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OSCILLATION OF SOLUTIONS FOR THIRD ORDER FUNCTIONAL DYNAMIC EQUATIONS

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Abstract. In this article we study the oscillation of solutions to the third order nonlinear functional dynamic equation

$$L_3(x(t)) + \sum_{i=0}^{n} p_i(t)\Psi_k\alpha_k i(x(h_i(t))) = 0,$$

on an arbitrary time scale $\mathbb{T}$. Here

$$L_0(x(t)) = x(t), \quad L_k(x(t)) = \Big(\frac{[L_{k-1}x(t)]^{\Delta}}{a_k(t)}\Big)^{\gamma_k k}, \quad k = 1, 2, 3$$

with a_1, a_2 positive rd-continuous functions on $\mathbb{T}$ and $a_3 \equiv 1$; the functions p_i are nonnegative rd-continuous on $\mathbb{T}$ and not all $p_i(t)$ vanish in a neighborhood of infinity; $\Psi_k c(u) = |u|^{c-1}u$, $c > 0$. Our main results extend known results and are illustrated by examples.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Stefan Hilger in his PhD dissertation [30], written under the direction of Bernd Aulbach. Since then a rapidly expanding body of literature has sought to unify, extend, and generalize ideas from discrete calculus, quantum calculus, and continuous calculus to arbitrary time scale calculus. Recall that a time scale $\mathbb{T}$ is a nonempty, closed subset of the reals, and the cases when this time scale is the reals or the integers represent the classical theories of differential and of difference equations. Many other interesting time scales exist, and they give rise to many applications (see [7]). Not only does the new theory of the so-called "dynamic equations" unify the theories of differential equations and difference equations, but also extends these classical cases to cases "in between", e.g., to the so-called q-difference equations when $\mathbb{T} = q^{\mathbb{N}_0}$ (which has important applications in quantum theory [31]) and can be applied on different types of time scales like $\mathbb{T} = h\mathbb{Z}$, $\mathbb{T} = \mathbb{N}_0^2$ and $\mathbb{T} = \mathbb{H}_n$ the space of harmonic numbers. In this work a knowledge and understanding of time scales and time scale notation is assumed; for an excellent introduction to the calculus on time scales, see Hilger [30], and Bohner and Peterson [7, 8].

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We are concerned with the oscillatory behavior of solutions to the third order nonlinear functional dynamic equation

$$L_3(x(t)) + \sum_{i=0}^{n} p_i(t)\Psi_k\alpha_i(x(h_i(t))) = 0, \tag{1.1}$$

on an arbitrary time scale $\mathbb{T}$, where $L_0(x(t)) = x(t)$,

$$L_k(x(t)) = (\frac{[L_{k-1}x(t)]^\Delta}{a_k(t)})^{\gamma_k k}, \quad k = 1, 2, 3,$$

with a_1, a_2 are positive rd-continuous functions on $\mathbb{T}$ and $a_3 \equiv 1$, and $\gamma_k k$, $k = 1, 2$ are quotients of odd positive integers, $\gamma_k 3 = 1$ and $\alpha_k 0 = \gamma_1\gamma_k 2$. We also assume $\Psi_k c(u) = |u|^{c-1}u$, $c > 0$ and p_i, $i = 0, 1, 2, \dots, n$ are nonnegative rd-continuous functions on $\mathbb{T}$ such that not all $p_i(t)$ vanish in a neighborhood of infinity. The functions $h_i : \mathbb{T} \to \mathbb{T}$ satisfy $\lim_{t\to\infty} h_i(t) = \infty$, $i = 0, 1, 2, \dots, n$. Since we are interested in the oscillatory behavior of solutions near infinity, we assume that $\sup \mathbb{T} = \infty$, and define the time scale interval $[t_0, \infty)_\mathbb{T}$ by $[t_0, \infty)_\mathbb{T} := [t_0, \infty) \cap \mathbb{T}$. By a solution of (1.1) we mean a nontrivial real-valued function $L_0x \in C^1_{rd}[T_x, \infty)_\mathbb{T}$, $T_x \geq t_0$ which has the property that $L_1x \in C^1_{rd}[T_x, \infty)_\mathbb{T}$, $L_2x \in C^1_{rd}[T_x, \infty)_\mathbb{T}$ and $x(t)$ satisfies equation (1.1) on $[T_x, \infty)_\mathbb{T}$, where C_{rd} is the space of rd-continuous functions. The solutions vanishing identically in some neighborhood of infinity will be excluded from our consideration. A solution x of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is nonoscillatory. Throughout the paper, we define, for $i = 0, 1, \dots, n$ and $k = 1, 2$,

$$A(s, u) := a_1(s)A_2^{1/\gamma_k 1}(s, u), \quad A_k(s, u) := \int_u^s a_k(u)\Delta u,$$

$$\varphi_k i(t, u) := A_1(h_i(t), u)\phi_{i,2}^{1/\gamma_k 1}(t), \quad \phi_k i, k(t) := \int_{h_i(t)}^\infty a_k(s)\Delta s,$$

$$h(t) := \min\{t,\ h_i(t);\ i = 0, 1, \dots, n\}, \quad (\Phi^\Delta(t))_+ := \max\{0, \Phi^\Delta(t)\}.$$

