of this artificial viscosity term leads to truncated the number of degrees of freedom in the long-time behavior of the solutions to these equations. This result suggests that the hyperviscous Navier-Stokes system is an interesting model for three-dimensional fluid turbulence.

1. Introduction

We regularize the Navier-Stokes equations by adding a higher-order viscosity term to the conventional system. In this paper we will restrict ourselves to periodic boundary conditions.

$$\begin{gathered}
\frac{du_\varepsilon}{dt} + \varepsilon(-\Delta)^l u_\varepsilon - \nu\Delta u_\varepsilon + (u_\varepsilon.\nabla)u_\varepsilon + \nabla p = f(x), \quad \text{in } \Omega \times (0,\infty) \\
\operatorname{div} u_\varepsilon = 0 \quad \text{in } \Omega \times (0,\infty), \\
p(x+Le_i,t) = p(x,t), \quad u(x+Le_i,t) = u(x,t) \quad i = 1,\dots,d \; t \in (0,\infty) \\
u_\varepsilon(x,0) = u_{\varepsilon_0}(x) \quad \text{in } \Omega,
\end{gathered} \tag{1.1}$$

Where $\Omega = (0,L)^d$ and $(e_1,\dots,e_d)$ is the natural basis of $\mathbb{R}^d$. Here $\varepsilon > 0$ is the artificial dissipation parameter and $\nu > 0$ is the kinematic viscosity of the fluid, $l > 1$. The function u_ε is the velocity vector field, p is the pressure, and f is a given force field. For $\varepsilon = 0$, the model is reduced to the Navier-Stokes system.

In Lions [25], the existence and uniqueness of weak solutions of the modified Navier-Stokes equations were established for all $l > 0$ if $l \geq (d+2)/4$, d is the space dimension.

This type of regularization was proposed by Ladyzhenskaya [20] and Lions [26] who added the artificial hyperviscosity $(-\Delta)^{l/2}$, $l > 2$ to the Navier-Stokes system.

Mathematical model for such fluid motion play an important role in theoretical and computational studies of bipolar fluids [7] and in the regularized Navier-Stokes

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equations (see [7, 26, 28] and the references therein). Hyperviscosity is introduced in the works [28, 30] to demonstrate global unique solvability of the Navier-Stokes equations in three dimensions. Hyperviscosity has been widely used for numerical simulations of turbulence [1, 3, 5, 6] and in computer simulations for oceanic and atmospheric flows (see [4, 23]) or to control the Navier–Stokes equations [31].

A well known example of such a result is the viscosity solution method for the Hamilton-Jacobi equations [26].

In this paper, we will study the effect of hyperviscosity on the Navier-Stokes turbulence. First, we show that the solutions of (1.1) converge strongly to the corresponding solutions of the Navier–Stokes equations for $d \leq 4$. This result can extend to each domain Ω with one finite size.

In this result, we show that the conjecture of Lions [25, Remarque 8.2. SecII] is true, for $d \leq 4$. In addition, it is an extension of a result due to Lions [26] (where only the weak convergence is proved). The results in this article can be seen as an improved version of the convergence results announced by Yuh-Roung and Sritharan [28, 29], in two different ways: On the one hand, we consider here a dimension $d \leq 4$, on the other hand the order viscosity term here is $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$.

Next, we consider the system (1.1) with $l = 2$; i.e., we modified the $3D$ Navier-Stokes system by adding a fourth order artificial viscosity term (Laplacian square) and we show the existence of absorbing sets. This fact implies that the system ($l = 2$) possesses a global attractor $\mathfrak{A}_\varepsilon$.

Finally, we obtain scale-invariant estimates on the Hausdorff and fractal dimensions of the global attractor $\mathfrak{A}_\varepsilon$ independent of ε in terms of the Landau–Lifschitz theory [22] of the number of degrees of freedom in turbulent flow [11, 32]. In fact such an estimate that improves on the Landau-Lifschitz estimates has already been done by Avrin [1] in which hyperviscous terms are spectrally added to the Navier-Stokes equations.

Thus we recover the improvement on the cubic power; i.e., get a bound proportional to $G^{\frac{p}{2}}$ for $p < 3$. The latter should be a possibility, as the attractor results in [1] were not intended to be optimal in this direction. We would then represent an overlapping result that is new as far as we know, although readers familiar with the attractor techniques used may anticipate that such a result is possible in the hyperviscous case given the existing results in [1] and the expected improvement in the Sobolev-space estimates in the fixed uniform hyperviscous case at hand.

In Section 2, we present the relevant mathematical framework for the paper. In Section 3, we show the convergence of the system (1.1) to the conventional Navier–Stokes equations. In Section 4, we consider the hyperviscous system ($l = 2$), we show the existence of a global attractor. In Section 5, we estimate the dimension of the attractor. Finally, we provide in Section 6, explicit upper bounds for the dimension of the global attractor of the modified Navier–Stokes in terms of the relevant physical parameters.

2. Notation and preliminaries

In this section we introduce notations and the definitions of standard functional spaces that will be used throughout the paper. We denote by $H^m(\Omega)$, the Sobolev space of L-periodic functions. These spaces are endowed with the inner product

$$(u, v) = \sum_{|\beta| \leq m} (D^\beta u, D^\beta v)_{L^2(\Omega)}$$

and the norm

$$\|u\|_m = \sum_{|\beta|\leq m} (\|D^\beta u\|^2_{L^2(\Omega)})^{1/2}.$$

$H^{-m}(\Omega)$ Denote the dual space of $H^m(\Omega)$.

We denote by $\dot{H}^m(\Omega)$ the subspace of $H^m(\Omega)$ with, zero average

$$\dot{H}^m(\Omega) = \{u \in H^m(\Omega); \int_\Omega u(x)dx = 0\}.$$

For $m = 0$, we have $\dot{H}^m(\Omega) = \dot{L}^2(\Omega)$.

We introduce the following solenoidal subspaces V_s, $s \in \mathbb{R}^+$ which are important to our analysis

$$\begin{gathered}
V_0(\Omega) = \{u \in \dot{L}^2(\Omega) : \operatorname{div} u = 0, u.n\big|_{\Sigma_i} = -u.n\big|_{\Sigma_{i+3}},\ i = 1, 2, 3\};\\
V_1(\Omega) = \{u \in \dot{H}^1(\Omega) : \operatorname{div} u = 0, \gamma_0 u\big|_{\Sigma_i} = \gamma_0 u\big|_{\Sigma_{i+3}},\ i = 1, 2, 3\};\\
V_2(\Omega) = \{u \in \dot{H}^2(\Omega) : \operatorname{div} u = 0, \gamma_0 u\big|_{\Sigma_i} = \gamma_0 u\big|_{\Sigma_{i+3}},\\
\gamma_1 u\big|_{\Sigma_i} = -\gamma_1 u\big|_{\Sigma_{i+3}},\ i = 1, 2, 3\};
\end{gathered}$$

see [32, Chapter III, Section 2]. We refer the reader to Temam [33] for details on these spaces. Here the faces of Ω are numbered as

$$\Sigma_i = \partial\Omega \cap \{x_i = 0\} \quad \text{and} \quad \Sigma_{i+3} = \partial\Omega \cap \{x_i = L\},\ i = 1, 2, 3.$$

Here γ_0, γ_1 are the trace operators and n is the unit outward normal on $\partial\Omega$.

- The space V_0 is endowed with the inner product $(u, v)_{L^2(\Omega)}$ and norm $\|u\| = (u, u)^{1/2}_{L^2(\Omega)}$.
- V_1 Is the Hilbert space with the norm $\|u\|_1 = \|u\|_{V_1}$. The norm induced by $\dot{H}^1(\Omega)$ and the norm $\|\nabla u\|$are equivalent in V_1.
- V_2 Is the Hilbert space with the norm $\|u\|_2 = \|u\|_{V_2}$. In V_2 the norm induced by $\dot{H}^2(\Omega)$ is equivalent to the norm $\|\Delta u\|$.

