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# EXISTENCE OF NONTRIVIAL SOLUTIONS FOR SINGULAR QUASILINEAR EQUATIONS WITH SIGN CHANGING NONLINEARITY

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Abstract. By an application of Bonanno's three critical point theorem, we establish the existence of a nontrivial solution to the problem

$$\begin{gathered} -\Delta_p u = \mu \frac{g(x)|u|^{p-2}u}{|x|^p} + \lambda a(x) f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

under some restrictions on $g, a$ and $f$ for certain positive values of $\mu$ and $\lambda$.

## 1. Introduction

Let us set up a problem which is used to give a brief introduction about previous research

$$\begin{gathered} -\Delta_p u = \mu \frac{g(x)|u|^{p-2}u}{|x|^p} + a(x) f_1(\lambda, u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega$ be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $0 \in \Omega$ and $f_1 : (0,\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Suppose there exists $M > 0$ such that $-M \leq g(x) \leq 1, a \in L^\infty(\Omega)$ and $0 \leq \mu < (\frac{N-p}{p})^p$. Let $\lambda$ be a positive parameter.

In the last few years, problem (1.1) with $\mu = 0$ and $a(x) \equiv 1$ has been extensively investigated for the case $p = 2$, (see, [8, 9, 16] and the references cited therein), where $f_1(\lambda, u) = \lambda f(u)$. In case $p = 2$, there are many publications dealing with the existence of solution to the problem (1.1) with $g = 1$ and $a = 1$. For convenience of the reader, we give a brief summary of these results. Ferrero and Gazzola [6] considered the problem (1.1), where $f_1(\lambda, u) = |u|^{2^*-2}u + \lambda u$. They established the existence of nontrivial solution by variational method for certain values of $\mu$ and $\lambda$. Ruiz and Willem [15] considered the aforesaid problem, where $f_1(\lambda, u) = |u|^{2^*-1}u + \lambda u$ and established the existence of positive solutions under various assumptions on the domain $\Omega$. Chen [2, 3] also studied the same problem and obtained multiple solutions by analyzing the exact growth order of the positive solutions near origin, where $f_1(\lambda, u) = u_+^{2^*-1} + \lambda u_+^q, 0 < q < 1$, $\lambda > 0$, $0 \leq \mu < (N-2)^2/4$. Recently,

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Kristály and Varga [10] obtained the existence of three solutions to the problem (1.1) with $g = 1$ and $a = 1$, by an application of Bonanno's three critical point theorem [1], where $f_1(\lambda, u) = \lambda f(u)$.

For a good amount of work concerning quasilinear equations with singularities, we refer the book of Drabek et al. [4] and for existence and multiplicity results concerning singular p-Laplacian, we refer the reader to [5, 13, 17] and reference cited therein. Montefusco [13] considered the problem (1.1) with $g = 1$ and $a = 1$, where $f_1(\lambda, u) = |u|^{q-2}u, 1 < p < q < p^*, 1 < p < N$. He established the existence of a nontrivial solution whenever $\mu \in (0, (\frac{(N-p)}{p})^p)$ is fixed. Faraci and Livrea [5] utilized Montefusco's result and gave some bifurcation results for singular p-Laplacian. By an application of Bonanno's three critical point theorem, Yang et al. [17] established the existence of three weak solutions to singular p–Laplacian type equation, which has singularity in the principal part of the operator.

In this study, our main purpose inspired by [10], is to see that the conditions introduced by [10] on $f$ can be extended for singular p-Laplacian with sign-changing nonlinearity also. It is worth noting that to establish the existence of solutions to the problem (1.1) is of more interest due to the presence of singular potential as well as sign changing nonlinearity. In this note, we establish the existence of two solutions to the problem (1.1) by Bonanno's theorem, where $f_1(\lambda, u) = \lambda f(u)$. More precisely, we give the existence of two solutions to the problem

$$\begin{gathered} -\Delta_p u = \mu \frac{g(x)|u|^{p-2}u}{|x|^p} + \lambda a(x) f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $f \in C(\mathbb{R}, \mathbb{R})$ and satisfies the following hypotheses:

- (H1) $\lim_{s\to 0} \frac{f(s)}{s^{p-1}} = 0$.
- (H2) $\lim_{|s|\to\infty} \frac{f(s)}{|s|^{p-1}} = 0$.
- (H3) Let $F(s) = \int_0^s f(t)dt$, we assume $\sup_{s\in\mathbb{R}} F(s) > 0$.

