Electronic Journal of Differential Equations, Vol. 2010(2010), No. 132, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

LOSS OF EXPONENTIAL STABILITY FOR A THERMOELASTIC SYSTEM WITH MEMORY

BRUNO FERREIRA ALVES, WALDEMAR DONIZETE BASTOS, CARLOS ALBERTO RAPOSO

Abstract. In this article we study a thermoelastic system considering the linearized model proposed by Gurtin and Pipkin [8] instead of the Fourier's law for the heat flux. We use theory of semigroups [9, 11] combining Pruss' Theorem [10] and the idea developed in [5] to show that the system is not exponentially stable.

1. Introduction

We study a partial differential equation that models an elastic string:

$$u_{tt} - u_{xx} + \theta_{xx} = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{1.1}$$

$$\theta_t - g * \theta_{xx} + c\, g * \theta - u_{xxt} = 0 \quad \text{in } (0, L) \times (0, \infty), \tag{1.2}$$

with initial data

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x)\,.$$

The function $u = u(x, t)$ is the small transversal vibration of the elastic string of reference configuration of length L, and $\theta = \theta(x, t)$ is the temperature difference from the material and natural ambient. To fix ideas we assume that the string is held fixed at both ends, $x = 0$ and $x = L$. We impose the boundary conditions

$$u(0, t) = u(L, t) = 0,$$
$$\theta(0, t) = \theta(L, t) = 0.$$

In this model, c is a positive constant, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is the relaxation function. We assume that g is differentiable and satisfies $g(0) > 0$, $g'(t) < 0$ and

$$1 - \int_0^\infty g(s)ds = \ell > 0.$$

2000 *Mathematics Subject Classification.* 35B40, 35Q80, 35L05, 47D06.
Key words and phrases. Heat conduction with memory; C_0-semigroup; decay of solutions; thermoelastic system.

Submitted February 23, 2010. Published September 14, 2010.
C. A. Raposo was supported by grants 573523/2008-8 - INCTMat from CNPq, and 620025/2006-9 from CNPq.

We introduce the convolution product

$$(g*u)(t) := \int_0^t g(t-\tau)u(\cdot,\tau)d\tau\,.$$

Now we observe that when $c=0$ the thermoelastic system has exponential decay, as can be seen in [4], when we replace $g*u$ by θ in (1.2) we also have exponential decay, see [3]. The similar situation is valid for thermoelastic plate, see [5] and [7].

The article is organized as follows, in the Section 2 we introduce the notation and the functional spaces, in the Section 3 we obtain the semigroup of solutions and finally, in the Section 4 we prove the loss of exponential stability for the thermoelastic system with memory.

2. Functional setting and notation

We use the standard Lebesgue spaces and Sobolev spaces with their usual proprieties as in [1]. Consider the positive operators A and B on $L^2(0,L)$ defined by $A=-(\cdot)_{xx}$ and $B=cI-(\,\cdot\,)_{xx}$, with domains $D(A)=D(B)=(H^2\cap H_0^1)(0,L)$. Now, for $r\in\mathbb{R}$, we introduce the scale of Hilbert spaces $H_r=D(A^{r/2})$ with the usual inner products $\langle v_1,v_2\rangle_{H_r}=\langle A^{r/2}v_1,A^{r/2}v_2\rangle$ and we have $H_{r_1}\hookrightarrow H_{r_2}$ are compact whenever $r_1>r_2$. Concerning the memory kernel g, we make the substitution $\mu(s)=-g(s)$ and we require

$$\mu\in C^1(\mathbb{R}^+)\cap L^1(\mathbb{R}^+),\quad \mu(s)>0,\quad \mu'(s)\le 0,\quad g(0)=\int_0^\infty \mu(s)ds>0. \tag{2.1}$$

Calling $\sigma_\infty=\sup\{s:\mu(s)>0\}$, we infer that, dual to (2.1), for each $\sigma>0$, there exists a set $\mathcal{O}_\sigma\subset(\sigma,\sigma_\infty)$ of positive Lebesgue measure such that $\mu'(s)<0$, in $\mathcal{O}_\sigma$. Now for $r\in\mathbb{R}$ consider the weighted Hilbert spaces:

$$\mathcal{M}_r=L^2_\mu(\mathbb{R}^+;H_r)$$

with the inner product

$$\langle\nu,\eta\rangle_{\mathcal{M}_r}=\int_0^\infty \mu(s)\langle B^{r/2}\nu(s),B^{r/2}\eta(s)\rangle\,ds \tag{2.2}$$

and we introduce as in [6] the linear operator T on $\mathcal{M}_1$ defined by $T\eta=-\eta_s$ with domain

$$D(T)=\{\eta\in\mathcal{M}_1:\eta_s\in\mathcal{M}_1,\,\eta(0)=0\},$$

where η_s is the distributional derivative of η with respect to the internal variable s, and then the operator T is the infinitesimal generator of a C_0-semigroup of contractions. In particular, there holds

$$\langle T\eta,\eta\rangle_{\mathcal{M}_1}=\int_0^\infty \mu'(s)\|B^{1/2}\eta(s)\|\,ds\le 0,\quad \text{for all } \eta\in D(T). \tag{2.3}$$