Let V_s' denote the dual space of V_s. We denote by A the Stokes operator

$$Au = -\Delta u \text{ for } u \in D(A).$$

We recall that the operator A is a closed positive self-adjoint unbounded operator, with $D(A) = \{u \in V_0, Au \in V_0\}$. We have in fact,

$$D(A) = \dot{H}^2(\Omega) \cap V_0 = V_2.$$

The eigenvalues of A are $\{\lambda_j\}_{j=1}^{j=\infty}$, $0 < \lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding orthonormal set of eigenfunctions $\{w_j\}_{j=1}^{j=\infty}$ is complete in V_0

$$Aw_j = \lambda_j w_j, \quad w_j \in D(A_1).$$

The spectral theory of A allows us to define the powers A^l of A for $l \geq 1$, A^l is an unbounded self-adjoint operator in V_0 with a domain $D(A^l)$ dense in $V_2 \subset V_0$. We set here

$$A^l u = (-\Delta)^l u \quad \text{for } u \in D(A^l) = V_{2l} \cap V_0.$$

The space $D(A^l)$ is endowed with the scalar product and the norm

$$(u, v)_{D(A^l)} = (A^l u, A^l v), \quad \|u\|_{D(A^l)} = \{(u, u)_{D(A^l)}\}^{1/2}.$$

Let us now define the trilinear form $b(\cdot,\cdot,\cdot)$ associated with the inertia terms

$$b(u,v,w)=\sum_{i,j=1}^{3}\int_{\Omega}u_i\frac{\partial v_j}{\partial x_i}w_j dx.$$

The continuity property of the trilinear form enables us to define (using Riesz representation Theorem) a bilinear continuous operator $B(u,v)$; $V_2\times V_2\to V_2'$ will be defined by

$$\langle B(u,v),w\rangle=b(u,v,w),\quad \forall w\in V_2. \tag{2.1}$$

Recall that for u satisfying $\nabla .u=0$, we have

$$b(u,u,u)=0,\quad b(u,v,w)=-b(u,w,v). \tag{2.2}$$

Hereafter, c_i for $i\in\mathbb{N}$, will denote a dimensionless scale invariant positive constant which might depend on the shape of the domain. Similarly, the trilinear form $b(u,v,w)$ satisfies the well-known inequalities (see, for instance, [30, Lemma 61.1] and [8, 33])

$$|b(u,v,u)|\le c_1\|u\|^{1/2}\|u\|_1^{3/2}\|v\|_1\quad \text{for all } u,v\in V. \tag{2.3}$$

We recall some well known inequalities that we will be using in what follows.

The Ladyzhenskaya inequality (cf.[19]) in $\mathbb{R}^3$

$$\|u\|_{L^\theta(\Omega)}\le c_2\|u\|_{L^2(\Omega)}^{\frac{6-\theta}{2\theta}}\|u\|_{H^1(\Omega)}^{\frac{3(\theta-2)}{2\theta}} \tag{2.4}$$

for every $u\in H^1(\Omega)$, $2\le\theta\le 6$.

Agmon inequality (see, e.g., [8])

$$\|u\|_\infty\le c_3\|u\|_1^{1/2}\|Au\|^{1/2}\quad \text{for all } u\in V_2 \tag{2.5}$$

Young's inequality

$$ab\le\frac{\sigma}{p}a^p+\frac{1}{q\sigma^{\frac{q}{p}}}b^q,\quad a,b,\sigma>0,\quad p>1,\quad q=\frac{p}{p-1}. \tag{2.6}$$

Poincaré inequality

$$\lambda_1\|u\|^2\le\|A^{1/2}u\|^2\quad \text{for all } u\in V. \tag{2.7}$$

To prove uniform bounds on different norms we use the uniform Gronwall Lemma; for a proof see [32, Lemma III 1.1].

Lemma 2.1 (The Uniform Gronwall Lemma)**.** *Let g, h, y be three positive locally integrable functions on $(t_{0,}+\infty)$ which satisfy*

$$\frac{dy}{dt}\le gy+h\quad \textit{for } t\ge t_0,$$
$$\int_t^{t+r}g(s)ds\le a_1,\quad \int_t^{t+r}h(s)ds\le a_2,\quad \int_t^{t+r}y(s)ds\le a_3\quad \textit{for } t\ge t_0,$$

where a_1, a_2, a_3 are positive constants. Then

$$y(t+r)\le(\frac{a_3}{r}+a_2)\exp(a_1)\quad \textit{for } t\ge t_{0.}$$

Denoting by G the dimensionless Grashoff number [10], this number measures the relative strength of the forcing and viscosity.

3. Strong convergence for the hyperviscous system

In this Section, we give a new Theorem which ensures the strong convergence of the solutions of the system (1.1) to the corresponding solutions of the Navier–Stokes equations for $d \leq 4$. This result can extend to each domain Ω with one finite size. Moreover, we show that $u_\varepsilon \in C(0,T;V_0)$.

Using the operators defined above, we can write the modified system (1.1) in the evolution form

$$\frac{du_\varepsilon}{dt} + \varepsilon A^l u_\varepsilon + \nu A u_\varepsilon + B(u_\varepsilon, u_\varepsilon) = f(x), \quad \text{in } \Omega \times (0,\infty) \tag{3.1}$$

$$u_{\varepsilon_0}(x) = u_{\varepsilon_0}, \quad \text{in } \Omega. \tag{3.2}$$

The existence and uniqueness results for initial value problem (1.1) can be found in [25], [26, Chap.1, Remarque 6.11]. The following theorem collects the main result in this work

Theorem 3.1. *For $l \geq (d+2)/4$, d is the space dimension, for $\varepsilon > 0$ fixed, $f \in L^2(0,T;V_0')$ and $u_{\varepsilon_0} \in V_0$ be given. There exists a unique weak solution of* (1.1) *which satisfies*

$$u_\varepsilon \in L^2(0,T;V_l) \cap L^\infty(0,T;V_0), \quad \forall T > 0.$$

Note that the conventional Navier-Stokes system can be written in the evolution form

$$\frac{du}{dt} + \nu A u + \hat{B}(u,u) = f(x) \quad \text{in } \Omega \times (0,\infty) \tag{3.3}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{3.4}$$

Theorem 3.2. *For $d \leq 4$, for $f \in L^2(0,T;V_0)$ and $u_0 \in V_0$ be given. There exists a weak solution of* (3.3)-(3.4) *which satisfies $u \in L^\infty(0,T;V_0) \cap L^2(0,T;V_1)$, for $T > 0$. For $d = 2$, u is unique (Lions* [25]*).*

We will establish various estimates uniform in ε for the solutions of the modified Navier Stokes. These bounds will be used to establish the limit of these solutions to the conventional Navier Stokes equations.

Proposition 3.3. *For $d \leq 4$ and for $\varepsilon > 0$ fixed, $f \in L^2(0,T;V_0)$ and $u_{\varepsilon_0} \in V_0$. The weak solution $u_\varepsilon(t)$ of the modified Navier-Stokes equations satisfy*

- (i) *u_ε is uniformly bounded in $L^\infty(0,T;V_0)$,*
- (ii) *u_ε is uniformly bounded in $L^2(0,T;V_1)$.*

We need the following Lemma proved in Temam [33, Lemma 4.1.ChIII,Sec4].

Lemma 3.4. *The form b is trilinear continuous on $V \times V \times V_s$ if $s \geq d/2$ and*

$$\|b(u,v,w)\| \leq c_4 \|u\| \|v\|_1 \|w\|_s.$$

Applying Lemma 3.4, we obtain the following result.

Lemma 3.5. *Let $u_\varepsilon(t)$ be a weak solution of the modified Navier-Stokes system. Then $B(u_\varepsilon)$ belongs to $L^2(0,T;V_l')$ for $l \geq d/2$.*

Proof. By the definition of the operator B and the above Lemma, we obtain

$$|\langle B(u(t),v)\rangle| = |b(u(t),u(t),v)| \leq c_4 \|u(t)\| \|u(t)\|_1 \|v\|_{V_l'}, \quad \forall v \in V_l.$$

Thus,

$$\|B(u(t))\|_{V_l'} \leq c_4 \|u(t)\| \|u(t)\|_1 \quad \text{for } 0 \leq t \leq T.$$

$\square$

Lemma 3.6. *If $f \in L^2(0,T;V_1')$, then, for any solution $u_\varepsilon(t)$ of problem* (1.1) *the time derivative $\frac{du_\varepsilon}{dt}$ is uniformly bounded in $L^2(0,T;V_l')$.*

Proof. Due to Lemma 3.5 $B(u_\varepsilon)$ belongs to $L^2(0,T;V_l')$, since $f - \varepsilon A^l u_\varepsilon - \nu A u_\varepsilon$ belongs to $L^2(0,T;V_l')$, this implies that $\frac{du_\varepsilon}{dt}$ belongs to $L^2(0,T;V_l')$. $\square$

Lemma 3.7. *The function u_ε is almost everywhere equal to a continuous function from $[0,T]$ to the space V_0.*

Proof. Since $u_\varepsilon \in L^2(0,T;V_1) \cap L^\infty(0,T;V_0)$ and $\frac{du_\varepsilon}{dt} \in L^2(0,T;V_l')$, the weak continuity in V_0 is a direct consequence of [33, Lemma 1.4.ChIII,Sec1].

Similarly, it follows that $u_\varepsilon(0)$ converges to $u(0)$ in V_0, and since u_{ε_0} converges to u_0 in V_l', we conclude that $u(0) = u_0$. $\square$

Now we prove the strong convergence. It follows from (ii) of Proposition 3.3 and from Lemma 3.6, that

$$u_{\varepsilon_n} \in \mathcal{X} = \{u_{\varepsilon_n} \in L^2(0,T;V_1),\ \frac{du_{\varepsilon_n}}{dt} \in L^2(0,T;V_l')\}$$

with bounds independent of ε_n. Hence (i) $u_{\varepsilon_n} \to u$ in $L^2(0,T;V_l)$ weakly; and (ii) $\frac{du_{\varepsilon_n}}{dt} \to \frac{du}{dt}$ in $L^2(0,T;V_l')$ weakly; These two properties allow us to establish the strong convergence result.

