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EXISTENCE AND UPPER SEMICONTINUITY OF GLOBAL ATTRACTORS FOR NEURAL FIELDS IN AN UNBOUNDED DOMAIN

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Abstract. In this article, we prove the existence and upper semicontinuity of compact global attractors for the flow of the equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + J * (f \circ u)(x,t) + h, \quad h > 0,$$

in L^2 weighted spaces.

1. Introduction

We consider here the non local evolution equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t) + J * (f \circ u)(x,t) + h, \quad h > 0, \tag{1.1}$$

where $u(x,t)$ is a real-valued function on $\mathbb{R}\times\mathbb{R}_+$, h is a positive constant, $J \in C^1(\mathbb{R})$ is a non negative even function supported in the interval $[-1,1]$, and, f is a non negative nondecreasing function. The $*$ above denotes convolution product, namely:

$$(J * u)(x) = \int_{\mathbb{R}} J(x-y)u(y)dy. \tag{1.2}$$

Equation (1.1) was derived by Wilson and Cowan, [18], to model a single layer of neurons in 1972. The function $u(x,t)$ denotes the mean membrane potential of a patch of tissue located at position $x \in (-\infty,\infty)$ at time $t \geq 0$. The connection function $J(x)$ determines the coupling between the elements at position x and position y. The non negative nondecreasing function $f(u)$ gives the neural firing rate, or averages rate at which spikes are generated, corresponding to an activity level u. The neurons at a point x are said to be active if $f(u(x,t)) > 0$. The parameter h denotes a constant external stimulus applied uniformly to the entire neural field, (see [1], [4], [6], [8], [9], [10], [15] and [16]).

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An equilibrium of (1.1) is a solution for (1.1) that is constant with respect to t. Thus, if u is an equilibrium for (1.1) then u satisfies

$$u(x) = J * (f \circ u)(x) + h. \tag{1.3}$$

In the literature, there are already several works dedicated to the analysis of this model. In [1] lateral inhibition type coupling is studied. Furthermore, when f is a Heaviside step function, [1] also treats the behavior of time dependent periodic solutions as well as traveling waves for systems of equations. Existence and uniqueness of monotone traveling waves was investigated in [6]. An another prove of existence of monotone travelling waves is given in [4]. In [8], the existence of a non-homogeneous stationary solution referred to as "bump" is proved. One link between the integral equations given by (1.3) and ODEs is given in [9]. In [10], the existence of a non-homogeneous stationary solution of the type "double-bump" is proved. In [15] is proved that solutions as "bump" can exist and be linearly stable in neural population models without recurrent excitation. In [16], assuming that f is Lipschitz and bounded, is proved the existence of global attractor, for the flow generated by (1.1), in weighted space.

We consider here the unique additional condition on f which will is used as hypothesis in our results when necessary.

(H1) The function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz, that is, there exists $k_1 > 0$ such that

$$|f(x) - f(y)| \leq k_1 |x - y|, \quad \forall\, x, y \in \mathbb{R}, \tag{1.4}$$

From (1.4), follows that there exists constant $k_2 \geq 0$ such that

$$|f(x)| \leq k_1 |x| + k_2. \tag{1.5}$$

This paper is organized as follows. In Section 2 we prove that, under hypothesis (H1), in the phase space $L^2(\mathbb{R}, \rho) = \{u \in L^1_{\text{loc}}(\mathbb{R}) : \int u^2 \rho(x) dx < \infty\}$, the Cauchy problem for (1.1) is well posed with globally defined solutions. In Section 3 we prove that the system is dissipative in the sense of [7], that is, it has a global compact attractor. Our proof is stronger of what the given one in [16] because we do not use no hypothesis of limitation on f. In our proof, we only use the Sobolev's compact embedding $H^1([-l, l]) \hookrightarrow L^2([-l, l])$ and some ideias from [12], where the equation $u_t = -u + \tanh(\beta J * u + h)$ is considered (see also [2], [11], [13] and [14] for related work). In Section 4, we prove an uniform estimate for the attractor and finally, in Section 5, after obtaining some estimates for the flow of (1.1), we prove the upper semicontinuity property of the attractors with respect to function J present in (1.1).