Finally, we define with the usual inner products, the following Hilbert spaces:

$$\mathcal{H}_r=H_{r+2}\times H_r\times H_r\times M_{r+1},\quad r\in\mathbb{R}.$$

3. The semigroup of solutions

To describe properly the solutions of the system (1.1)-(1.2) by means of a C_0-semigroup of linear operators acting on the phase-space $\mathcal{H}_0$, we will follow the ideas of [1]. In this direction we introduce an additional variable, namely, the summed past history of θ defined as

$$\eta^t(s) = \int_0^s \theta(t-y)dy, \quad \text{with } t, s \geq 0.$$

Observe that we have formally $(\frac{d}{dt} + \frac{d}{ds})(\eta^t(s)) = \theta$ in $(0, L)$ subject to the boundary and initial conditions $\eta^t(0) = 0$ in $(0, L)$, $t \geq 0$,

$$\eta^0(s) = \int_0^s \theta(-y)dy, \quad s \geq 0.$$

For the rest of this article, we consider the vectors $U(t) = (u(t), v(t), \theta(t), \eta^t)^T$ and $U(0) = (u_0, v_0, \theta_0, \eta_0)^T \in \mathcal{H}_0$. We obtain the linear evolution equation, in $\mathcal{H}_0$,

$$U_t - L\,U = 0 \tag{3.1}$$

$$U(0) = U_0 \tag{3.2}$$

where the linear operator L is defined as

$$L\,U = \begin{pmatrix} v \\ u_{xx} - \theta_{xx} \\ u_{xx} - \int_0^\infty g(s)[c\theta(t-s) - \theta_{xx}(t-s)]ds \\ \eta \end{pmatrix}.$$

with domain $D(L) = \{(u, v, \theta, \eta)^T \in \mathcal{H}_0\}$ such that $v \in H_2$, $u_{xx} - \theta_{xx} \in H_0$,

$$u_{xx} - \int_0^\infty g(s)[c\theta(t-s) - \theta_{xx}(t-s)]ds \in H_0, \quad \eta \in D(T).$$

Theorem 3.1. *System* (3.1) *defines a C_0-semigroup of contractions $S(t) = e^{tL}$ on the phase-space $\mathcal{H}_0$.*

The proof is done by using the Lumer - Phillips theorem [9, Theorem 4.3].

4. Loss of exponential stability

To prove the loss of exponential stability we use the following result.

Theorem 4.1. *Let $S(t) = e^{tL}$ be a C_0-semigroup of contractions in a Hilbert space. Then $S(t)$ is exponentially stable if and only if,*

$$i\mathbb{R} = \{i\beta : \beta \in \mathbb{R}\} \subset \rho(L) \tag{4.1}$$

and

$$\|(\lambda I - L)^{-1}\| \leq C, \quad \textit{for every } \lambda \in i\mathbb{R}. \tag{4.2}$$

The proof of the above theorem can be found in [10] and in [11].

We note that (3.1)-(3.2) is dissipative, because (2.3) implies

$$\langle LU, U\rangle_{\mathcal{H}_0} = \langle T\eta, \eta\rangle_{\mathcal{M}_1} \leq 0, \quad \text{for all } U \in D(L), \tag{4.3}$$

and it is standard matter to show that $(I - L)$ maps $D(L)$ onto $\mathcal{H}_0$, see [3], where a similar case is treated.

Then, using $\langle Tu, u\rangle < 0$ for all nonzero u in $D(T)$, one can show that the solution of thermoelastic system (1.1)-(1.2) decays to zero as time approaches ∞.

Now we are in position of to show our main result.