In the previous few years, there has been an increasing interest in obtaining sufficient conditions for the oscillation/nonoscillation of solutions of different classes of dynamic equations, we refer the reader to the papers [1, 2, 3, 4, 6, 9, 10, 11, 12, 13, 14, 28, 29, 15, 16, 17, 18, 23, 25, 27] and the references cited therein. Regarding third order dynamic equations, [19, 20, 26, 21, 22] considered the third order dynamic equation (1.1), in particular case and under quite restrictive conditions, for example, Erbe et al [19] studied the dynamic equation (1.1), when $n = 0$, $\gamma_1 = \gamma_k 2 = 1$, $\alpha_k 0 = 1$ and $h_0(t) \equiv t$ and established sufficient conditions which ensure the solution of equation (1.1) is either oscillatory or tends to zero and Erbe et al [20] extended the results which established in [19], when $\gamma_k 2 \geq 1$ is the quotient of odd positive integers. Also, Hassan [26] generalized the results which were established in [20, 19], for the equation (1.1), when $n = 0$, $\gamma_k 1 = 1$, $\alpha_k 0 = \gamma_k 2$, $\gamma_2 > 0$ is the quotient of odd positive integers, and $h_0(t) \leq t$ and $h_0^\Delta(t) \geq 0$, for $t \in \mathbb{T}$ and $h_0 \circ \sigma = \sigma \circ h_0$. A number of sufficient conditions for oscillation were obtained for the cases when, for $k = 1, 2$,

$$\int_{t_0}^\infty a_k(t)\Delta t = \infty.$$

Erbe, Hassan and Peterson [21] solved this problem for the third order functional dynamic equation (1.1), when $n = 0$, $\gamma_k 1 = 1$, $\alpha_k 0 = \gamma_k 2$, $\gamma_k 2 > 0$ is the quotient of odd positive integers and for both cases, for $k = 1, 2$,

$$\int_{t_0}^{\infty} a_k(t)\Delta t = \infty, \tag{1.2}$$

and

$$\int_{t_0}^{\infty} a_k(t)\Delta t < \infty. \tag{1.3}$$

Recently, Erbe, Hassan and Peterson [22] extended previous results for the third order dynamic equation (1.1) under some restrictive conditions, $n = 2$ and $h \circ \sigma = \sigma \circ h$.

The fact that the condition $h \circ \sigma = \sigma \circ h$ is not satisfied for some time scales, see [15]. The purpose of this paper is to extend the oscillation criteria which are established by [22], for the more general third order functional dynamic equation with mixed arguments (1.1), for several terms n and without restrictive condition $h\circ\sigma = \sigma\circ h$ and for both of the cases (1.2) and (1.3). We will still assume $\gamma_1, \gamma_k 2 > 0$ are the quotient of odd positive integers and, hence our results will improve and extend results in the [19, 20, 26, 21, 22], and many known results on nonlinear oscillation.

2. Main Results

Throughout this paper, we assume that $\alpha_k 1 > \alpha_k 2 > \cdots > \alpha_k m > \alpha_k 0 > \alpha_k m + 1 > \cdots > \alpha_k n > 0$. Before stating our main results, we begin with the following lemmas which will play an important role in the proof of our main results.

Lemma 2.1. *For each n-tuple $(\alpha_k 1, \alpha_k 2, \dots, \alpha_k n)$, there exists $(\eta_k 1, \eta_k 2, \dots, \eta_n)$ with $0 < \eta_k i < 1$ satisfying*

$$\sum_{i=1}^{n} \alpha_k i \eta_k i = \alpha_k 0, \quad \sum_{i=1}^{n} \eta_i = 1. \tag{2.1}$$

The proof of the above lemma is the same as [29, Lemma 2.1].

Lemma 2.2. *Let a_1 be nondecreasing and delta differentiable on $[t_0, \infty)_{\mathbb{T}}$ and x be a positive solution of (1.1) such that*

$$x^{\Delta}(t) > 0 \quad \text{and} \quad (L_1(x(t)))^{\Delta} > 0, \quad \text{for } t \geq t_0. \tag{2.2}$$

Then, if

$$\sum_{i=0}^{n} \int_{t_0}^{\infty} p_i(t) h_i^{\alpha_k i}(t)\Delta t = \infty, \tag{2.3}$$

we have

$$x^{\Delta\Delta}(t) > 0, \quad \big(\frac{x(t)}{t}\big)^{\Delta} < 0 \quad \text{on } [t_0, \infty)_{\mathbb{T}}.$$

Proof. Let x be as in the statement of this lemma. Since

$$(L_1(x(t)))^{\Delta} = \frac{((x^{\Delta}(t))^{\gamma_k 1})^{\Delta}}{a_1^{\gamma_1 \sigma}(t)} - \frac{(x^{\Delta}(t))^{\gamma_1}(a_1^{\gamma_k 1}(t))^{\Delta}}{a_1^{\gamma_k 1}(t) a_1^{\gamma_k 1 \sigma}(t)} > 0, \quad \text{for } t \geq t_0. \tag{2.4}$$

Using the Pötzsche chain rule [7, Theorem 1.90], we have

$$\begin{aligned}((x^{\Delta}(t))^{\gamma_k 1})^{\Delta} &= \gamma_k 1 \int_0^1 [x^{\Delta}(t) + h\mu(t)x^{\Delta\Delta}(t)]^{\gamma_k 1-1}\, dh\, x^{\Delta\Delta}(t) \\ &= \gamma_k 1 x^{\Delta\Delta}(t) \int_0^1 [hx^{\Delta\sigma}(t) + (1-h)x^{\Delta}(t)]^{\gamma_k 1-1} dh.\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}(a_1^{\gamma_k 1}(t))^{\Delta} &= \gamma_k 1 \int_0^1 [a_1(t) + h\mu(t)a_1^{\Delta}(t)]^{\gamma_1-1}\, dh\, a_1^{\Delta}(t) \\ &= \gamma_k 1 a_1^{\Delta}(t) \int_0^1 [ha_1^{\sigma}(t) + (1-h)a_1(t)]^{\gamma_k 1-1} dh.\end{aligned} \tag{2.6}$$

Using that a_1 is a nondecreasing and x is increasing, we obtain, from (2.4), (2.5) and (2.6) that $x^{\Delta\Delta}(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Next, we show that $\big(\frac{x(t)}{t}\big)^{\Delta} < 0$. To see this, let $U(t) := x(t) - tx^{\Delta}(t)$, then $U^{\Delta}(t) = -\sigma(t)x^{\Delta\Delta}(t) < 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$. This implies that $U(t)$ is strictly decreasing on $[t_0,\infty)_{\mathbb{T}}$. We claim $U(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Assume not, then there exists $t_1 \geq t_0$ such that $U(t) < 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore,

$$\big(\frac{x(t)}{t}\big)^{\Delta} = \frac{tx^{\Delta}(t) - x(t)}{t\sigma(t)} = -\frac{U(t)}{t\sigma(t)} > 0, \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{2.7}$$