The proof of the following theorem can be found in Temam [33, Theorem 2.1, Chapter III, Sec 2].

Theorem 3.8. *The injection of $\mathcal{X} = \{u \in L^2(0,T;V_1),\ \frac{du_\varepsilon}{dt} \in L^2(0,T;V_l')\}$ into $\mathcal{Y} = \{u \in L^2(0,T;V_0)\}$ is compact.*

By virtue of the above estimates and the compactness Theorem 3.8. We can now state our first result.

Theorem 3.9. *For $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$ and for $d \leq 4$, the weak solution u_ε of the modified Navier-Stokes equations* (1.1) *given by Theorem 3.1 converges strongly in $L^2(0,T;V_0)$ as $\varepsilon \to 0$ to u the weak solution of the system* (3.1)-(3.2).

Proof. Theorem 3.1 and Lemma 3.4 are satisfied for $l \geq \sup(\frac{d}{2}, \frac{d+2}{4})$. We use part *ii)* of Proposition 3.3 and Lemma 3.6 we can deduce that the weak solutions $u_{\varepsilon_n} \in \mathcal{X} = \{u_{\varepsilon_n} \in L^2(0,T;V_1),\ \frac{du_{\varepsilon_n}}{dt} \in L^2(0,T;V_l')\}$. Hence, the compactness Theorem 3.8 implies the strong convergence in $L^2(0,T;V_0)$. $\square$

The following proposition is a consequence of Proposition 3.3.

Proposition 3.10. *For all $w \in L^2(0,T;V_1)$, $\forall \frac{dw}{dt} \in L^2(0,T;V_1')$*

(a) $\lim_{n\to\infty} \int_0^T (\frac{du_{\varepsilon_n}(t)}{dt}, w)dt = \int_0^T (\frac{du(t)}{dt}, w(t))dt,$
(b) $\lim_{n\to\infty} \int_0^T (\nabla u_{\varepsilon_n}(t), \nabla w(t))dt = \int_0^T (\nabla u(t), \nabla w(t))dt,$
(c) $\lim_{n\to\infty} \int_0^T b(u_{\varepsilon_n}(t), u_{\varepsilon_n}(t), w(t))dt = \int_0^T b(u(t), u(t), w(t))dt.$

Let us now establish the limit of the equations (3.1) as $\varepsilon_n \to 0$. Taking the inner product of (3.1) with a test function $\varphi \in \mathcal{D}(0,T;\mathcal{D}(A^{l/2}))$ then integrate by parts and using the convergence Proposition 3.10 we can pass to the limit as $\varepsilon_n \to 0$, we get $-\int_0^T (u,\varphi')dt + \nu\int_0^T(\nabla u,\nabla\varphi)dt + \int_0^T b(u,u,\varphi)dt = \int_0^T \langle f,\varphi\rangle dt$.

Here the term $\varepsilon_n \int_0^T (A^{l/2}u_{\varepsilon_n}(t), A^{l/2}\varphi(t))dt$ approaches 0 as $\varepsilon_n \to 0$. Since the weak solution u_{ε_n} is in $L^2(0,T;V_1)$ with a uniform bound in ε_n and we obtain

$$\varepsilon_n \int_0^T |(A^{l/2}u_{\varepsilon_n}, A^{l/2}\varphi)|dt \le \varepsilon_n \int_0^T |(u_{\varepsilon_n}, A^l\varphi)|dt \le c\varepsilon_n.$$

Since $u \in L^2(0,T;V_1) \cap L^\infty(0,T;V_0)$, we can conclude that u is indeed the weak solution for the conventional Navier-Stokes equations.

4. The hyperviscous Navier-Stokes system and attractors

Now, we consider modifications of the 3D Navier-Stokes system by adding a fourth order artificial viscosity term (Laplacian square) depending on a small parameter ε to the conventional system.

$$\begin{gathered}
\frac{du_\varepsilon}{dt} + \varepsilon A^2 u_\varepsilon + \nu A u_\varepsilon + B(u_\varepsilon,u_\varepsilon) = f(x), \quad \text{in } \Omega\times(0,\infty) \\
\operatorname{div} u_\varepsilon = 0, \quad \text{in } \Omega\times(0,\infty), u_\varepsilon(x,0) = u_{\varepsilon_0}(x) \quad \text{in } \Omega, \\
p(x+Le_i,t) = p(x,t), \quad u(x+Le_i,t) = u(x,t) \quad i=1,2,3.\ t\in(0,\infty)
\end{gathered} \tag{4.1}$$

where $\Omega = (0,L)^3$. In this section we will show the existence of the compact global attractor $\mathfrak{A}_\varepsilon$ associated with the semigroup $S_\varepsilon(t)$ generated by the problem (4.1). For the theory of global attractors see [2, 8, 14, 18, 27, 30, 32].

For $\varepsilon = 0$ weak solutions of problem are known to exist by a basic result by Leray from 1934 [24], only the uniqueness of weak solutions remains as an open problem. Then the known theory of global attractors of infinite dimensional dynamical systems is not applicable to the 3D Navier–Stokes system.

The theory of trajectory attractors for evolution partial differential equations was developed in [30], which the uniqueness theorem of solutions of the corresponding initial-value problem is not proved yet, e.g. for the 3D Navier–Stokes system (see, for instance,[14, 30]). Such trajectory attractor is a classical global attractor but in the space of weak solutions.

The problem of upper semicontinuity of global attractors for the 2D with periodic boundary conditions was discussed by Yuh-Roung Ou and S. S. Sritharan in [28]. For related results which use the theory has been introduced by Foias, Sell, and Temam in [12, 32] to show that the system (1.1) possesses an inertial manifold (see [1, 29, 32]).

The existence and uniqueness results for initial value problem (4.1) are consequence of Theorem 3.9 for $l = 2$ and $d = 3$.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^3$, and let $f \in L^2(0,T;V_2')$ and $u_{\varepsilon_0} \in V_0$ be given. Then there exists a unique weak solution of (4.1) which satisfies $u_\varepsilon \in C([0,T];V_0) \cap L^2(0,T;V_2)$, for all $T > 0$. Then as $\varepsilon \to 0$, the solution u_ε converges to the weak solution of the Navier-Stokes equations.*

Now, we show that the semigroup $S_\varepsilon(t)$ has an absorbing ball in V_0 and an absorbing ball in V_1. Then we show that $S_\varepsilon(t)$ admits a compact attractor in V_0 for each $\varepsilon \ge 0$.

We take the inner product of (4.1) with u_ε, we obtain the energy equality

$$\frac{d}{dt}\|u_\varepsilon\|^2 + 2\varepsilon\|Au_\varepsilon\|^2 + 2\nu\|\nabla u_\varepsilon\|^2 = 2(f, u_\varepsilon).$$

Here we have used the fact that $b(u_\varepsilon, u_\varepsilon, u_\varepsilon) = 0$. By applying Young's inequality and the Poincaré Lemma, we get

$$\frac{d}{dt}\|u_\varepsilon\|^2 + 2\varepsilon\|Au_\varepsilon\|^2 + \nu\|\nabla u_\varepsilon\|^2 \le \frac{\|f\|^2}{\nu\lambda_1}, \tag{4.2}$$

we drop the term $2\varepsilon\|Au_\varepsilon\|^2$, we obtain

$$\frac{d}{dt}\|u_\varepsilon\|^2 + \nu\lambda_1\|u_\varepsilon\|^2 \le \frac{\|f\|^2}{\nu\lambda_1},$$

by integrating the above inequality from 0 to t,we get

$$\|u_\varepsilon(t)\|^2 \le \|u_{\varepsilon_0}\|^2 e^{-\nu\lambda_1 t} + \rho_0^2(1 - e^{-\nu\lambda_1 t}),\ t > 0, \tag{4.3}$$

where $\rho_0 = \frac{1}{\nu\lambda_1}\|f\|$. Hence for any ball $B_{R_0} = \{u_{\varepsilon_0} \in V_0; \|u_{\varepsilon_0}\| \le R_0\}$ there is a ball $B(0, \delta_0)$ in V_0 centered at origin with radius $\delta_0 > \rho_0$ ($R_0 > \delta_0$) such that

$$S_\varepsilon(t)B_{R_0} \subset B_{r_0} \quad \text{for } t \ge t_0(B_{R_0}) = \frac{1}{\nu\lambda_1}\log\frac{R_0^2 - \rho_0^2}{\delta_0^2 - \rho_0^2}. \tag{4.4}$$

The ball B_{δ_0} is said to be absorbing and invariant under the action of $S_\varepsilon(t)$.

Taking the limit in (4.3) we obtain

$$\limsup_{t\to\infty}\|u_\varepsilon(t)\| \le \rho_0. \tag{4.5}$$

We integrate (4.2) from t to $t + r$, we obtain for $u_{\varepsilon 0} \in B_{R_0}$,

$$\int_t^{t+r}\|u_\varepsilon\|_1^2 ds \le \frac{1}{\nu}(\frac{r\|f\|^2}{\nu\lambda_1} + \|u_\varepsilon(t)\|^2), \forall r > 0,\ \forall t \ge t_0(B_{R_0}). \tag{4.6}$$

With the use of (4.5) we conclude that

$$\limsup_{t\to\infty}\int_t^{t+r}\|u_\varepsilon\|_1^2 ds \le \frac{r}{\nu^2\lambda_1}\|f\|^2 + \frac{\|f\|^2}{\nu^3\lambda_1^2}, \tag{4.7}$$

from which we obtain

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|u_\varepsilon\|_1^2 ds \le \frac{\|f\|^2}{\nu^2\lambda_1}, \tag{4.8}$$

this verifies that the left-hand side is finite.