We state now the theorem we will prove in Section 4.

**Theorem 1.1.** *Let $f \in C(\mathbb{R}, \mathbb{R})$ which satisfies the hypotheses* (H1)–(H3)*. Let there exists $M > 0$ such that $-M \le g(x) \le 1, a \in L^\infty(\Omega)$. Then for every $\mu \in [0, (\frac{N-p}{p})^p)$ there exist an open interval $\Lambda_\mu \subset (0, \infty)$ and a real number $\eta_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$, the problem* (1.2) *has one non-trivial weak solution $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}(\Omega)} \le \eta_\mu$.*

We remark that in proving the above theorem ideas from [10] are used. We organize this paper as follows: Section 2 deals with the preliminaries. Section 3 deals with some lemmas which have been used in the main theorem. The main result is proved in Section 4. In the last section, we construct some examples for the illustration of main result.

## 2. Preliminaries

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$ and $0 \in \Omega$. The space $W_0^{1,p}(\Omega)$ is endowed by the norm

$$\|u\|_{W_0^{1,p}} = \Big( \int_\Omega |\nabla u|^p \Big)^{1/p}.$$

Let $1 < p < N$, we recall classical Hardy's inequality, which says that

$$\int_\Omega \frac{|u(x)|^p}{|x|^p} dx \le \frac{1}{C_{N,p}} \int_\Omega |\nabla u|^p, \quad u \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $C_{N,p} = (\frac{N-p}{p})^p$. For a detail about Hardy inequality and related problem, we refer the reader to [7]. The Hardy inequality proves that embedding of $W_0^{1,p}(\Omega)$ in $L^p(\Omega, \frac{1}{|x|^p})$ is continuous but is not compact as for the Sobolev embeddding. The Sobolev embedding constant of the compact embedding $W_0^{1,p}(\Omega) \circlearrowleft L^q(\Omega), q \in [1, p^*)$, will be denoted by $c(N,p) > 0$; i.e.,$\|u\|_{W_0^{1,p}} \ge c(N,p)\|u\|_{L^p}$, for every $u \in W_0^{1,p}$. Let us define $F(s) = \int_0^s f(t)dt$. We introduce the energy functional $E_{\mu,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ associated with (1.2),

$$E_{\mu,\lambda} = \Phi_\mu(u) - \lambda J(u), u \in W_0^{1,p}(\Omega),$$

where

$$\Phi_\mu(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p}\int_\Omega \frac{g(x)|u(x)|^p}{|x|^p} dx, J(u) = \int_\Omega a(x)F(u(x))dx.$$

It is easy to see that the critical points of $E_{\mu,\lambda}$ are exactly the weak solutions of (1.2). Therefore, it is sufficient to give the existence of multiple critical points of $E_{\mu,\lambda} : W_0^{1,p}(\Omega) \to \mathbb{R}$ for certain values of $\mu$ and $\lambda$. To establish the existence of critical points of $E_{\mu,\lambda}$, we use Bonanno's three critical point theorem. Since Bonanno's result [1] is a special case of Ricceri's three critical point theorem [14], so for the reader's convenience we give a brief sketch.

Ricceri [14] proved the following result.