Pick $t_2 \in [t_1,\infty)_{\mathbb{T}}$ so that $h_i(t) \geq t_1$, for $t \geq t_2$ and $i = 0,1,\dots,n$. Then, for $t \geq t_2$ and $i = 0,1,\dots,n$, we have from (2.7) that

$$\frac{x(h_i(t))}{h_i(t)} \geq \frac{x(t_1)}{t_1} =: c > 0, \quad \text{for } t \geq t_2 \text{ and } i = 0,1,\dots,n,$$

so $x(h_i(t)) \geq ch_i(t)$ for $t \geq t_2$ and $i = 0,1,\dots,n$. Now by integrating both sides of the dynamic equation (1.1) from t_2 to t, we have

$$L_2(x(t_2)) - L_2(x(t)) = \sum_{i=0}^{n} \int_{t_2}^{t} p_i(s)\Psi_k\alpha_i(x(h_i(s)))\Delta s.$$

This implies, from (2.2) that

$$L_2(x(t_2)) \geq \sum_{i=0}^{n} \int_{t_2}^{t} p_i(s)\Psi_k\alpha_k i(x(h_i(s)))\Delta s \geq \sum_{i=0}^{n} c^{\alpha_k i} \int_{t_2}^{t} p_i(s)h_i^{\alpha_i}(s)\Delta s. \tag{2.8}$$

Letting $t \to \infty$ we obtain a contradiction to assumption (2.3). Hence $U(t) = x(t) - tx^{\Delta}(t) > 0$ on $[t_0,\infty)_{\mathbb{T}}$. Consequently,

$$\big(\frac{x(t)}{t}\big)^{\Delta} = -\frac{U(t)}{t\sigma(t)} < 0, \quad t \in [t_0,\infty)_{\mathbb{T}}. \tag{2.9}$$

This completes the proof of Lemma 2.2. □

First, we establish oscillation criteria for (1.1) when (1.2) holds.

Theorem 2.3. *Let $\alpha_k 0 = \gamma_k 1\gamma_k 2$ and $h_i(t)$, $i = 0,1,2,\dots,n$ be nondecreasing functions on $[t_0,\infty)_{\mathbb{T}}$. Assume that* (1.2) *and*

$$\int_{t_0}^{\infty} a_1(t)\Big(\int_t^{\infty} a_2(s)\Big(\sum_{i=0}^{n} P_i(s)\Big)^{1/\gamma_k 2}\Delta s\Big)^{1/\gamma_k 1}\Delta t = \infty, \tag{2.10}$$

where $P_i(s) := \int_s^\infty p_i(u)\Delta u$, $i = 0, 1, \dots, n$. Furthermore, suppose that, for all sufficiently large T_1, there is $T > T_1$ such that $h(T) > T_1$ and

$$\limsup_{t\to\infty} Q_1(t)\Big(\int_{T_1}^{h(t)} A(s, T_1)\Delta s\Big)^{\alpha_k 0} > 1, \tag{2.11}$$

where

$$Q_1(t) := P_0(t) + \prod_{i=1}^{n} (\eta_k i^{-1} P_i(t))^{\eta_k i}.$$

Then every solution of (1.1) *is either oscillatory or tends to zero.*

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(h_i(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, for $i = 0, 1, 2, \dots, n$ and not all of the $p_i(t)$'s are identically zero on $[t_1, \infty)_{\mathbb{T}}$. From (1.1), we have

$$L_3(x(t)) = -\sum_{i=0}^{n} p_i(t)\Psi_k\alpha_i(x(h_i(t))) < 0, \tag{2.12}$$

on $[t_1, \infty)_{\mathbb{T}}$. Then $L_2(x(t))$ is strictly decreasing on $[t_1, \infty)_{\mathbb{T}}$. Therefore, $x^\Delta(t)$ and $(L_1(x(t)))^\Delta$ are eventually of one sign. Then, there exists a sufficiently large $t_2 \ge t_1$ so that $x^\Delta(t)$ and $L_1^\Delta(x(t))$ are of fixed sign for $t \ge t_2$. Therefore, we consider the following four cases:

(i) $x^\Delta(t) < 0$ and $(L_1(x(t)))^\Delta < 0$;
(ii) $x^\Delta(t) > 0$ and $(L_1(x(t)))^\Delta < 0$;
(iii) $x^\Delta(t) < 0$ and $(L_1(x(t)))^\Delta > 0$;
(iv) $x^\Delta(t) > 0$ and $(L_1(x(t)))^\Delta > 0$, on $[t_2, \infty)_{\mathbb{T}}$.

For case (i), we have

$$x(t) = x(t_2) + \int_{t_2}^{t} a_1(s)L_1^{1/\gamma_1}(x(s))\Delta s \le x(t_2) + L_1^{1/\gamma_1}(x(t_2)) \int_{t_2}^{t} a_1(s)\Delta s.$$

Hence by (1.2), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of (1.1). For the case (ii), from (2.12) we have

$$\begin{aligned} L_1(x(t)) &= L_1(x(t_2)) + \int_{t_2}^{t} a_2(s)L_2^{1/\gamma_k 2}(x(s))\Delta s \\ &\le L_1(x(t_2)) + L_2^{1/\gamma_k 2}(x(t_2)) \int_{t_2}^{t} a_2(s)\Delta s. \end{aligned}$$