2. Well-posedness

In this section we consider the flow generated by (1.1) in the space $L^2(\mathbb{R}, \rho)$ defined by

$$L^2(\mathbb{R}, \rho) = \big\{u \in L^1_{\text{loc}}(\mathbb{R}) : \int_{\mathbb{R}} u^2(x) \rho(x) dx < +\infty\big\},$$

with norm $\|u\|_{L^2}(\mathbb{R}, \rho) = \big(\int_{\mathbb{R}} u^2(x) \rho(x) dx\big)^{1/2}$. Here ρ is an integrable positive even function with $\int_{\mathbb{R}} \rho(x) dx = 1$. Note that in this space the constant function equal to 1 has norm 1. The corresponding higher-order Sobolev space $H^k(\mathbb{R}, \rho)$ is the space of functions $u \in L^2(\mathbb{R}, \rho)$ whose distributional derivatives up to order k are also in $L^2(\mathbb{R}, \rho)$, with norm

$$\|u\|_{H^k(\mathbb{R},\rho)} = \Big(\sum_{i=1}^{k} \| \frac{\partial^i u}{\partial x^i} \|^2_{L^2(\mathbb{R},\rho)} \Big)^{1/2}.$$

To obtain some convenient estimates we will need the following additional hypothesis on the function ρ.

(H2) There exists constant $K > 0$ such that

$$\sup\{\rho(x) : x \in \mathbb{R},\ y-1 \le x \le y+1\} \le K\rho(y), \quad \forall\, y \in \mathbb{R}.$$

Remark 2.1. When $\rho(x) = \frac{1}{\pi}(1+x^2)^{-1}$, the hypothesis (H2), is verified with $K = 3$, (see, [12]).

Lemma 2.2. *Suppose that* (H2) *holds. Then*

$$\|J * u\|_{L^2(\mathbb{R},\rho)} \le \sqrt{K} \|J\|_{L^1} \|u\|_{L^2(\mathbb{R},\rho)}.$$

Proof. Since J is bounded and compact supported, $(J * u)(x)$ is well defined for $u \in L^1_{\rm loc}(\mathbb{R})$. Thus, using (1.2) and Holder's inequality (see [3]), we obtain

$$\begin{aligned}
\|J * u\|^2_{L^2(\mathbb{R},\rho)} &= \int_{\mathbb{R}} |(J*u)(x)|^2 \rho(x)dx \\
&\le \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} (J(x-y))^{1/2} (J(x-y))^{1/2} |u(y)| dy \Big)^2 \rho(x) dx \\
&\le \int_{\mathbb{R}} \Big(\Big[\int_{\mathbb{R}} J(x-y)dy \Big]^{1/2} \Big[\int_{\mathbb{R}} J(x-y)|u(y)|^2 dy \Big]^{1/2} \Big)^2 \rho(x) dx \\
&= \|J\|_{L^1} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} J(x-y) |u(y)|^2 dy \Big) \rho(x) dx \\
&= \|J\|_{L^1} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} J(x-y) \rho(x) dx \Big) |u(y)|^2 dy \\
&\le \|J\|_{L^1} \int_{\mathbb{R}} \Big(\int_{x=y-1}^{x=y+1} J(x)\rho(x)dx \Big) |u(y)|^2 dy \\
&\le \|J\|_{L^1} \int_{\mathbb{R}} \Big(K\rho(y) \int_{x=y-1}^{x=y+1} J(x)dx \Big) |u(y)|^2 dy \\
&\le K \|J\|^2_{L^1} \int_{\mathbb{R}} |u(y)|^2 \rho(y) dy \\
&= K \|J\|^2_{L^1} \|u\|^2_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

It conclude the result. $\square$

Remark 2.3. Under hypothesis (H1), for each $u \in L^2(\mathbb{R}, \rho)$, we have

$$|J * (f \circ u)(x)| \le k_1 (J * |u|)(x) + k_2 \|J\|_{L^1}. \tag{2.1}$$

In fact, using (1.5) we obtain

$$\begin{aligned}
|J * (f \circ u)(x)| &\le \int_{\mathbb{R}} J(x-y)[k_1|u(y)| + k_2] dy \\
&= k_1 \int_{\mathbb{R}} J(x-y)|u(y)|dy + k_2 \int_{\mathbb{R}} J(x-y) dy \\
&= k_1 J * |u|(x) + k_2 \|J\|_{L^1}.
\end{aligned}$$

Proposition 2.4. *Suppose that the hypotheses* (H1) *and* (H2) *hold. Then the function*

$$F(u) = -u + J * (f \circ u) + h$$

is globally Lipschitz in $L^2(\mathbb{R}, \rho)$.

Proof. From triangle inequality and Lemma 2.2, it follows that

$$\begin{aligned} \|F(u) - F(v)\|_{L^2(\mathbb{R},\rho)} &\le \|v - u\|_{L^2(\mathbb{R},\rho)} + \|J * (f \circ u) - J * (f \circ v)\|_{L^2(\mathbb{R},\rho)} \\ &\le \|v - u\|_{L^2(\mathbb{R},\rho)} + \sqrt{K}\|J\|_{L^1}\|(f \circ u) - (f \circ v)\|_{L^2(\mathbb{R},\rho)}. \end{aligned}$$