**Theorem 2.1.** *Let $X$ be a separable and reflexive real Banach space, $I \subset \mathbb{R}$ an interval, and $g : X \times I \to \mathbb{R}$ a continuous function satisfying the following conditions:*

(i) *for each $x \in X, g(x,.)$ is continuous and concave;*

(ii) *for each $\lambda \in I, g(.,\lambda)$ is sequentially weakly lower semicontinuous and Gâteaux differentiable and*

$$\lim_{\|x\|\to\infty} g(x,\lambda) = +\infty;$$

(iii) *there exists a continuous concave function $h : I \to \mathbb{R}$ such that*

$$\sup_{\lambda\in I} \inf_{x\in X} (g(x,\lambda) + h(\lambda)) < \inf_{x\in X} \sup_{\lambda\in I} (g(x,\lambda) + h(\lambda)).$$

*Then there exist an open interval $\Lambda \subset I$ and a positive real number $\eta$, such that for each $\lambda \in \Lambda$, the equation*

$$g'_x(x,\lambda) = 0$$

*admits at least two solutions in $X$ whose norms are less than $\eta$.*

*If, in addition, the function $g$ is continuous in $X \times I$, and for each $\lambda \in I$, the function $g(.,\lambda)$ is $C^1$ and satisfies the Palais-Smale condition, then the above conclusion holds with "three solutions" instead of "two solutions".*

As a special case of the above theorem, Bonanno [1] gave the following

**Theorem 2.2.** *Let $X$ be a separable and reflexive real Banach space and $\Phi, J : X \to \mathbb{R}$ be two continuously Gâteau differentiable functionals. Assume that there exists $x_0 \in X$ such that $\Phi(x_0) = 0 = J(x_0)$ and $\Phi(x) \ge 0$ for every $x \in X$ and suppose there exist $x_1 \in X$ and $r > 0$ such that*

(i) $r < \Phi(x_1)$;
(ii) $\sup_{\Phi(x)<r} J(x) < r\frac{J(x_1)}{\Phi(x_1)}$.

*Further, put*

$$\bar{a} = \frac{hr}{r\frac{J(x_1)}{\Phi(x_1)} - \sup_{\Phi(x)<r} J(x)},$$

*with $h > 1$, and assume that the functional $\Phi - \lambda J$ is sequentially weakly lower semicontinuous, satisfies Palais-Smale condition and*

(iii) $\lim_{\|x\|\to+\infty}(\Phi(x) - \lambda J(x)) = +\infty$ *for every* $\lambda \in [0, \bar{a}]$.

*Then there exist an open interval $\Lambda \subseteq [0, \bar{a}]$ and a positive real number $\eta$ such that for each $\lambda \in \Lambda$, the equation*

$$\Phi'(x) - \lambda J'(x) = 0$$

*admits at least three solutions in $X$ whose norms are less than $\eta$.*

We remark that in view of Ricceri's theorem [14], if we drop the Palais-Smale condition and continuous Gâteau differentiability of the functional $g(., \lambda) = \Phi(.) - \lambda J(.)$ from Theorems 2.1 and 2.2, we have the existence of two solutions. This fact is carried out in Theorem 2.1.

## 3. Auxiliary lemmas

In this section, we sate some lemmas to be used in the proof of main theorem.

**Lemma 3.1.** *For every $\mu \in [0, C_{N,p})$ and $\lambda \in \mathbb{R}$, the functional $E_{\mu,\lambda}$ is coercive.*

*Proof.* Let us fix $\mu \in [0, C_{N,p})$ and $\lambda \in \mathbb{R}$ be arbitrary. By (H2), for any given $\lambda \in \mathbb{R}$, there exists $\delta = \delta(\mu, \lambda) > 0$ such that

$$|f(s)| < \big(1 - \frac{\mu}{C_{N,p}}\big)\frac{c(N,p)^{-p}}{(1 + \|a\|_{L^\infty})}(1 + |\lambda|)^{-1}|s|^{p-1},$$

whenever $|s| > \delta$. This implies

$$|f(s)| < \big(1 - \frac{\mu}{C_{N,p}}\big)\frac{c(N,p)^{-p}}{(1 + \|a\|_{L^\infty})}(1 + |\lambda|)^{-1}|s|^{p-1} + \max_{|t|\le\delta}|f(t)|, \forall\, s \in \mathbb{R}.$$