Then, by (1.2), we have $\lim_{t\to\infty} L_1(x(t)) = -\infty$, which contradicts $x^\Delta(t) > 0$, for $t \ge t_2$. For the case (iii), we have $\lim_{t\to\infty} x(t) = c_0 \ge 0$ and $\lim_{t\to\infty} L_1(x(t)) = c_1 \le 0$. If we assume $c_0 > 0$, then $\Psi_k\alpha_k i(x(h_i(t))) \ge K$, for $i = 0, 1, 2, \dots, n$ and $t \ge t_3 \ge t_2$, where $K := \min\{c_0^{\alpha_i} :\ i = 0, 1, 2, \dots, n\}$. Integrating equation (1.1) from t to ∞, we obtain

$$\begin{aligned} -L_2(x(t)) &< -\sum_{i=0}^{n} \int_t^\infty p_i(s)\Psi_k\alpha_k i(x(h_i(s)))\Delta s \\ &\le -c_0 \sum_{i=0}^{n} \int_t^\infty p_i(s)\Delta s = -c_0 \sum_{i=0}^{n} P_i(t), \end{aligned}$$

which implies

$$-(L_1(x(t)))^{\Delta} < -C_0 a_2(t)\Big(\sum_{i=0}^{n} P_i(t)\Big)^{1/\gamma_k 2},$$

where $C_0 := K^{1/\gamma_k 2} \geq 0$. Integrating this inequality from t to ∞, we obtain

$$\begin{aligned} L_1(x(t)) &< -C_0 \int_t^{\infty} a_2(s)\Big(\sum_{i=0}^{n} P_i(s)\Big)^{1/\gamma_2} \Delta s + c_1 \\ &\leq -C_0 \int_t^{\infty} a_2(s)\Big(\sum_{i=0}^{n} P_i(s)\Big)^{1/\gamma_k 2} \Delta s. \end{aligned}$$

Finally, integrating the last inequality from t_3 to t, we obtain

$$x(t) < -c_0 \int_{t_3}^{t} a_1(s)\Big(\int_s^{\infty} a_2(u)(\sum_{i=0}^{n} P_i(u))^{1/\gamma_2} \Delta u\Big)^{1/\gamma_k 1} \Delta s + x(t_3).$$

Hence by (2.10), we have $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that x is a positive solution of (1.1). Thus, we conclude that $\lim_{t\to\infty} x(t) = 0$. For the case (iv), integrating both sides of the dynamic equation (1.1) from t to ∞ and then using the facts that $x(t)$ is strictly increasing and $h_i(t)$, $i = 0, 1, 2, \dots, n$ are nondecreasing, we obtain

$$\sum_{i=0}^{n} \Psi_k \alpha_k i(x(h_i(t)))P_i(t) \leq L_2(x(t)). \tag{2.13}$$

Since $L_2(x(t))$ is strictly decreasing on $[t_2, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned} L_1(x(t)) > L_1(x(t)) - L_1(x(t_2)) &= \int_{t_2}^{t} a_2(s) L_2^{1/\gamma_k 2}(x(s))\Delta s \\ &\geq L_2^{1/\gamma_k 2}(x(t)) \int_{t_2}^{t} a_2(s)\Delta s = L_2^{1/\gamma_k 2}(x(t))A_2(t, t_2). \end{aligned}$$

Hence

$$x^{\Delta}(t) \geq a_1(t) A_2^{1/\gamma_k 1}(t, t_2) L_2^{1/\alpha_0}(x(t)).$$

Similarly, we see that

$$x(t) \geq L_2^{1/\alpha_k 0}(x(t)) \int_{t_2}^{t} a_1(s) A_2^{1/\gamma_k 1}(s, t_2)\Delta s,$$

and so

$$x^{\alpha_k 0}(t) \geq L_2(x(t))\Big(\int_{t_2}^{t} A(s, t_2)\Delta s\Big)^{\alpha_k 0}. \tag{2.14}$$

Pick $t_3 > t_2$, sufficiently large, so that $h(t) > t_2$, for $t \geq t_3$. Then from (2.14), for $t \geq t_3$, we obtain

$$\begin{aligned} \Psi_k \alpha_k 0(x(h(t))) &\geq L_2(x(h(t)))\Big(\int_{t_2}^{h(t)} A(s, t_2)\Delta s\Big)^{\alpha_k 0} \\ &\geq L_2(x(t))\Big(\int_{t_2}^{h(t)} A(s, t_2)\Delta s\Big)^{\alpha_k 0}. \end{aligned} \tag{2.15}$$

Using (2.15) in (2.13), for $t \ge t_3$, we find that

$$\sum_{i=0}^{n} \Psi_k \alpha_k i(x(h_i(t)))P_i(t) \le \Psi_{\alpha_k 0}(x(h(t)))\Big(\int_{t_2}^{h(t)} A(s,t_2)\Delta s\Big)^{-\alpha_k 0},$$

which yields from the fact $x^{\Delta}(t) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ that

$$P_0(t) + \sum_{i=1}^{n} \Psi_k \alpha_k i - \alpha_0(x(h(t)))P_i(t) < \Big(\int_{t_2}^{h(t)} A(s,t_2)\Delta s\Big)^{-\alpha_k 0}. \tag{2.16}$$

Using the arithmetic-geometric mean inequality see [5, Page 17]

$$\sum_{i=1}^{n} \eta_k i u_i \ge \prod_{i=1}^{n} u_i^{\eta_i}, \quad \text{where } u_i \ge 0.$$

From Lemma 2.1, for $t \ge T$, we obtain

$$\begin{aligned}\sum_{i=1}^{n} \Psi_k \alpha_k i - \alpha_k 0(x(h(t)))P_i(t) &= \sum_{i=1}^{n} \eta_k i(\eta_k i^{-1} \Psi_k \alpha_k i - \alpha_0(x(h(t)))P_i(t)) \\ &\ge \prod_{i=1}^{n} (\eta_k i^{-1} P_i(t))^{\eta_k i} \Psi_k \eta_k i(\alpha_i - \alpha_k 0)(x(h(t))) \\ &\overset{(2.1)}{=} \prod_{i=1}^{n} (\eta_k i^{-1} P_i(t))^{\eta_k i},\end{aligned} \tag{2.17}$$

Using (2.17) in (2.16), we have

$$Q_1(t)\Big(\int_{t_2}^{h(t)} A(s,t_2)\Delta s\Big)^{\alpha_k 0} < 1, \quad \text{for } t \ge t_3.$$

Then

$$\limsup_{t\to\infty} Q_1(t)\Big(\int_{t_2}^{h(t)} A(s,t_2)\Delta s\Big)^{\alpha_k 0} \le 1,$$

which leads to a contradiction to (2.11). □

Theorem 2.4. *Assume that* (1.2) *and* (2.10) *hold. Furthermore, suppose that, for all sufficiently large* $T \in [t_0,\infty)_{\mathbb{T}}$, *such that* $h_i(T) > t_0$, $i = 0,1,\dots,n$,

$$\sum_{i=0}^{n} \int_{t_0}^{\infty} p_i(t) A_1^{\alpha_k i}(h_i(t),t_0)\Delta t = \infty. \tag{2.18}$$