To show that the semigroup $S_\varepsilon(t)$ has an absorbing set in V_1, we consider the strong solutions and take the inner product of (4.1) with Au_ε, we obtain

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}u_\varepsilon\|^2 + \varepsilon\|A^{3/2}u_\varepsilon\|^2 + \nu\|Au_\varepsilon\|^2 = -b(u_\varepsilon, u_\varepsilon, Au_\varepsilon) + (f, Au_\varepsilon). \tag{4.9}$$

By applying Young's inequality, we obtain

$$(f, Au_\varepsilon) \le \|f\|\|Au_\varepsilon\| \le \frac{\nu}{4}\|Au_\varepsilon\|^2 + \frac{1}{\nu}\|f\|^2.$$

By using the Agmon's inequality (2.5) and Young's inequality we can estimate the last term in the left-hand side of (4.9) as follows

$$\begin{aligned}|b(u_\varepsilon, u_\varepsilon, Au_\varepsilon)| &\le \|u_\varepsilon\|_\infty \|u_\varepsilon\|_1 \|Au_\varepsilon\| \\ &\le c_4 \|u_\varepsilon\|_1^{3/2} \|Au_\varepsilon\|^{3/2} \\ &\le \frac{\nu}{4}\|Au_\varepsilon\|^2 + c_4\|u_\varepsilon\|_1^6.\end{aligned}$$

Hence we obtain from (4.9)

$$\frac{d}{dt}\|u_\varepsilon\|_1^2 + 2\varepsilon\|A^{3/2}u_\varepsilon\|^2 + \nu\|Au_\varepsilon\|^2 \le \frac{2}{\nu}\|f\|^2 + 2c_5\|u_\varepsilon\|_1^6.$$

Dropping the positive terms associated with ε we have

$$\frac{d}{dt}\|u_\varepsilon\|_1^2 + \nu\|A_1u_\varepsilon\|^2 \le \frac{2\|f\|^2}{\nu} + 2c_4\|u_\varepsilon\|_1^6 \tag{4.10}$$

we apply the uniform Gronwall Lemma to (4.10) with

$$g = 2c_4\|u_\varepsilon\|_1^4, \quad h = \frac{2\|f\|^2}{\nu}, \quad y = \|u_\varepsilon\|_1^2.$$

Thanks to (4.3)-(4.7) we estimate the quantities a_1, a_2, a_3 in Gronwall Lemma by

$$a_1 = 2c_4a_3^2, \quad a_2 = \frac{2r\|f\|^2}{\nu}, \quad a_3 = \frac{r\|f\|^2}{\nu^2\lambda_1} + \frac{\|f\|^2}{\nu^3\lambda_1^2}$$

and we obtain

$$\|u_\varepsilon(t)\|_1^2 \le (\frac{a_3}{r} + a_2)\exp(a_1) = R_1^2 \quad \text{for } t \ge t_0,\ t_0 \text{ as in (4.4)}.$$

Hence, for any ball B_{R_1}, there exists a ball B_{δ_1}, in V_1 centered at origin with radius $R_1 > \delta_1 > \rho_1$ such that

$$S_\varepsilon(t)B_{R_1} \subset B_{\delta_1} \quad \text{for } t \ge t_1(B_{R_0}) = t_0(B_{R_0}) + 1 + \frac{1}{\nu\lambda_1}\log\frac{R_1^2 - \rho_1^2}{\delta_1^2 - \rho_1^2}.$$

The ball B_{δ_1} is said to be absorbing and invariant for the semigroup $S_\varepsilon(t)$.

Furthermore, if B is any bounded set of V_0, then $S_\varepsilon(t)B \subset B_{\delta_1}$ for $t \ge t_1(B, R_0)$, this shows the existence of an absorbing set in V_1. Since the embedding of V_1 in V_0 is compact, we deduce that $S_\varepsilon(t)$ maps a bounded set in V_0 into a compact set in V_0. In addition, the operators $S_\varepsilon(t)$ are uniformly compact for $t \ge t_1(B, R_0)$. That is,

$$\cup_{t\ge t_1} S_\varepsilon(t, 0, B_{R_0})$$

is relatively compact in V_0.

Due to a the standard procedure (cf., for example, [32, Theorem I.1.1] for details), one can prove that there is a global compact attractor $\mathfrak{A}_\varepsilon$ for the operators $S_\varepsilon(t)$ for $\varepsilon \ge 0$. Note that the global attractor $\mathfrak{A}_\varepsilon$ must be contained in the absorbing balls V_0 and V_1

$$\mathfrak{A}_\varepsilon = \cap_{t_1\ge 0}\overline{\cup_{t\ge t_1} B_{\delta_1}(t)} \subset B_{\delta_0} \cap B_{\delta_1}. \tag{4.11}$$

Notice that all the above bounds are independent of ε.

5. Estimates of Dimensions of the Global Attractor

Our aim in this section is to study the finite dimensionality of the global attractor. In the first part we will prove the differentiability property of $S_\varepsilon(t)$ and in the second part we will provide estimates of the fractal and Hausdorff dimensions of their global attractors $\mathfrak{A}_\varepsilon$.

Using the trace formula [32, Chapters V and VI], we estimate the Hausdorff and the fractal dimensions of the global attractor $\mathfrak{A}_\varepsilon$ in V.

For a solution $u_\varepsilon(t) = S_\varepsilon(t)u_{\varepsilon_0}$, $t \geq 0$, lying on the attractor $u_{\varepsilon_0} \in \mathfrak{A}_\varepsilon$, we see from (4.1) that the linearized flow around u_ε is given by the equation

$$\begin{gathered} U_\varepsilon' + \varepsilon A^2 U_\varepsilon + \nu A U_\varepsilon + B(u_\varepsilon, U_\varepsilon) + B(U_\varepsilon, u_\varepsilon) = 0, \quad \text{in } V' \\ U_\varepsilon(0) = \xi, \quad \text{in } V. \end{gathered} \tag{5.1}$$

We show the differentiability of the semigroup S_ε with respect to the initial data in the space V.

Theorem 5.1. *For any $t > 0$, the function $u_{\varepsilon_0} \to u_\varepsilon(t) = S_\varepsilon(t)u_{\varepsilon_0}$ is Fréchet differentiable on the attractor $\mathfrak{A}_\varepsilon$. Its differential is the linear operator*

$$D(S_\varepsilon(t)u_{\varepsilon_0}) = L(t, u_{\varepsilon_0}) : \xi \in V \to U_\varepsilon(t) \in V,\ t \in [0, T],$$

where $U_\varepsilon(t)$ is the solution of (5.1).

Proof. Let

$$\eta(t) = v_\varepsilon(t) - u_\varepsilon(t) - U_\varepsilon(t), \quad U_\varepsilon(0) = \xi = v_{\varepsilon_0} - u_{\varepsilon_0}.$$

Clearly, η satisfies

$$\eta_t + \varepsilon A^2\eta + \nu A\eta + B(\eta, v_\varepsilon) + B(v_\varepsilon, \eta) - B(w_\varepsilon, w_\varepsilon) = 0, \quad \eta(0) = 0$$

where $w_\varepsilon = v_\varepsilon - u_\varepsilon$. Taking the inner product of the last equation with η and using the identity $B(v_\varepsilon, \eta, \eta) = 0$ we obtain

$$\frac{d\|\eta\|^2}{dt} + 2\varepsilon\|A\eta\|^2 + 2\nu\|\eta\|_1^2 = 2b(\eta, v_\varepsilon, \eta) - 2b(w_\varepsilon, w_\varepsilon, \eta). \tag{5.2}$$

By (2.3) the first term in the right-hand side of (5.2) has the estimate

$$\begin{aligned} |2b(\eta, v_\varepsilon, \eta)| &\leq 2c_1\|\eta\|^{1/2}\|\eta\|_1^{\frac{3}{2}}\|v_\varepsilon\|_1 \\ &\leq 2c_1R_1\|\eta\|^{1/2}\|\eta\|_1^{3/2} \\ &\leq \frac{c_1^4R_1^4}{\nu^3}\|\eta\|^2 + \frac{3\nu}{4}\|\eta\|_1^2. \end{aligned}$$

Employing the inequalities (2.3) we estimate the second term in the right hand side of (5.2) as follows

$$2b(w_\varepsilon, w_\varepsilon, \eta) \leq 2c_2\|\eta\|_1\|w_\varepsilon\|_1^2 \leq \frac{2c_2^2}{\nu}\|w_\varepsilon\|_1^4 + \frac{\nu}{2}\|\eta\|_1^2.$$