An integration yields,

$$|F(s)| < \frac{1}{p}\big(1 - \frac{\mu}{C_{N,p}}\big)\frac{c(N,p)^{-p}}{(1 + \|a\|_{L^\infty})}(1 + |\lambda|)^{-1}|s|^{p} + \max_{|t|\le\delta}|f(t)||s|, \forall s \in \mathbb{R}. \quad (3.1)$$

Since we have

$$E_{\mu,\lambda}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p}\int_\Omega \frac{g(x)|u(x)|^p}{|x|^p}dx - \lambda\int_\Omega a(x)F(u(x))dx,$$

so by Hardy inequality, for every $u \in W_0^{1,p}$ and using the fact that $-M \le g(x) \le 1$, we have

$$\begin{aligned}
E_{\mu,\lambda}(u) &\ge \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p}\int_\Omega \frac{|u(x)|^p}{|x|^p}|dx - \lambda\int_\Omega a(x)F(u(x))dx \\
&\ge \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{C_{N,p}\,p}\int_\Omega |\nabla u|^p dx - \lambda\int_\Omega a(x)F(u(x))dx \\
&\ge \frac{1}{p}\big(1-\frac{\mu}{C_{N,p}}\big)\int_\Omega |\nabla u|^p dx - |\lambda|\int_\Omega |a(x)||F(u(x))|dx \\
&\ge \frac{1}{p}\big(1-\frac{\mu}{C_{N,p}}\big)\int_\Omega |\nabla u|^p dx \qquad (3.2)\\
&\quad - \frac{|\lambda|}{(1+|\lambda|)p}c(N,p)^{-p}\int_\Omega |u|^p dx - |\lambda| c(N,1)\max_{|t|\le\delta}|f(t)|\|u\|_{W_0^{1,p}} \\
&\ge \frac{1}{p}\big(1-\frac{\mu}{C_{N,p}}\big)\big(\frac{1}{1+|\lambda|}\big)\int_\Omega |\nabla u|^p dx - |\lambda| c(N,1)\max_{|t|\le\delta}|f(t)|\|u\|_{W_0^{1,p}} \\
&\ge \frac{1}{p}\big(1-\frac{\mu}{C_{N,p}}\big)\big(\frac{1}{1+|\lambda|}\big)\|u\|^p_{W_0^{1,p}} - |\lambda| c(N,1)\max_{|t|\le\delta}|f(t)|\|u\|_{W_0^{1,p}},
\end{aligned}$$

where we have used (3.1). Now if $\|u\|_{W_0^{1,p}} \to \infty$, one can conclude that $E_{\mu,\lambda} \to \infty$ and hence $E_{\mu,\lambda}$ is coercive. □

**Lemma 3.2.** *Assume that $\mu \in [0, C_{N,p}]$, then $\Phi_\mu(u)$ is a sequentially weakly lower semicontinuous functional on $W_0^{1,p}(\Omega)$.*

*Proof.* Montefusco [13], proved the sequentially weakly lower semicontinuity of the functional

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\mu}{p}\int_\Omega \frac{|u(x)|^p}{|x|^p}dx,$$

using the ideas from Lions [11, 12]. Since for $-M \le g(x) \le 1$, the proof of this lemma is similar to the proof of [13, Theorem 3.2], so we omit the details. □

**Lemma 3.3.** *For every $\mu \in [0, C_{N,p})$ and $\lambda \in \mathbb{R}$, the functional $E_{\mu,\lambda}$ is sequentially weakly lower semicontinuous functional on $W_0^{1,p}(\Omega)$.*