Then every solution of equation (1.1) *is either oscillatory or tends to zero.*

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0,\infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0,\infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(h_i(t)) > 0$ on $[t_1,\infty)_{\mathbb{T}}$, for $i = 0,1,2,\dots,n$ and not all $p_i(t)$ are identically zero on $[t_1,\infty)_{\mathbb{T}}$. Therefore, as in the proof of Theorem 2.3, we obtain there exists $t_2 \ge t_1$ so that

$$(L_2(x(t)))^{\Delta} < 0, \quad (L_1(x(t)))^{\Delta} > 0, \quad \text{on } [t_2,\infty)_{\mathbb{T}},$$

and either $L_1(x(t)) > 0$ on $[t_2,\infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Assume $L_1(x(t)) > 0$ on $[t_2,\infty)_{\mathbb{T}}$. Since $(L_1(x(t)))^{\Delta} > 0$ on $[t_2,\infty)_{\mathbb{T}}$, then

$$L_1(x(t)) \ge L_1(x(t_2)) =: c > 0.$$

Thus
$$x(t) \geq x(t) - x(t_2) \geq C \int_{t_2}^{t} a_1(s)\Delta s,$$
where $C := c^{1/\gamma_k 1}$. Pick $t_3 \geq t_2$ such that $h_i(t) > t_2$, for $t \geq t_3$ and $i = 0, 1, \dots, n$, then, for $i = 0, 1, \dots, n$,
$$x(h_i(t)) > CA_1(h_i(t), t_2) \quad \text{on } [t_3, \infty)_{\mathbb{T}}. \tag{2.19}$$
It follows from (1.1) and (2.19) that
$$-L_3(x(t)) = \sum_{i=0}^{n} p_i(t)\Psi_k \alpha_i(x(h_i(t))) \geq \sum_{i=0}^{n} p_i(t)[CA_1(h_i(t), t_2)]^{\alpha_k i}.$$
Integrating both sides of the last inequality from t_3 to t, we have
$$\begin{aligned} L_2(x(t_3)) &> \sum_{i=0}^{n} \int_{t_3}^{t} p_i(s)[CA_1(h_i(s), t_2)]^{\alpha_k i}\Delta s + L_2(x(t)) \\ &> \sum_{i=0}^{n} \int_{t_3}^{t} p_i(s)[CA_1(h_i(s), t_2)]^{\alpha_k i}\Delta s, \end{aligned}$$
which contradicts (2.18). This completes the proof. □

Example 2.5. Consider the third order dynamic equation (1.1), for $t_0 \geq 1$, where $0 < \gamma_k 1 \leq 1$ and $\gamma_k 2 = \frac{1}{\gamma_k 1}$ are the quotient of odd positive integers and $\alpha_k i$, $i = 0, 1, \dots, n$ are positive constants and we assume $h_i(t) \leq t^{\gamma_k 2}$, $i = 0, 1, \dots, n$. Let
$$a_1(t) = 1, \quad a_2(t) = \frac{1}{t^{\gamma_k 1}}, \quad p_i(t) = \frac{1}{th_i(t)}, \quad i = 0, 1, \dots, n.$$
It is clear that conditions (1.2) hold, since
$$\int_{t_0}^{\infty} \Delta t = \infty, \quad \int_{t_0}^{\infty} \frac{\Delta t}{t^{\gamma_k 1}} = \infty, \quad \text{for } 0 < \gamma_k 1 \leq 1,$$
by [8, Example 5.60]. Note that
$$\int_{s}^{\infty} p_i(u)\Delta u = \int_{s}^{\infty} \frac{\Delta u}{uh_i(u)} \geq \int_{s}^{\infty} \frac{\Delta u}{u^{\gamma_k 2+1}} \geq \frac{1}{\gamma_k 2} \int_{s}^{\infty} (\frac{-1}{u^{\gamma_2}})^{\Delta}\Delta u = \frac{1}{\gamma_k 2}\frac{1}{s^{\gamma_2}},$$
$$\int_{t}^{\infty} a_2(s)(\sum_{i=0}^{n} P_i(s))^{1/\gamma_k 2}\Delta s \geq \gamma_k 0 \int_{t}^{\infty} \frac{\Delta s}{s^{\gamma_k 1+1}} \geq \frac{\gamma_k 0}{\gamma_k 1} \int_{t}^{\infty} (\frac{-1}{s^{\gamma_k 1}})^{\Delta}\Delta s = \frac{\gamma_k 0}{\gamma_1 t^{\gamma_k 1}},$$
where $\gamma_k 0 := (\frac{n+1}{\gamma_k 2})^{1/\gamma_k 2}$ and
$$\int_{t_0}^{\infty} a_1(t)\Big(\int_{t}^{\infty} a_2(s)(\sum_{i=0}^{n} P_i(s))^{1/\gamma_k 2}\Delta s\Big)^{1/\gamma_k 1}\Delta t \geq \gamma \int_{t_0}^{\infty} \frac{\Delta t}{t} = \infty,$$
where $\gamma := (\gamma_k 0/\gamma_k 1)^{1/\gamma_1}$, so that condition (2.10) holds. To apply Theorem 2.4, it remains to prove that condition (2.18) holds. To see this, note that
$$\sum_{i=0}^{n} \int_{t_0}^{\infty} p_i(t)\Big[\int_{t_0}^{h_i(t)} a_1(s)\Delta s\Big]^{\alpha_i}\Delta t \geq \int_{t_0}^{\infty} (\frac{1}{t} - \frac{t_0}{th_0(t)})\Delta t = \infty,$$
for those time scales, where $\int_{t_0}^{\infty} \frac{\Delta t}{th_0(t)} < \infty$. Then, by Theorem 2.4, every solution of equation (1.1) is either oscillatory or tends to zero.