Hence, we obtain from (5.2)

$$\frac{d\|\eta\|^2}{dt} + 2\varepsilon\|A\eta\|^2 + \frac{3\nu}{4}\|\eta\|_1^2 \leq \frac{c_1^4R_1^4}{\nu^3}\|\eta\|^2 + \frac{2c_1^2}{\nu}\|w_\varepsilon\|_1^4$$

we drop the positive terms $2\varepsilon\|A\eta\|^2$ and $\frac{3\nu}{4}\|\eta\|_1^2$ we get

$$\frac{d\|\eta\|^2}{dt} \leq \frac{c_1^4R_1^4}{\nu^3}\|\eta\|^2 + \frac{2c_1^2}{\nu}\|w_\varepsilon\|_1^4. \tag{5.3}$$

From the classical Gronwall Lemma (see [33]), (5.3) gives

$$\|\eta\|^2 \leq \frac{2c_1^2}{\nu}\int_0^t \|w_\varepsilon\|_1^4 \exp(\int_s^t \frac{c_1^4 R_1^4}{\nu^3} d\tau) ds.$$

Thus

$$\|\eta\|^2 \leq C_o \int_0^t \|w_\varepsilon\|_1^4 ds, \quad C_o = \frac{2c_1^2}{\nu}\exp(\frac{Tc_1^4 R_1^4}{\nu^3}). \tag{5.4}$$

The difference

$$w_\varepsilon(t) = v_\varepsilon(t) - u_\varepsilon(t) = S_\varepsilon(t)v_{\varepsilon_0} - S_\varepsilon(t)u_{\varepsilon_0}$$

satisfies the equation

$$\frac{dw_\varepsilon}{dt} + \varepsilon A^2 w_\varepsilon + \nu A w_\varepsilon + B(w_\varepsilon, v_\varepsilon) + B(v_\varepsilon, w_\varepsilon) - B(w_\varepsilon, w_\varepsilon) = 0,$$
$$w_\varepsilon(0) = v_{\varepsilon_0} - u_{\varepsilon_0} = w_{\varepsilon_0}.$$

Taking the inner product of the last equation with w_ε, we obtain

$$\frac{d}{dt}\|w_\varepsilon\|^2 + 2\varepsilon\|Aw_\varepsilon\|^2 + 2\nu\|w_\varepsilon\|_1^2 = 2b(w_\varepsilon, w_\varepsilon, v_\varepsilon). \tag{5.5}$$

By using inequalities (2.3), and Young's inequality we obtain

$$|2b(w_\varepsilon, v_\varepsilon, w_\varepsilon)| \leq 2c_1\|v_\varepsilon\|\|w_\varepsilon\|_1^{3/2}\|w_\varepsilon\|^{1/2} \leq \frac{c_1^4 R^4}{\nu^3}\|w_\varepsilon\|^2 + \frac{3\nu}{4}\|w_\varepsilon\|_1^2.$$

Substituting the above result into (5.5), we obtain

$$\frac{d}{dt}\|w_\varepsilon\|^2 + 2\varepsilon\|Aw_\varepsilon\|^2 + \frac{5\nu}{4}\|w_\varepsilon\|_1^2 \leq \frac{c_1^4 R^4}{\nu^3}\|w_\varepsilon\|^2. \tag{5.6}$$

We drop the positive terms $2\varepsilon\|Aw_\varepsilon\|^2$ and $\frac{5\nu}{4}\|w_\varepsilon\|_1^2$ to obtain the following differential inequality

$$\frac{d}{dt}\|w_\varepsilon\|^2 \leq \frac{c_1^4 R^4}{\nu^3}\|w_\varepsilon\|^2. \tag{5.7}$$

Using the classical Gronwall Lemma we deduce from (5.7) that

$$\|w_\varepsilon\|^2 \leq \|w_\varepsilon(0)\|^2 \exp(\frac{Tc_1^4 R^4}{\nu^3}). \tag{5.8}$$

From (5.8) we deduce that

$$\int_0^t \|u_\varepsilon(t) - v_\varepsilon(t)\|^2 dt \leq C_1\|u_{\varepsilon 0} - v_{\varepsilon_0}\|^2, \quad C_1 = T\exp(\frac{Tc_1^4 R^4}{\nu^3}), \tag{5.9}$$

with (5.4) we conclude that

$$\|\eta\|^2 \leq C_o C_1^2 \|u_{\varepsilon 0} - v_{\varepsilon_0}\|^4,$$

then we deduce from (5.4) and (5.9) that

$$\|\eta\|^2 \leq C_2\|w_\varepsilon(0)\|^4, \quad \text{where } C_2 = \frac{2c_1^2 T^2}{\nu}\exp(\frac{Tc_1^4(2R^4 + R_1^4)}{\nu^3}) \tag{5.10}$$

this shows that

$$\frac{\|v_\varepsilon(t) - u_\varepsilon(t) - U_\varepsilon(t)\|^2}{\|v_{\varepsilon_0} - u_{\varepsilon_0}\|^2} \leq C_2\|v_{\varepsilon_0} - u_{\varepsilon_0}\|^2 \to 0 \quad \text{as } \|v_{\varepsilon_0} - u_{\varepsilon 0}\|_1 \to 0 \text{ on } \mathfrak{A}_\varepsilon.$$

The differentiability of $S_\varepsilon(t)$ is proved. $\square$

From Theorem 5.1 the function $S_\varepsilon(t)$ is Fréchet differentiable on $\mathfrak{A}_\varepsilon$ for $t>0$. For $\xi \in V_0$, there exists a unique solution U_ε of (5.1) satisfies

$$U_\varepsilon \in C([0,T];V_0)\cap L^2(0,T;V_2) \quad \forall T>0.$$

With the differentiability ensured in Theorem 4.1 we can then define a linear map $L(t;u_{\varepsilon_0}):\xi\in V_0 \to U_\varepsilon(t)\in V_0$ where U_ε is the solution of (5.1).

We can apply the trace formula (see [8] and [32, Section V. 3]) to find a bound on the dimension of the global attractor $\mathfrak{A}_\varepsilon$. We consider the trace $TrF'(u_\varepsilon)$ of the linear operator $F'(u_\varepsilon)$ and for $m\in\mathbb{N}$, the number

$$q_m = \limsup_{t\to\infty} \sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon} \sup_{\substack{\xi_1\in V_0,\,|\xi_1|\le 1\\ i=1,\dots,m}} \frac{1}{t}\int_0^t TrF'(S_\varepsilon(\tau)u_{\varepsilon_0})\circ Q_m(\tau)d\tau$$

where $Q_m(\tau)=Q_m(\tau,u_{\varepsilon_0};\xi_1,\dots,\xi_m)$ is the orthogonal projector in V_0 onto the space spanned by $U_\varepsilon^1(\tau),\dots,U_\varepsilon^m(\tau)$. where $U_\varepsilon^j(\tau)=L(\tau,u_{\varepsilon_0}).\xi_j$, $j=1,\dots,m$, $t\ge 0$, are m solutions of (5.1), corresponding to $\xi=\xi_1,\dots,\xi_m\in V_1$. Let $\varphi_j(\tau)$, $j=1,\dots,m$, $\tau\ge 0$, be an orthonormal basis of for $\tilde{Q}_m(\tau)V_0$ =span $\{U_\varepsilon^1(\tau),\dots,U_\varepsilon^m(\tau)\}$, $\varphi_j(t)\in V_1$ for $j=1,\dots,m$, since $U_\varepsilon^1(\tau),\dots,U_\varepsilon^m(\tau)\in V_1$, $\tau\in\mathbb{R}^+$.

From the general result in [32, Section V.3.41], we have that if $q_m<0$ for some $m\in N$ then the global attractor has finite Hausdorff and fractal dimensions estimated respectively as

$$\dim_H(\mathfrak{A}_\varepsilon)\le m, \tag{5.11}$$

$$\dim_F(\mathfrak{A}_\varepsilon)\le m\Big(1+\max_{1\le j\le m-1}\frac{(q_j)_+}{\|q_m\|}\Big). \tag{5.12}$$

Then we have

$$\begin{aligned} TrF'(S_\varepsilon(\tau)u_{\varepsilon 0})\circ Q_m(\tau) &= \sum_{j=1}^{\infty}(TrF'(u_\varepsilon(\tau))\circ Q_m(\tau)\varphi_j(\tau),\varphi_j(\tau))\\ &= \sum_{j=1}^{m}(F'(u_\varepsilon(\tau))\varphi_j(\tau),\varphi_j(\tau)), \end{aligned}$$

Recall that $(\cdot,\cdot)$ denotes the scalar product in V_0, we write using (2.1) and (2.2),

$$\begin{aligned} &Tr(F'(u_\varepsilon(\tau))\varphi_j(\tau),\varphi_j(\tau))\\ &=\sum_{j=1}^{m}(-\varepsilon A^2\varphi_j-\nu A\varphi_j-B(\varphi_j,u_\varepsilon)-B(u_\varepsilon,\varphi_j),\varphi_j)\\ &=\sum_{j=1}^{m}(-\varepsilon\|A\varphi_j\|^2-\nu\|A^{\frac12}\varphi_j\|^2-b(u_\varepsilon,\varphi_j,\varphi_j)-b(\varphi_j,u_\varepsilon,\varphi_j)); \end{aligned}$$

thus

$$Tr(F'(u_\varepsilon(\tau))\varphi_j(\tau),\varphi_j(\tau))=\sum_{j=1}^{m}(-\varepsilon\|\varphi_j\|_2^2-\nu\|\varphi_j\|_1^2-b(\varphi_j,u_\varepsilon,\varphi_j)). \tag{5.13}$$