*Proof.* By Lemma 3.2, $\Phi_\mu(u)$ is a sequentially weakly lower semicontinuous functional on $W_0^{1,p}(\Omega)$, for all $\mu \in [0, C_{N,p})$. By (H2), there exists $C > 0$ such that

$$|f(s)| \le C(1 + |s|^{p-1}), \quad s \in \mathbb{R}. \tag{3.3}$$

Now the sequentially weak continuity of $J$ is obtained by a classical way. So, this proves the lemma. □

**Lemma 3.4.** *For every $\mu \in [0, C_{N,p})$,*

$$\lim_{\xi\to 0^+} \frac{\sup\{J(u) : \Phi_\mu(u) < \xi\}}{\xi} = 0.$$

*Proof.* We fix $\mu \in [0, C_{N,p})$. By (H1), for any given $\epsilon > 0$ there exists a $\delta(\epsilon)$ such that

$$|f(s)| < \frac{\epsilon}{2}\big(1-\frac{\mu}{C_{N,p}}\big)\frac{c(N,p)^{-p}}{(1+\|a\|_{L^\infty})}|s|^{p-1}, \quad \text{whenever } |s| < \delta. \tag{3.4}$$

We fix a $\gamma_1 \in (p, p^*)$ and combining (3.3) and (3.4) yields

$$|F(s)| \le \frac{\epsilon}{2p}\big(1 - \frac{\mu}{C_{N,p}}\big)\frac{c(N,p)^{-p}}{(1+\|a\|_{L^\infty})}|s|^p + C\frac{(1+\delta)}{(1+\|a\|_{L^\infty})}\delta^{1-\gamma_1}|s|^{\gamma_1}, \tag{3.5}$$

for all $s \in \mathbb{R}$. For $\xi > 0$, we define the sets

$$A_\xi = \{u \in W_0^{1,p} : \Phi_\mu(u) < \xi\}; \quad B_\xi = \{u \in W_0^{1,p} : \big(1 - \frac{\mu}{C_{N,p}}\big)\|u\|^p_{W_0^{1,p}} < \xi\, p\}.$$

By an application of (2.1), one can observe that $A_\xi \subseteq B_\xi$. By (3.5),for every $u \in A_\xi$ and hence $u \in B_\xi$ we have

$$\begin{aligned}
&J(u)\\
&\le \frac{\epsilon}{2p}\big(1 - \frac{\mu}{C_{N,p}}\big)c(N,p)^{-p}\int_\Omega |u|^p dx + C(1+\delta)\delta^{1-\gamma_1}\int_\Omega |u(x)|^{\gamma_1} dx\\
&\le \frac{\epsilon}{2p}\big(1 - \frac{\mu}{C_{N,p}}\big)\int_\Omega |\nabla u|^p dx + C(1+\delta)^{1-\gamma_1}c(N,\gamma_1)^{\gamma_1}p^{\gamma_1/p}\xi^{\gamma_1/p}\Big(1 - \frac{\mu}{C_{N,p}}\Big)^{-\gamma_1/p}\\
&\le \frac{\epsilon}{2}\xi + C(1+\delta)^{1-\gamma_1}c(N,\gamma_1)^{\gamma_1}p^{\gamma_1/p}\xi^{\gamma_1/p}\Big(1 - \frac{\mu}{C_{N,p}}\Big)^{-\gamma_1/p}\\
&\le \frac{\epsilon}{2}\xi + C_1\xi^{\gamma_1/p},
\end{aligned} \tag{3.6}$$

where

$$C_1 = C(1+\delta)^{1-\gamma_1}c(N,\gamma_1)^{\gamma_1}p^{\gamma_1/p}\Big(1 - \frac{\mu}{C_{N,p}}\Big)^{-\gamma_1/p}.$$

Thus there exists $\xi(\epsilon) > 0$ such that for every $0 < \xi < \xi(\epsilon)$,

$$0 \le \frac{\sup_{u\in A_\xi} J(u)}{\xi} \le \frac{\sup_{u\in B_\xi} J(u)}{\xi} \le \frac{\epsilon}{2} + C_1\xi^{\frac{\gamma_1 - p}{p}} < \epsilon,$$

which proves the lemma. $\square$

Now we are ready to sketch the proof of the main result.