By using Lemma 2.2, we obtain the following oscillation criterion for (1.1).

Theorem 2.6. *Let $\alpha_k 0 = \gamma_k 1 \gamma_k 2$ and a_1 be nondecreasing and delta differentiable on $[t_0, \infty)_{\mathbb{T}}$. Assume that (1.2), (2.3) and (2.10) hold. Furthermore, suppose that there exists a positive $\Delta-$differentiable function $\phi(t)$ and, for all sufficiently large T_1, there is $T > T_1$ such that*

$$\limsup_{t\to\infty} \int_T^t \Big[\phi(s)Q_2(s) - \frac{((\phi^\Delta(s))_+)^{\alpha_k 0+1}}{(\alpha_k 0+1)^{\alpha_k 0+1}(\phi(s)A(s,T_1))^{\alpha_k 0}}\Big]\Delta s = \infty, \tag{2.20}$$

where

$$Q_2(t) := p_0(t)H_0(t) + \prod_{i=1}^n (\eta_k i^{-1} p_i(t)H_i(t))^{\eta_k i},$$

and, for $i = 0, 1, \dots, n$,

$$H_i(t) := \begin{cases} 1, & h_i(t) \ge t \\ (\frac{h_i(t)}{t})^{\alpha_k i}, & h_i(t) \le t. \end{cases}$$

Then every solution of (1.1) is either oscillatory or tends to zero.

Proof. Assume (1.1) has a non-oscillatory solution x on $[t_0, \infty)_{\mathbb{T}}$. Then, without loss of generality, there is a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, sufficiently large, such that $x(t) > 0$ and $x(h_i(t)) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, for $i = 0, 1, 2, \dots, n$ and not all of the $p_i(t)$'s are identically zero on $[t_1, \infty)_{\mathbb{T}}$. Therefore, as in the proof of Theorem 2.3, we obtain there exists $t_2 \ge t_1$ so that

$$(L_2(x(t)))^\Delta < 0, \quad (L_1(x(t)))^\Delta > 0, \quad \text{on } [t_2, \infty)_{\mathbb{T}},$$

and either $L_1(x(t)) > 0$ on $[t_2, \infty)_{\mathbb{T}}$ or $\lim_{t\to\infty} x(t) = 0$. Assume $L_1(x(t)) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Consider the Riccati substitution

$$w(t) = \phi(t)\frac{L_2(x(t))}{x^{\alpha_0}(t)}.$$

By the product rule and then the quotient rule

$$\begin{aligned} w^\Delta(t) &= \frac{\phi(t)}{x^{\alpha_k 0}(t)}(L_2(x(t)))^\Delta + \big(\frac{\phi(t)}{x^{\alpha_k 0}(t)}\big)^\Delta L_2^\sigma(x(t)) \\ &= \phi(t)\frac{L_3(x(t))}{x^{\alpha_k 0}(t)} + \Big(\frac{\phi^\Delta(t)}{x^{\alpha_k 0\sigma}(t)} - \frac{\phi(t)(x^{\alpha_k 0}(t))^\Delta}{x^{\alpha_k 0}(t)x^{\alpha_k 0\sigma}(t)}\Big)L_2^\sigma(x(t)). \end{aligned} \tag{2.21}$$

From (2.21) and the definition of $w(t)$, we have, for $t \ge t_2$

$$w^\Delta(t) = -\phi(t)\sum_{i=0}^n p_i(t)\frac{x^{\alpha_k i}(h_i(t))}{x^{\alpha_k 0}(t)} + \frac{\phi^\Delta(t)}{\phi^\sigma(t)}w^\sigma(t) - \frac{\phi(t)(x^{\alpha_k 0}(t))^\Delta}{\phi^\sigma(t)x^{\alpha_k 0}(t)}w^\sigma(t).$$

Now, for a fixed i and let t be a fixed point in $[t_2, \infty)_{\mathbb{T}}$. Then either $h_i(t) \le t$ or $h_i(t) \ge t$. First, consider the case when $h_i(t) \ge t$. Then, by using the fact that x is strictly increasing, we obtain $x(h_i(t)) \ge x(t)$. Next, consider the case when $h_i(t) \le t$. Then, in view of Lemma 2.2, we obtain $x(h_i(t)) \ge \frac{h_i(t)}{t}x(t)$. It follows from the definition of $H_i(t)$ that, for $t \ge t_2$,

$$w^\Delta(t) \le -\phi(t)\sum_{i=0}^n p_i(t)H_i(t)x^{\alpha_k i-\alpha_k 0}(t) + \frac{\phi^\Delta(t)}{\phi^\sigma(t)}w^\sigma(t) - \frac{\phi(t)(x^{\alpha_k 0}(t))^\Delta}{\phi^\sigma(t)x^{\alpha_k 0}(t)}w^\sigma(t).$$

As in the proof of Theorem 2.3, we obtain, for $t \ge t_2$,

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\phi(t)(x^{\alpha_k 0}(t))^{\Delta}}{\phi^{\sigma}(t)x^{\alpha_k 0}(t)}w^{\sigma}(t).$$

Then, by the Pötzsche chain rule [7, Theorem 1.90], we obtain

$$\begin{aligned}
(x^{\alpha_k 0}(t))^{\Delta} &= \alpha_k 0\int_0^1 [x(t) + h\mu(t)x^{\Delta}(t)]^{\alpha_k 0-1} dh\, x^{\Delta}(t) \\
&= \alpha_k 0\int_0^1 [(1-h)x(t) + hx^{\sigma}(t)]^{\alpha_k 0-1} dh\, x^{\Delta}(t) \\
&\ge \begin{cases} \alpha_k 0 x^{(\alpha_k 0-1)\sigma}(t)\, x^{\Delta}(t), & 0 < \alpha_k 0 \le 1 \\ \alpha_k 0 x^{\alpha_k 0-1}(t)\, x^{\Delta}(t), & \alpha_k 0 \ge 1. \end{cases}
\end{aligned}$$