Using (1.4), we have

$$\|(f \circ u) - (f \circ v)\|^2_{L^2(\mathbb{R},\rho)} \le \int_{\mathbb{R}} k_1^2 |u(x) - v(x)|^2 \rho(x) dx = k_1^2 \|u - v\|^2_{L^2(\mathbb{R},\rho)}.$$

Then

$$\|F(u) - F(v)\|_{L^2(\mathbb{R},\rho)} \le (1 + \sqrt{K}\|J\|_{L^1} k_1)\|u - v\|_{L^2(\mathbb{R},\rho)}.$$

Therefore, F is globally Lipschitz in $L^2(\mathbb{R}, \rho)$. □

Remark 2.5. Since the right-hand side of (1.1) defines a Lipschitz map in $L^2(\mathbb{R}, \rho)$, from standard results of ODEs in Banach spaces, follows that the Cauchy problem for (1.1) is well posed in $L^2(\mathbb{R}, \rho)$ with globally defined solutions, (see [3] and [5]).

3. Existence of a global attractor

In this section, we prove the existence of a global maximal invariant compact set $\mathcal{A} \subset L^2(\mathbb{R}, \rho)$ for the flow of (1.1), which attracts each bounded set of $L^2(\mathbb{R}, \rho)$ (the global attractor, see [7] and [17]).

To obtain the existence of a global attractor we will need the following additional hypothesis on the function J.

(H3) The function J satisfies $k_1\sqrt{K}\|J\|_{L^1} < 1$.

Remark 3.1. In the particular case that $\rho(x) = \frac{1}{\pi}(1 + x^2)^{-1}$ and $f = \tanh$, whenever $\|J\|_{L^1} < \frac{1}{\sqrt{3}}$, the hypothesis (H3) is satisfied.

In what follows, we denote by $S(t)$ the flow generated by (1.1).

We recall that a set $\mathcal{B} \subset L^2(\mathbb{R}, \rho)$ is an absorbing set for the flow $S(t)$ in $L^2(\mathbb{R}, \rho)$ if, for any bounded set $B \subset L^2(\mathbb{R}, \rho)$, there is a $t_1 > 0$ such that $S(t)B \subset \mathcal{B}$ for any $t \ge t_1$, (see [17]).

Lemma 3.2. *Assume that* (H1), (H2), (H3) *hold. Let*

$$R = \frac{2(k_2\|J\|_{L^1} + h)}{1 - k_1\sqrt{K}\|J\|_{L^1}}.$$

Then the ball with center at the origin of $L^2(\mathbb{R}, \rho)$ *and radius* R *is an absorbing set for the flow* $S(t)$.

Proof. Let $u(x, t)$ be the solution of (1.1), then

$$\begin{aligned} &\frac{d}{dt} \int_{\mathbb{R}} |u(x, t)|^2 \rho(x) dx \\ &= \int_{\mathbb{R}} 2u(x, t) \frac{d}{dt} u(x, t) \rho(x) dx \\ &= -2 \int_{\mathbb{R}} u^2(x, t) \rho(x) dx + 2 \int_{\mathbb{R}} u(x, t)[J * (f \circ u)(x, t) + h] \rho(x) dx. \end{aligned}$$

Using Holder inequalit's, (2.1) and Lemma 2.2, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} u(x,t)[J*(f\circ u)(x,t)+h]\rho(x)dx\\
&\le \Big(\int_{\mathbb{R}} u(x,t)^2\rho(x)dx\Big)^{1/2}\Big(\int_{\mathbb{R}} |J*(f\circ u)(x,t)+h|^2\rho(x)dx\Big)^{1/2}\\
&\le \|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}\Big(\int_{\mathbb{R}}[k_1 J*|u(x,t)|+k_2\|J\|_{L^1}+h]^2\rho(x)dx\Big)^{1/2}\\
&= \|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}\|k_1 J*|u(\cdot,t)|+k_2\|J\|_{L^1}+h\|_{L^2(\mathbb{R},\rho)}\\
&\le k_1\sqrt{K}\|J\|_{L^1}\|u(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)}+(k_2\|J\|_{L^1}+h)\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

Hence

$$\frac{d}{dt}\int_{\mathbb{R}}|u(x,t)|^2\rho(x)dx\le 2\|u(\cdot,t)\|^2_{L^2(\mathbb{R})}\big[-1+k_1\sqrt{K}\|J\|_{L^1}+\frac{(k_2\|J\|_{L^1}+h)}{\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}}\big].$$

Since $k_1\sqrt{K}\|J\|_{L^1}<1$, let $\varepsilon=1-k_1\sqrt{K}\|J\|_{L^1}>0$. Then, while $\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}>\frac{2(k_2\|J\|_{L^1}+h)}{\varepsilon}$, we have

$$\frac{d}{dt}\|u(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)}\le 2\|u(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)}(-\varepsilon+\frac{\varepsilon}{2})=-\varepsilon\|u(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)}.$$

Therefore,

$$\begin{aligned}
\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)}&\le e^{-\varepsilon t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)}\\
&=e^{-(1-k_1\sqrt{K}\|J\|_{L^1})t}\|u(\cdot,0)\|_{L^2(\mathbb{R},\rho)}.
\end{aligned}$$

This concludes the proof. □

Remark 3.3. From Lemma 3.2, follows that the ball of center in the origin and radius R is invariant set under flow $S(t)$.