We estimate the nonlinear term as follows

$$|\sum_{j=1}^{m}b(\varphi_j,u,\varphi_j)| = |\sum_{j=1}^{m}\int_\Omega\sum_{k,l=1}^{3}\varphi_{jk}\frac{\partial u_l}{\partial x_k}(x)\varphi_{jl}dx|$$

whence for almost every $x \in \Omega$ we have

$$|\sum_{j=1}^{m}\sum_{k,l=1}^{3}\varphi_{jk}\frac{\partial u_l}{\partial x_k}(x)\varphi_{jl}dx| \le \|u\|_1\|\rho\|$$

where

$$\|u(x)\|_1 = (\sum_{k,l=1}^{3}\|D_i u_k(x)\|^2)^{\frac{1}{2}} \text{ and } \rho(x) = \sum_{j=1}^{m}\sum_{i=1}^{3}(\varphi_{ji}(x))^2.$$

Therefore,

$$\big|\sum_{j=1}^{m} b(\varphi_j, u, \varphi_j)\big| \le \int_\Omega \rho(x)\|u(x)\|_1 dx \tag{5.14}$$

with the Schwarz inequality

$$|\sum_{j=1}^{m} b(\varphi_j, u, \varphi_j)| \le \|u(x)\|_1\|\rho(x)\|. \tag{5.15}$$

Applying the weighted Sobolev-Lieb-Thirring inequality [32, Theorem A.3.1], there exists c_5 independent of the family φ_j, m and of ε such that

$$\|\rho\|^2 \le c_5\sum_{j=1}^{m}\|\varphi_j(x)\|_1^2. \tag{5.16}$$

Insert (5.16) into (5.15) to find

$$|\sum_{j=1}^{m} b(\varphi_j, u, \varphi_j)| \le \|u\|_1(c_5\sum_{j=1}^{m}\|\varphi_j(x)\|_1^2)^{1/2},$$

using the Young inequality we obtain

$$|\sum_{j=1}^{m} b(\varphi_j, u, \varphi_j)| \le \frac{\nu}{2}\sum_{j=1}^{m}\|\varphi_j(x)\|_1^2 + \frac{c_5}{2\nu}\|u\|_1^2.$$

By using the Sobolev embedding Theorem $V_2 \subset V_1$, we have

$$c_6\|\varphi_j\|_1^2 \le \|\varphi_j(x)\|_2 \tag{5.17}$$

for an absolute constant c_6. Using the inequalities above (5.13) gives

$$TrF'(u_\varepsilon(\tau)) \circ Q_m(\tau) \le -\varepsilon c_6\sum_{j=1}^{m}\|\varphi_j\|_1^2 - \frac{\nu}{2}\sum_{j=1}^{m}\|\varphi_j(x)\|_1^2 + \frac{c_5}{2\nu}\|u\|_1^2.$$

We now use the estimate for ρ. In fact it is $\lambda_m \sim c\lambda_1 m^{2/3}$ in 3D, which can be found for example in [11] or [32, Lemma VI.2.1], there exists a constant c_7 such that

$$\sum_{j=1}^{m}\|\varphi_j(x)\|_1^2 \ge \lambda_1 + \dots + \lambda_m \ge c_7\lambda_1 m^{5/3},$$

use (5.17) to estimate $TrF'(u_\varepsilon(\tau)) \circ Q_m(\tau)$ as follows

$$TrF'(u_\varepsilon(\tau)) \circ Q_m(\tau) \le -(\varepsilon c_7 + \frac{\nu}{2})c_7\lambda_1 m^{5/3} + \frac{c_5}{2\nu}\|u_\varepsilon\|_1^2. \tag{5.18}$$

Kolmogorov's mean rate of dissipation of energy in turbulent flow (see e.g. [11, 16, 32, VI.(3.20)]) is defined as

$$\epsilon = \lambda_1^{3/2}\nu \limsup_{t\to\infty} \sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon} \frac{1}{t}\int_0^t \|u_\varepsilon(\tau)\|_1^2 d\tau \tag{5.19}$$

the maximal mean rate of dissipation of energy on the attractor, which is finite thanks to (4.8). Hence

$$\frac{1}{t}\int_0^t Tr(F'(S_\varepsilon(\tau)u_{\varepsilon_0})\circ Q_m(\tau))d\tau \le -(\varepsilon c_7 + \frac{\nu}{2})c_7\lambda_1 m^{\frac{5}{3}} + \frac{c_5}{2\nu}\frac{1}{t}\int_0^t \|u_\varepsilon(\tau)\|_1^2 d\tau.$$

Using (5.19) we can estimate the quantities q_m

$$q_m \le -\kappa_1 m^{5/3} + \kappa_2,$$

with

$$\kappa_1 = (\varepsilon c_6 + \frac{\nu}{2})c_7\lambda_1 \quad \text{and} \quad \kappa_2 = \frac{c_5}{2\nu^2\lambda_1^{3/2}}\epsilon.$$

Therefore, if $m' \in \mathbb{N}$ is defined by

$$m' - 1 < (\frac{2\kappa_2}{\kappa_1})^{3/5} = (\frac{2c_5}{\nu^2\lambda_1^{5/2}(2\varepsilon c_6 + \nu)c_7})^{3/5}\epsilon^{3/5} \le m',$$

then $q_{m'} \le 0$, setting $c_8^\varepsilon = (\frac{2c_5}{\nu^2\lambda_1^{5/2}(2\varepsilon c_6+\nu)c_7})^{3/5}$so that from (5.11)-(5.12) this m' is an upper bound for the dimension of the global attractor,

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_8^\varepsilon \epsilon^{3/5}.$$

Using (4.8) we can estimate the energy dissipation flux ϵ by

$$\epsilon \le \frac{\lambda_1^{1/2}\|f\|^2}{\nu}. \tag{5.20}$$

To make the dimension estimate more explicit, we can estimate the energy dissipation flux ϵ in terms of G by

$$\epsilon \le \lambda_1^2\nu^3 G^2. \tag{5.21}$$

Therefore, using (5.21) we prove the following Proposition.

Proposition 5.2. *The global attractor $\mathfrak{A}_\varepsilon$ of the regularized 3D Navier-Stokes* (4.1)*, is finite dimensional, in V_0 has finite Hausdorff and fractal dimensions, which can be estimated in terms of the Grashoff number by*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_9 G^{6/5}$$

where $c_9 = c_8^\varepsilon \nu^{9/5}\lambda_1^{6/5}$.

We can estimate c_8^ε as follow

$$c_8^\varepsilon \le (\frac{2c_5}{\nu^3\lambda_1^{5/2}c_7})^{3/5} = c_8^0.$$

Then there exists a constant $c_{10} = c_8^0\nu^{9/5}\lambda_1^{6/5}$ independent of ε. Hence

Theorem 5.3. *The Hausdorff and fractal dimensions of the global attractor $\mathfrak{A}_\varepsilon$ of the regularized 3D Navier-Stokes* (4.1)*, $\dim_F(\mathfrak{A}_\varepsilon)$ and $\dim_H(\mathfrak{A}_\varepsilon)$ respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{10} G^{6/5}.$$

6. Numbers of degrees of freedom in turbulent flows

In this Section, we estimate the effects of hyperviscosity on the turbulent flow. An argument from the classical theory of turbulence (see, L. Landau and Lifshitz [22]) suggests that there are finitely many degrees of freedom in turbulent flows. Heuristic physical arguments are used to justify this assertion and to provide an estimate for this number of degrees of freedom by dividing a typical length scale of the flow, $l_0 = \lambda_1^{-1/2}$, by the Kolmogorov dissipation length scale l_ϵ; i.e., $l_\epsilon = \frac{\nu^3}{\epsilon}$ where ϵ is Kolmogorov's mean rate of dissipation of energy in turbulent flow and taking the third power in 3D.

We will express our primary attractor results in terms of the Kolmogorov length-scale l_ϵ and the Landau-Lifschitz estimates [22] of the number of degrees of freedom in turbulent flow [11, 32] and we can easily observe such compatibility that exists between these estimates and the number of degrees of freedom in turbulence (see also [22]). Such estimates will give us useful information about the capability of (4.1) to approximate Navier-Stokes equations dynamics. We will show that the corresponding number of degrees of freedom is proportional to the dimension of the global attractor.