If $0 < \alpha_k 0 \le 1$, we have

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha_k 0\phi(t)w^{\sigma}(t)}{\phi^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\Big(\frac{x^{\sigma}(t)}{x(t)}\Big)^{\alpha_k 0},$$

whereas if $\alpha_k 0 \ge 1$, we have

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha_k 0\phi(t)w^{\sigma}(t)}{\phi^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}\frac{x^{\sigma}(t)}{x(t)}.$$

Using that $x(t)$ is strictly increasing on $[t_2,\infty)_{\mathbb{T}}$, we obtain that, for $\alpha_k 0 > 0$,

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{\phi^{\Delta}(t)}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha_k 0\phi(t)w^{\sigma}(t)}{\phi^{\sigma}(t)}\frac{x^{\Delta}(t)}{x^{\sigma}(t)}. \tag{2.22}$$

Then using that $L_2(x(t))$ is strictly decreasing on $[t_2,\infty)_{\mathbb{T}}$, we obtain that, for $t \ge t_2$,

$$\begin{aligned}
L_1(x(t)) > L_1(x(t)) - L_1(x(t_2)) &= \int_{t_2}^{t} a_2(s)L_2^{1/\gamma_k 2}(x(s))\Delta s \\
&\ge L_2^{1/\gamma_k 2}(x(t))\int_{t_2}^{t} a_2(s)\Delta s \ge L_2^{\sigma/\gamma_k 2}(x(t))A_2(t,t_2).
\end{aligned} \tag{2.23}$$

From (2.22) and (2.23), we obtain

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{(\phi^{\Delta}(t))_+}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha_k 0\phi(t)A(t,t_2)}{\phi^{\alpha\sigma}(t)}w^{\alpha\sigma}(t), \tag{2.24}$$

where $\alpha := \frac{\alpha_k 0+1}{\alpha_k 0}$. Define $X \ge 0$ and $Y \ge 0$ by

$$X^{\alpha} := \frac{\alpha_k 0\phi(t)A(t,t_2)}{\phi^{\alpha\sigma}(t)}w^{\alpha\sigma}(t), \quad Y^{\alpha-1} := \frac{(\phi^{\Delta}(t))_+}{\alpha(\alpha_0\phi(t)A(t,t_2))^{1/\alpha}}.$$

Then, using the inequality, see [24],

$$\alpha XY^{\alpha-1} - X^{\alpha} \le (\alpha-1)Y^{\alpha}, \tag{2.25}$$

we obtain

$$\frac{(\phi^{\Delta}(t))_+}{\phi^{\sigma}(t)}w^{\sigma}(t) - \frac{\alpha_k 0\phi(t)A(t,t_2)}{\phi^{\alpha\sigma}(t)}w^{\alpha\sigma}(t) \le \frac{((\phi^{\Delta}(t))_+)^{\alpha_k 0+1}}{(\alpha_k 0+1)^{\alpha_k 0+1}(\phi(t)A(t,t_2))^{\alpha_k 0}}.$$

From the above inequality and (2.25), we obtain

$$w^{\Delta}(t) \le -\phi(t)Q_2(t) + \frac{((\phi^{\Delta}(t))_+)^{\alpha_k 0+1}}{(\alpha_k 0+1)^{\alpha_0+1}(\phi(t)A(t,t_2))^{\alpha_k 0}}.$$

Integrating both sides from t_2 to t,

$$\int_{t_2}^{t}\Big[\phi(s)Q_2(s)-\frac{((\phi^{\Delta}(s))_+)^{\alpha_k0+1}}{(\alpha_k0+1)^{\alpha_0+1}(\phi(s)A(s,t_2))^{\alpha_k0}}\Big]\Delta s\le w(t_2)-w(t)\le w(t_2),$$

which leads to a contradiction to (2.20). $\square$

Example 2.7. Consider the third order nonlinear dynamic equation (1.1), for $t_0\ge 1$, where γ_k1 and γ_k2 are the quotient of odd positive integers and α_ki, $i=0,1,\dots,n$ are positive constants with $\alpha_k1>\alpha_k2>\dots>\alpha_km>\alpha_k0>\alpha_{m+1}>\dots>\alpha_kn>0$ and also, we assume $h_i(t)\ge t$, $i=0,1,\dots,n$ such that (2.3) holds. Note that since $h_i(t)\ge t$, $i=0,1,\dots,n$ it follows that $H_i(t)\equiv 1$, $i=0,1,\dots,n$. Let

$$a_1(t)=t^{1/\alpha_k0},\quad a_2(t)=\frac{1}{t^{1-1/\gamma_k2}},$$
$$p_0(t)=\frac{\beta_k0}{t^{\alpha_k0+2}},\quad p_i(t)=\frac{\beta_ki}{t^{\eta_ki+2}},\quad i=1,2,\dots,n.$$

where η_ki, $i=1,2,\dots,n$ and β_ki, $i=0,1,\dots,n$ are positive constants such that $\alpha_k0\ge\eta_ki$, $i=1,2,\dots,n$. It is clear that conditions (1.2) hold, since

$$\int_{t_0}^{\infty}a_1(t)\Delta t=\int_{t_0}^{\infty}t^{1/\alpha_k0}\Delta t=\infty,$$

and

$$\int_{t_0}^{\infty}a_2(t)\Delta t=\int_{t_0}^{\infty}\frac{\Delta t}{t^{1-1/\gamma_k2}}=\infty,$$

by [7, Example 5.60]. Therefore, we can find $T>T_1$ such that $A_2(s,T_1)\ge 1$, for $s\ge T$. Also, using the Pötzsche chain rule,

$$\begin{aligned}P_0(s)&=\int_s^{\infty}p_0(u)\Delta u=\beta_0\int_s^{\infty}\frac{\Delta u}{u^{\alpha_k0+2}}\\&\ge\frac{\beta_k0}{\alpha_k0+1}\int_s^{\infty}(\frac{-1}{u^{\alpha_k0+1}})^{\Delta}\Delta u=\frac{\beta_k0}{\alpha_k0+1}\frac{1}{s^{\alpha_k0+1}},\end{aligned}$$

and

$$\begin{aligned}P_i(s)&=\int_s^{\infty}p_i(u)\Delta u=\beta_i\int_s^{\infty}\frac{\Delta u}{u^{\eta_ki+2}}\\&\ge\frac{\beta_ki}{\eta_ki+1}\int_s^{\infty}(\frac{-1}{u^{\eta_ki+1}})^{\Delta}\Delta u\\&=\frac{\beta_ki}{\eta_ki+1}\frac{1}{s^{\eta_ki+1}}\\&\ge\frac{\beta_ki}{\alpha_k0+1}\frac{1}{s^{\alpha_k0+1}},\quad i=1,2,\dots,n.\end{aligned}$$