## 4. Proof of Theorem 1.1

*Proof.* Let $t_0 \in \mathbb{R}$ such that $F(t_0) > 0$, by (H3). We choose $R_0 > 0$ such that $R_0 < \operatorname{dist}(0, \partial\Omega)$. For $\eta \in (0,1)$ as already defined in [10], we also define

$$u_\eta(x) = \begin{cases} 0, & \text{if } x \in \mathbb{R}^N \setminus B_N(0,R_0);\\ t_0, & \text{if } x \in B_N(0,\eta R_0);\\ \frac{t_0}{R_0(1-\eta)}(R_0 - |x|), & \text{if } x \in B_N(0,R_0) \setminus B_N(0,\eta R_0), \end{cases}$$

where $B_N(0,r)$ denotes the $N$-dimensional open ball with center 0 and radius $r > 0$. It is easy to see that $u_\eta \in W_0^{1,p}$. Let $V_N$ denote the volume of the $N$-dimensional unit ball in $\mathbb{R}^N$, one can compute

$$\|u_\eta\|^p_{W_0^{1,p}} = t_0^p R_0^{N-p}(1-\eta)^{-p}V_N(1-\eta^N) \tag{4.1}$$

and

$$J(u_\eta) \ge [F(t_0)\eta^N - \max_{|t|\le|t_0|}|F(t)|(1-\eta^N)]V_N R_0^N. \tag{4.2}$$

For $\eta$ close enough to 1, the right hand side of the last inequality becomes strictly positive, so we choose such a number, say $\eta_0$. We fix $\mu \in [0, C_{N,p})$. By Lemma 3.4 and in view of (4.1), we may choose $\xi_0$ such that

$$p\xi_0 < \big(1 - \frac{\mu}{C_{N,p}}\big)\|u_{\eta_0}\|^p_{W_0^{1,p}},$$
$$\sup\{J(u) : \Phi_\mu(u) < \xi_0\} < \frac{p[F(t_0)\eta^N - \max_{|t|\le|t_0|}|F(t)|(1-\eta^N)]V_N R_0^N}{\|u_{\eta_0}\|^p_{W_0^{1,p}}}.$$

By choosing $x_1 = u_{\eta_0}$, hypotheses of Theorem 2.2 are satisfied. Define

$$\bar{A} = \bar{A}_\mu = \frac{1+\xi_0}{\frac{J(u_{\eta_0})}{\Phi_\mu(u_{\eta_0})} - \frac{\sup\{J(u):\Phi_\mu(u)<\xi_0\}}{\xi_0}}. \tag{4.3}$$

In view of Lemmas 3.1, 3.3, all the hypotheses of Theorem 2.2 are satisfied after putting $x_0 = 0$. An application of Theorem 2.2 implies that there exist an open interval $\Lambda_\mu \subset [0, \bar{A}_\mu]$ and a number $\eta_\mu > 0$ such that for each $\lambda \in \Lambda_\mu$, the equation $E'_{\mu,\lambda} \equiv \Phi'_\mu(u) - \lambda J'(u) = 0$, admits at least two solutions in $W_0^{1,p}$ which have $W_0^{1,p}$-norm less than $\eta_\mu$. Since (H1) implies that $f(0) = 0$, so (H1) admits one trivial solution and hence there exists a nontrivial solution to (1.2), which completes the proof. $\square$

**Remark 4.1.** Let $g(x) \equiv 1 \equiv a(x)$ and $p = 2$ in (1.2), then the proof of this corollary is given by Kristály and Varga [10]. In fact, they obtained the existence of three solutions. Since in the present study, $E_{\mu,\lambda}$ fails to satisfy the Palais-Smale condition, so we get the existence of two solutions.