By Holder's inequality the right hand side of (5.14) can be estimated as follow

$$
\begin{aligned}
\int_\Omega \|u(x)\|_1 \|\rho(x)\| dx &\le \|\rho(x)\|_{L^{5/3}(\Omega)} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)} \\
&\le \Big(c_5 \sum_{j=1}^m \|\varphi_j\|_1^2\Big)^{3/5} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega).}
\end{aligned}
$$

By Young's inequality we obtain

$$
\int_\Omega \|u(x)\|_1 \|\rho(x)\| dx \le \frac{\nu}{2} \sum_{j=1}^m \|\varphi_j\|_1^2 + \frac{c_5}{\nu^{3/2}} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)}^{5/2}. \tag{6.1}
$$

Using (5.17), (6.1) we can majorize $TrF'(u_\varepsilon(\tau)) \circ \tilde{Q}_m(\tau)$ as follows

$$
\begin{aligned}
&TrF'(u_\varepsilon(\tau)) \circ \tilde{Q}_m(\tau) \\
&\le -\varepsilon c_6 \sum_{j=1}^m \|\varphi_j(\tau)\|_1^2 - \frac{\nu}{2} \sum_{j=1}^m \|\varphi_j(x)\|_1^2 + \frac{c_5}{\nu^{3/2}} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)}^{5/2}.
\end{aligned} \tag{6.2}
$$

Note that in the 3D case we have $\lambda_j \ge c_{11} L^{-2} j^{\frac{2}{3}}$ for some positive universal constant (see, for example [32, Lemma VI 2.1]). Therefore,

$$
\sum_{j=1}^m \|\varphi_j(x)\|_1^2 \ge \lambda_1 + \dots + \lambda_m \ge c_{12} \lambda_1 m^{5/3}. \tag{6.3}
$$

Taking into account (6.2) and (6.7) then yields

$$
\begin{aligned}
&TrF'(u_\varepsilon(\tau)) \circ Q_m(\tau) d\tau \\
&\le -\varepsilon c_6 c_{12} \lambda_1^2 m^{5/3} - \frac{\nu}{2} c_{12} \lambda_1 m^{5/3} + \frac{c_5}{\nu^{3/2}} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)}^{5/2}. \\
&\le (-\varepsilon c_6 - \frac{\nu}{2}) c_{12} \lambda_1 m^{5/3} + \frac{c_5}{\nu^{3/2}} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)}^{5/2}. \\
&\le -(\varepsilon c_6 + \frac{\nu}{2}) c_{12} \lambda_1 m^{5/3} + \frac{c_5}{\nu^{3/2}} \|A^{1/2} u_\varepsilon(x)\|_{L^{5/2}(\Omega)}^{5/2}.
\end{aligned}
$$

Thanks to (2.4) with $\theta = 5/2$, we have

$$\|A^{1/2}u_\varepsilon(x)\|_{L^{5/2}(\Omega)} \le c_2\|A^{1/2}u_\varepsilon(x)\|^{1/2}\|A^{1/2}u_\varepsilon(x)\|_1^{1/5}$$

and hence

$$\|A^{1/2}u_\varepsilon(x)\|^{5/2}_{L^{\frac{5}{2}}(\Omega)} \le c_2^{5/2}\|A^{1/2}u_\varepsilon(x)\|^{\frac{5}{4}}\|A^{1/2}u_\varepsilon(x)\|_1^{1/2}. \tag{6.4}$$

In fact, the norm $\|A_1^{1/2}u_\varepsilon\|_1$ is equivalent to the norm $\|u_\varepsilon\|_2$ in V_2. This means

$$\|Au_\varepsilon(x)\|d_2 \le \|A^{1/2}u_\varepsilon(x)\|_1 \le d_1\|Au_\varepsilon(x)\|. \tag{6.5}$$

Notice that d_1 and d_2 do not depend on ε. Then, from the above and using Hölder's inequality we obtain

$$\begin{aligned} &\limsup_{t\to\infty}\sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon}\frac{1}{t}\int_0^t\|A^{1/2}u_\varepsilon(\tau,x)\|^{5/2}_{L^{5/2}(\Omega)}d\tau \\ &\le C_3\limsup_{t\to\infty}\sup_{u_{\varepsilon 0}\in\mathfrak{A}_\varepsilon}\frac{1}{t}\int_0^t\|A^{\frac{1}{2}}u_\varepsilon(x)\|^{5/4}d\tau \end{aligned} \tag{6.6}$$

where $C_3 = c_2^{5/2}d_1M^{1/2}$ and

$$M = \sup_{t\in[0,T]}\sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon\cap D(A)}\|Au_\varepsilon(x)\| \tag{6.7}$$

it is clear that M is finite.

On the other hand, using (5.19) we have

$$\begin{aligned} &\sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon}(\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|A^{1/2}u_\varepsilon(x)\|^{5/4}d\tau) \\ &\le \sup_{u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon}(\limsup_{t\to\infty}\frac{1}{t}\int_0^t\|A^{1/2}u_\varepsilon(x)\|^2d\tau)^{5/8} \le \big(\frac{\epsilon}{\lambda_1^{3/2}\nu}\big)^{5/8}. \end{aligned} \tag{6.8}$$

For $u_{\varepsilon_0}\in\mathfrak{A}_\varepsilon$, we can estimate the quantities $q_m(t)$, q_m

$$q_m = \limsup_{t\to\infty}q_m(t) \le -\kappa_1 m^{5/3} + \kappa_2,$$

where

$$\kappa_1 = (\varepsilon c_6 + \frac{\nu}{2})c_{12}\lambda_1, \quad \kappa_2 = C_3\frac{c_5}{\nu^{3/2}}\big(\frac{\epsilon}{\lambda_1^{3/2}\nu}\big)^{5/8}.$$

Therefore, if $m'\in\mathbb{N}$ is defined by

$$m' - 1 < \big(\frac{2\kappa_2}{\kappa_1}\big)^{3/5} = \Big(\frac{4C_3c_5}{(2\varepsilon c_6+\nu)\lambda_1^{\frac{31}{16}}\nu^{\frac{17}{8}}c_{12}}\Big)^{3/5}\epsilon^{3/8} < m', \tag{6.9}$$

Setting $l_\epsilon = (\frac{\nu^3}{\epsilon})^{1/4}$ the dissipation length scale, and $l_0 = \lambda_1^{-1/2}$ the macroscopical length by setting. Then we can rewrite (6.9) in the form

$$m' - 1 < c_{13}\big(\frac{l_0}{l_\epsilon}\big)^{3/2} < m', \tag{6.10}$$

where

$$c_{13} = \big(\frac{4C_3c_5}{\lambda_1^{\frac{11}{16}}\nu^{\frac{17}{8}}c_{12}}\big)^{3/5}c_{14}^\varepsilon, \quad c_{14}^\varepsilon = \big(\frac{1}{2\varepsilon c_6+\nu}\big)^{3/5}. \tag{6.11}$$

Thus, we have proved the following Proposition

Proposition 6.1. *The Hausdorff and fractal dimensions of the global attractor* $\mathfrak{A}_\varepsilon$ *of the regularized 3D Navier-Stokes* (4.1)*,* $\dim_F(\mathfrak{A}_\varepsilon)$ *and* $\dim_H(\mathfrak{A}_\varepsilon)$ *respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{13}(\frac{l_0}{l_\epsilon})^{3/2}. \tag{6.12}$$

The exponent on l_0/l_ϵ is significantly less than the Landau–Lifschitz predicted value of 3, less than the results in [9] for the 3D Camassa–Holm equations, or simply NS-α model and less than the Avrin exponent (for $\alpha = l = 2$) [1, Theorem 1].

This, in a sense, suggests that in the absence of boundary effects (e.g., in the case of periodic boundary conditions) the modified 3D Navier-Stokes represent, very well, the averaged equation of motion of turbulent flows.

Since the Grashoff number $G = \|f\|/(\nu^2\lambda_1^{3/4})$ in 3D, (see e.g. [1, 11, 33]) is an upper bound for $(\frac{l_0}{l_\epsilon})^2$, expressing the above estimates in terms of G is straightforward. The above Proposition becomes

Proposition 6.2. *The Hausdorff and fractal dimensions of the global attractor* $\mathfrak{A}_\varepsilon$ *of the regularized 3D Navier-Stokes* (4.1)*,* $\dim_F(\mathfrak{A}_\varepsilon)$ *and* $\dim_H(\mathfrak{A}_\varepsilon)$ *respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{13}G^{3/4}. \tag{6.13}$$

Thus we recover the improvement on the cubic power; i.e., get a bound proportional to $G^{p/2}$ for $p < 3$, in (6.13) $p = 3/2$. This improvement suggesting to very good agreement with the conventional theory of turbulence.

For $\alpha = l = 2$, motivated by the Chapman–Enskog expansion, we recover (6.13). This result can be seen as an improved version of the results announced by Joel Avrin [1, Theorem 2].

We can estimate (6.12) independent of ε.