Theorem 4.2. *The semigroup $S(t) = e^{tL}$ on $\mathcal{H}_0$ defined by* (3.1)-(3.2) *is not exponentially stable.*

Proof. For $i\lambda \in \rho(L)$ and $V = (0,0,0,\eta)^T \in \mathcal{H}_0$, consider the complex equation

$$(i\lambda\, I - L)U = V \tag{4.4}$$

that when written explicitly reads

$$i\lambda u - v = 0 \tag{4.5}$$

$$i\lambda v - u_{xx} + \theta_{xx} = 0 \tag{4.6}$$

Consider an orthonormal basis $\{w_j\}_{n\in\mathbb{N}}$ of eigenvectors of the operator A and the respective eigenvalues $\{\alpha_n\}_{n\in\mathbb{N}}$. We recall that $\alpha_n \to \infty$ as $n \to \infty$. We set

$$\eta_n(s) = \frac{w_n}{\sqrt{c+\alpha_n}}$$

and

$$V_n = (0,0,0,\eta_n)^T.$$

Notice that, using (2.1) and (2.2) we have

$$\|V_n\|_{\mathcal{H}_0} = \|\eta_n\|_{\mathcal{M}_1} = \frac{1}{(c+\alpha_n)}$$

$$\begin{aligned}\int_0^\infty \mu(s)\|B^{1/2}w_n(s)\|^2 ds &= \frac{1}{(c+\alpha_n)}\int_0^\infty \mu(s)(c+\alpha_n)\|w_n(s)\|^2 ds\\ &= \int_0^\infty \mu(s)ds = g(0).\end{aligned}$$

Now we build a sequence of λ_n such that the corresponding solution U_n of

$$(i\lambda_n I - L)U_n = V_n \tag{4.7}$$

satisfies $\|U_n\|_{\mathcal{H}_0} \to \infty$ as $n \to \infty$. In this direction we look for a solution $U_n = (w_n, w_n, s_n w_n, w_n)$ where $s_n \in \mathbb{C}$. Then, from (4.5) and (4.6) we have

$$-\lambda_n^2 - \alpha_n + s_n\alpha_n = 0 \tag{4.8}$$

that implies

$$s_n = 1 + \frac{\lambda_n^2}{\alpha_n}.$$

Choosing $\lambda_n = |\alpha_n|$ we finally have

$$\|U_n\|_{\mathcal{H}_0} \ge \|s_n\, w_n\|_{H_0} = |s_n| \ge \frac{\lambda_n^2}{|\alpha_n|} = |\alpha_n| \to \infty \quad \text{as } n \to \infty.$$

which yields the conclusion. $\square$

References

[1] R. A. Adams; *Sobolev Spaces*, Academic Press, New York, (1975).
[2] C. M. Dafermos; *Asymptotic stability in viscoelasticity*, Arch. Ration. Mech. Anal. 37 (1970), 297308.
[3] M. Fabrizio, B. Lazzari, J. E. Muñoz Rivera; *Asymptotic Behavior in Linear Thermoelasticity*, Journal of Mathematical Analysis and Applications. 232 (1999) 138-165.
[4] L. H. Fatore, J. E. Muñoz Rivera; *Energy decay for hyperbolic thermoelastic systems of memory type*. Quartely Of Applied Mathematics. 59(3) (2001) 441-458.
[5] M. Grasselli, J. E. Muñoz Rivera, V. Pata; *On the energy decay of the linear thermoelastic plate with memory*. J. Math. Anal. Appl. 309 (2005) 114.
[6] M. Grasselli, V. Pata; *Uniform attractors of nonautonomous systems with memory*, in: A. Lorenzi, B. Ruf (Eds.), Evolution Equations, Semigroups and Functional Analysis, in: Progr. Nonlinear Differential Equations Appl., vol. 50, Birkhuser, Boston, (2002).
[7] C. Giorgi, V. Pata; *Stability of linear thermoelastic systems with memory*. Math. Models Methods Appl. Sci. 11 (2001) 627644.
[8] M. E. Gurtin, A. C. Pipkin; *A general theory of heat conduction with finite wave speeds*. Arch. Rational Mech. Anal. 31 (1968) 113-126.
[9] A. Pazy; *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer- Verlag, New York, (1983).
[10] J. Prss; *On the spectrum of C0-semigroups* Trans. Amer. Math. Soc. 284 (1984) 847857.
[11] Z. Liu, S. Zheng; *Semigroups Associated with Dissipative Systems*. Chapman & Hall/CRC Press, Boca Raton, FL, (1999).

Bruno Ferreira Alves
Departament of Mathematics, Federal University of S. J. del-Rey, Praça Frei Orlando, 170, 36300-000, São João del Rei, MG, Brazil
E-mail address: bruno.fa@ibest.com.br

Waldemar Donizete Bastos
Departament of Mathematics, São Paulo State University, UNESP
Rua Cristóvão Colombo, 2265, 15054-000, São José do Rio Preto, SP, Brazil
E-mail address: waldemar@ibilce.unesp.br

Carlos Alberto Raposo
Departament of Mathematics, Federal University of S. J. del-Rey, Praça Frei Orlando, 170, 36300-000, São João del Rei, MG, Brazil
E-mail address: raposo@ufsj.edu.br