**Remark 4.2.** As in [10], we also give the explicit estimation of the interval $\Lambda_\mu$, $\mu \in [0, C_{N,p})$. We fix $t_0, R_0, \eta_0$ as in the previous section. In view of Lemma 3.4, we have

$$\frac{\sup\{J(u) : \Phi_\mu(u) < \xi_0\}}{\xi_0} < \frac{J(u_{\eta_0})}{2\Phi_\mu(u_{\eta_0})}.$$

Then by (4.3), one can see that

$$\Lambda_\mu \subset \big[0, \frac{4}{p}\big(1 - \frac{\mu}{C_{N,p}}\big)\big(\frac{t_0}{R_0}\big)^p \frac{(1-\eta_0)^{-p}(1-\eta_0^N)}{[F(t_0)\eta_0^N - \max_{|t|\le|t_0|}|F(t)|(1-\eta_0^N)]}\big].$$

## 5. Examples

In this section, we construct some examples for the illustrations of main theorem.

**Example 5.1.** Consider (1.2) with $g(x) = 1 - e^{-|x|^2}, a(x) = \sin|x|$ and $\mu \in [0, (\frac{N-p}{p})^p)$. Suppose there exist $c > p-1$ and $S > 0$ such that

$$f(s) = \begin{cases} 0, & s \le 0; \\ s^c, & 0 < s \le S; \\ e^{-s} - e^{-S} + S^c, & S < s. \end{cases}$$

Then it is easy to see that $g, a$ and $f$ satisfy the hypotheses of Theorem 1.1. An application of Theorem 1.1 gives the existence of an open interval $\Lambda_\mu \subset (0, \infty)$ and a real number $\eta_\mu > 0$ such that for every $\lambda \in \Lambda_\mu$, the problem (1.2) has one non-trivial weak solution $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}(\Omega)} \le \eta_\mu$.

**Example 5.2.** Consider (1.2) with $g(x) = \frac{1+\sin|x|}{2}$, $a(x) = (1+|x|)^{-\alpha}\cos|x|$, where $\alpha > 0$ and $\mu \in [0, (\frac{N-p}{p})^p)$. Suppose there exist $\beta > 0, c > 0$ such that $\beta < p-1 < c$ and $S > 0$ such that

$$f(s) = \begin{cases} 0, & s \le 0; \\ e^{(s^c)} - 1, & 0 < s \le S; \\ e^{(S^c)} - S^\beta + s^\beta - 1, & S < s. \end{cases}$$

Then it is not difficult to see that $g, a$ and $f$ satisfy the hypotheses of Theorem 1.1. By Theorem 1.1, (1.2) has one non-trivial weak solution $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}(\Omega)} \le \eta_\mu$ for every $\lambda \in \Lambda_\mu$, where the existence of $\Lambda_\mu \subset (0,\infty)$ and a real number $\eta_\mu > 0$ are guaranteed by Theorem 1.1.

**Example 5.3.** Consider (1.2) with $g(x) = 1$, $a(x) = e^{\sin|x|}$ and $\mu \in [0, (\frac{N-p}{p})^p)$. Suppose there exist $\beta > 0, c > 0$ such that $\beta < p-1 < c$ and $S > 0$ such that

$$f(s) = \begin{cases} |s|^\beta - |S|^\beta, & s < -S; \\ 0, & -S \le s \le 0; \\ s^c(\sin s + e^{-s}), & 0 < s \le S; \\ S^c(\sin S + e^{-S}), & S < s. \end{cases}$$

Then it is easy to see that $g, a$ and $f$ satisfy the hypotheses of Theorem 1.1 and hence (1.2) has one non-trivial weak solution $u \in W_0^{1,p}(\Omega)$ such that $\|u\|_{W_0^{1,p}(\Omega)} \le \eta_\mu$ for every $\lambda \in \Lambda_\mu$, where the existence of $\Lambda_\mu \subset (0,\infty)$ and a real number $\eta_\mu > 0$ are guaranteed by Theorem 1.1.

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