From (6.10) we have $c_{14}^\varepsilon = 1/(2\varepsilon c_6 + \nu)^{3/5} \le 1/\nu^{3/5} = c_{14}^0$. Then there exists a constant c_{15}, which is independent of ε, such that

$$c_{15} = (\frac{4C_3c_5}{\lambda_1^{11/16}\nu^{25/8}c_{12}})^{3/5}.$$

The following estimates are independent of ε and with them we finish stating our main results

Theorem 6.3. *The Hausdorff and fractal dimensions of the global attractor* $\mathfrak{A}_\varepsilon$ *of the regularized 3D Navier-Stokes* (4.1)*,* $\dim_F(\mathfrak{A}_\varepsilon)$ *and* $\dim_H(\mathfrak{A}_\varepsilon)$ *respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{15}(\frac{l_0}{l_\epsilon})^{3/2}.$$

This upper bound is much smaller than what one would expect for three-dimensional models; i.e., $(l_0/l_\epsilon)^3$. This improves significantly on previous bounds have demonstrated that hyperviscosity can have profound effects on the number of degree freedom. The modifying effects are well understood, which makes the use of hyperviscosity an efficient tool for numerical studies and suggests that the regularized 3D Navier-Stokes has a great potential to become a good sub-gridscale large-eddy simulation model of turbulence. The results obtained agree very well with those provided in numerical studies of turbulence; see [1, 9, 13, 15, 21].

The present results explain some fundamental differences between the theory use instead a hyper-viscous term to approximate Navier-Stokes equations and which hyperviscous terms are added spectrally to the standard incompressible Navier-Stokes equations [1]. It would be interesting to obtain estimates for (1.1) in this context in 3D and to see how the estimates depend on l for $l \geq 3/2$.

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7. Addendum posted on September 27, 2011

The author wants to correct some misprints and present a new proof of the estimates for the dimension of the attractor, using the Lieb-Thirring inequality.

p5, formula (3.3): Replace $\hat{B}$ by B.

p5, Lemma 3.5: Define $B(u) = B(u,u)$.

p6, Theorem 3.9: Replace 'u is the weak solution' by 'u is a weak solution'; also in the last line of Section 3; and in Theorem 4.1.

p6, Theorem 3.9: Replace (3.1)–(3.2) by (3.3)–(3.4);

P9: We can just interpolate,

$$\|u\|_{H^1}^4 \le \|u\|_{H^2}^2 \|u\|_{L^2}^2, \tag{7.1}$$

which then gives $u_\varepsilon \in L^4\big(0,T;H^1(\Omega)\big)$ if $u_\varepsilon \in L^2\big(0,T;H^2(\Omega)\big) \cap L^\infty\big(0,T;L^2(\Omega)\big)$ (and gives an explicit estimate on the norm), thus $a_4 = L^4(0,T;V_1(\Omega))$.

p11, entire page: Replace R by R_1.

P15, Section 6: We present a new and rigorous proof for estimates of the dimensions of the attractors using Lieb-Thirring inequality [32, Theorem A4.1].

By Holder's inequality the right hand side of (5.14) can be estimated as

$$\int_\Omega \|u_\varepsilon(x)\|_1 \rho(x) dx \le \|\rho(x)\|_{L^{7/3}(\Omega)} \|A^{1/2} u_\varepsilon(x)\|_{L^{7/4}(\Omega)}. \tag{7.2}$$

Applying Young's inequality with $p = 7/3$, $q = 7/4$, $\sigma = 7\varepsilon/(6\kappa)$, we obtain

$$\int_\Omega \|u_\varepsilon(x)\|_1 \rho(x)\, dx \le \frac{\varepsilon}{2\kappa} \|\rho(x)\|_{L^{7/3}(\Omega)}^{7/3} + c_5 \|A^{1/2} u_\varepsilon(x)\|_{L^{7/4}(\Omega)}^{7/4}, \tag{7.3}$$

where $c_5 = \frac{4}{7}(\frac{7\varepsilon}{6\kappa})^{-3/4}$. Using the above inequality, we have the estimate

$$\begin{aligned} TrF'\big(u_\varepsilon(\tau)\big) \circ \tilde{Q}_m(\tau) \le{} & -\nu \sum_{j=1}^m \|\varphi_j(x)\|_1^2 - \varepsilon \sum_{j=1}^m \|\varphi_j(\tau)\|_2^2 \\ & + \frac{\varepsilon}{2\kappa} \|\rho(x)\|_{L^{7/3}(\Omega)}^{7/3} + c_5 \|A^{1/2} u_\varepsilon(x)\|_{L^{7/4}(\Omega)}^{7/4}. \end{aligned} \tag{7.4}$$

Applying the 3D Lieb-Thirring inequality with $m = l$ as developed in [32, Theorem A4.1] and using the Sobolev embedding $V_2 \subset V_1$, we obtain

$$TrF'\big(u_\varepsilon(\tau)\big) \circ \tilde{Q}_m(\tau) \le -c_7 \sum_{j=1}^{m} \|\varphi_j(x)\|_1^2 + c_5 \|A^{1/2} u_\varepsilon(x)\|_{L^{7/4}(\Omega)}^{7/4}, \tag{7.5}$$

where $c_7 = \frac{\nu}{2} + \frac{\varepsilon}{2c_6}$. Therefore, $\sum_{j=1}^{m} \|\varphi_j(x)\|_1^2 \ge c_9 \lambda_1 m^{5/3}$ and we have by Holder's inequality

$$\|A^{1/2} u_\varepsilon(x)\|_{L^{7/4}(\Omega)}^{7/4} \le c_{10} \|A^{1/2} u_\varepsilon(x)\|^{7/4}, \quad c_{10} = |\Omega|^{1/8}. \tag{7.6}$$

Taking into account this inequality, we have

$$TrF'\big(u_\varepsilon(\tau)\big) \circ Q_m(\tau) d\tau \le -c_7 c_9 \lambda_1 m^{5/3} + c_5 c_{10} \|A^{1/2} u_\varepsilon(x)\|^{7/4}. \tag{7.7}$$

By Hölder's inequality,

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{u_{\varepsilon 0}\in \mathfrak{A}_\varepsilon} \frac{1}{t} \int_0^t \|A^{1/2} u_\varepsilon(\tau, x)\|^{7/4} d\tau \\ &\le \limsup_{t\to\infty} \Big(\sup_{u_{\varepsilon 0}\in \mathfrak{A}_\varepsilon} \frac{1}{t} \int_0^t \|A^{1/2} u_\varepsilon(\tau, x)\|^2 d\tau \Big)^{7/8}. \end{aligned} \tag{7.8}$$

On the other hand, using (5.19) we obtain

$$\limsup_{t\to\infty} \sup_{u_{\varepsilon 0}\in \mathfrak{A}_\varepsilon} \frac{1}{t} \int_0^t \|A^{1/2} u_\varepsilon(\tau, x)\|^{7/4} d\tau \le \big(\frac{\epsilon}{\lambda_1^{3/2}\nu}\big)^{7/8}. \tag{7.9}$$

For $u_{\varepsilon 0} \in \mathfrak{A}_\varepsilon$, we can estimate the quantities $q_m(t)$ and q_m:

$$q_m = \limsup_{t\to\infty} q_m(t) \le -\kappa_1 m^{5/3} + \kappa_2, \tag{7.10}$$

where $\kappa_1 = c_7 c_9 \lambda_1$ and

$$\kappa_2 = c_5 c_{10} \big(\frac{\epsilon}{\lambda_1^{3/2}\nu}\big)^{7/8}.$$

Therefore, if $m' \in \mathbb{N}$ is defined by

$$m' - 1 < \big(\frac{2\kappa_2}{\kappa_1}\big)^{3/5} = \big(\frac{2c_5 c_{10}}{c_7 c_9 \lambda_1^{37/16} \nu^{7/8}}\big)^{3/5} \epsilon^{21/40} < m', \tag{7.11}$$

Then we can rewrite (7.11) in the form

$$m' - 1 < c_{11} \big(\frac{l_0}{l_\epsilon}\big)^{21/10} < m', \quad c_{11} = \big(\frac{2c_5 c_{10}}{c_7 c_9 \lambda_1^{37/16} \nu^{7/8}}\big)^{3/5} (\nu^{63/40}) \lambda_1^{21/20}. \tag{7.12}$$

Thus, Proposition 6.1. and Proposition 6.2. can be reformulated as follows.

Proposition 7.1. *The Hausdorff and fractal dimensions of the global attractor $\mathfrak{A}_\varepsilon$ of the regularized 3D Navier-Stokes (4.1), $\dim_F \mathfrak{A}_\varepsilon)$ and $\dim_H(\mathfrak{A}_\varepsilon)$ respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{11} \big(\frac{l_0}{l_\epsilon}\big)^{21/10}. \tag{7.13}$$

Proposition 7.2. *The Hausdorff and fractal dimensions of the global attractor $\mathfrak{A}_\varepsilon$ of the regularized 3D Navier-Stokes* (4.1)*, $\dim_F(\mathfrak{A}_\varepsilon)$ and $\dim_H(\mathfrak{A}_\varepsilon)$ respectively, satisfy*

$$\dim_H(\mathfrak{A}_\varepsilon) \le \dim_F(\mathfrak{A}_\varepsilon) \le c_{11} G^{21/20}. \tag{7.14}$$

Thus we recover the improvement on the cubic power; i.e. get a bound proportional to $G^{p/2}$ for $p < 3$, in (7.14) $p = 21/10$.

End of addendum.

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