Hence

$$\begin{aligned}\Big(\int_t^{\infty}a_2(s)\Big(\sum_{i=0}^{n}P_i(s)\Big)^{1/\gamma_k2}\Delta s\Big)^{1/\gamma_k1}&\ge\beta\Big(\int_t^{\infty}\frac{\Delta s}{s^{\gamma_k1+1}}\Big)^{1/\gamma_k1}\\&\ge\frac{\beta}{\gamma_k1^{1/\gamma_k1}}\Big(\int_t^{\infty}(\frac{-1}{s^{\gamma_k1}})^{\Delta}\Delta s\Big)^{1/\gamma_k1}\\&=\frac{\beta}{\gamma_k1^{1/\gamma_k1}}\frac{1}{t},\end{aligned}$$

where $\beta := (\frac{1}{\alpha_k 0+1}\sum_{i=0}^{n}\beta_k i)^{1/\alpha_k 0}$. Then

$$\begin{aligned}
&\int_{t_0}^{\infty} a_1(t)\Big(\int_t^{\infty} a_2(s)(\sum_{i=0}^{n} P_i(s))^{1/\gamma_k 2}\Delta s\Big)^{1/\gamma_k 1}\Delta t\\
&= \frac{\beta}{\gamma_k 1^{1/\gamma_k 1}}\int_{t_0}^{\infty}\frac{1}{t^{1-1/\alpha_k 0}}\Delta t = \infty,
\end{aligned}$$

so that condition (2.10) holds. Let us take $\phi(t) = t^{\alpha_k 0+1}$, then, by the Pötzsche chain rule

$$\phi^{\Delta}(t) = (\alpha_k 0+1)\int_0^1 (t+h\mu(t))^{\alpha_0}dh \le (\alpha_k 0+1)(\sigma(t))^{\alpha_k 0}.$$

Now, we assume $\mathbb{T}$ is a time scale satisfying $\sigma(t) \le kt$, for some $k > 0$, $t \ge T_k > T$. Note that

$$\begin{aligned}
&\limsup_{t\to\infty}\int_T^t [\phi(s)Q_2(s) - \frac{((\phi^{\Delta}(s))_+)^{\alpha_k 0+1}}{(\alpha_k 0+1)^{\alpha_k 0+1}(\phi(s)A(s,T_1))^{\alpha_k 0}}]\Delta s\\
&\ge \limsup_{t\to\infty}\int_{T_k}^t [\frac{\beta_k 0}{s} - \frac{k^{\alpha_k 0(\alpha_k 0+1)}}{s}]\Delta s\\
&\ge (\beta_k 0 - k^{\alpha_k 0(\alpha_k 0+1)})\limsup_{t\to\infty}\int_{T_k}^t \frac{\Delta s}{s} = \infty,
\end{aligned}$$

if $\beta_k 0 > k^{\alpha_k 0(\alpha_k 0+1)}$ and hence (2.20) holds. We conclude that if $[T,\infty)_{\mathbb{T}}$ is a time scale where $\sigma(t) \le kt$, for some $k > 0$, $t \ge T_k$, then, by Theorem 2.6, every solution of (1.1) is either oscillatory or tends to zero if $\beta_k 0 > k^{\alpha_k 0(\alpha_k 0+1)}$.

In the following, we assume that (1.3) holds and establish sufficient conditions which ensure that every solution $x(t)$ of (1.1) is either oscillatory or tends to zero. By Theorems 2.3 and 2.6 and [22, Theorem 2.1], we obtain the following oscillation criteria for equation (1.1).

Corollary 2.8. *Let $\alpha_k 0 = \gamma_k 1\gamma_k 2$ and $h_i(t)$, $i = 0,1,2,\dots,n$ be nondecreasing functions on $[t_0,\infty)_{\mathbb{T}}$. Assume that* (1.3) *and* (2.10) *hold. Furthermore, suppose that, for all sufficiently large $T_1 \in [t_0,\infty)_{\mathbb{T}}$, there is $T > T_1$ such that $h_i(T) > T_1$, $i = 1,2,\dots,n$,*

$$\int_T^{\infty} a_1(t)\Big[\int_T^t a_2(s)[\sum_{i=1}^{n}\int_T^s p_i(u)\phi_k i,1^{\alpha_k i}(u)\Delta u]^{1/\gamma_k 2}\Delta s\Big]^{1/\gamma_k 1}\Delta t = \infty, \quad (2.26)$$

and

$$\int_T^{\infty} a_2(t)\Big[\sum_{i=1}^{n}\int_T^t p_i(u)\varphi_i^{\alpha_k i}(u,T_1)\Delta u\Big]^{1/\gamma_2}\Delta t = \infty. \quad (2.27)$$

If condition (2.11) *holds, then every solution of* (1.1) *is either oscillatory or tends to zero.*

Corollary 2.9. *Assume that* (1.3), (2.10), (2.26) *and* (2.27) *hold. If* (2.18) *holds, then every solution of* (1.1) *is either oscillatory or tends to zero.*

Corollary 2.10. *Let $\alpha_k 0 = \gamma_k 1\gamma_k 2$ and a_1 be nondecreasing and delta differentiable on $[t_0,\infty)_{\mathbb{T}}$. Assume that* (1.3), (2.3), (2.10), (2.26) *and* (2.27) *hold. If* (2.20) *holds, then every solution of* (1.1) *is either oscillatory or tends to zero.*

Remark 2.11. If (2.10) is not satisfied, we have sufficient conditions which ensure that every solution $x(t)$ of (1.1) oscillates or $\lim_{t\to\infty} x(t)$ exists as a finite number.

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