Lemma 3.4. *Besides the assumptions from Lemma 3.2 we also suppose that the functions J and ρ satisfy the relation $J(x)\le C\rho(x)$, $\forall x\in[-1,1]$, for some constant $C>0$. Let $R=\frac{2(k_2\|J\|_{L^1}+h)}{1-k_1\sqrt{K}\|J\|_{L^1}}$ be, then, for any $\eta>0$, there exists t_η such that $S(t_\eta)B(0,R)$ has a finite covering by balls of $L^2(\mathbb{R},\rho)$ with radius smaller than η.*

Proof. From Lemma 3.2, it follows that $B(0,R)$ is invariant. Now, the solutions of (1.1) with initial condition $u_0\in B(0,R)$ is given, by the variation of constant formula, by

$$u(x,t)=e^{-t}u_0(x)+\int_0^t e^{-(t-s)}[(J*(f\circ u))(x,s)+h]ds.$$

Write

$$v(x,t)=e^{-t}u_0(x),\quad w(x,t)=\int_0^t e^{-(t-s)}[(J*(f\circ u))(x,s)+h]ds.$$

Let $\eta>0$ given. We may find $t(\eta)$ such that if $t\ge t(\eta)$ then $\|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)}\le\frac{\eta}{2}$. In fact,

$$\|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)}=e^{-t}\|u_0\|_{L^2(\mathbb{R},\rho)},$$

then for $t>\ln(\frac{2\|u_0\|_{L^2(\mathbb{R},\rho)}}{\eta})$, we have $\|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)}<\frac{\eta}{2}$ for any $u_0\in B(0,R)$.

Now, from (H1) it follows that

$$
\begin{aligned}
|J*(f\circ u)(x,s)| &\le k_1\int J(x-y)|u(y,s)|dy + k_2\int J(x-y)dy\\
&= k_1\int J(y-x)|u(y,s)|dy + k_2\|J\|_{L^1}\\
&= k_1\int_{y=x-1}^{y=x+1} J(y)|u(y,s)|dy + k_2\|J\|_{L^1}.
\end{aligned}
$$

Since that ρ is a positive function, J is supported in the interval $[-1,1]$ and $J(x)\le C\rho(x)$, $\forall\, x\in[-1,1]$, we obtain

$$
\begin{aligned}
|J*(f\circ u)(x,s)| &\le Ck_1\int_{y=x-1}^{y=x+1}\rho(y)|u(y,s)|dy + k_2\|J\|_{L^1}\\
&\le Ck_1\int \rho(y)|u(y,s)|dy + k_2\|J\|_{L^1}\\
&= Ck_1\int \rho^{1/2}(y)|u(y,s)|\rho^{1/2}(y)dy + k_2\|J\|_{L^1}\\
&\le Ck_1\Big(\int \rho(y)|u(y,s)|^2dy\Big)^{1/2}\Big(\int\rho(y)dy\Big)^{1/2} + k_2\|J\|_{L^1}.
\end{aligned}
$$

Then

$$
|J*(f\circ u)(x,s)| \le Ck_1\|u(\cdot,s)\|_{L^2(\mathbb{R},\rho)} + k_2\|J\|_{L^1}. \tag{3.1}
$$

Thus, using (3.1) and that $\|u(\cdot,s)\|_{L^2(\mathbb{R},\rho)}\le R$, results

$$
\begin{aligned}
|w(x,t)| &\le \int_0^t e^{-(t-s)}[|J*(f\circ u)(x,s)|+h]ds\\
&\le \int_0^t e^{-(t-s)}(Ck_1R + k_2\|J\|_{L^1}+h).
\end{aligned}
$$

Hence

$$
|w(x,t)| \le Ck_1R + k_2\|J\|_{L^1} + h. \tag{3.2}
$$

Now, since

$$
\begin{aligned}
J'*|u|(x,s) &= \int_{x-1}^{x+1} J'(x-y)|u(y,s)|ds\\
&\le \Big(\int_{x-1}^{x+1}|J'(x-y)|^2dy\Big)^{1/2}\Big(\int_{x-1}^{x+1}|u(y,s)|^2dy\Big)^{1/2}\\
&\le \|J'\|_{L^2}\Big(\int_{x-1}^{x+1}|u(y,s)|^2dy\Big)^{1/2},
\end{aligned}
$$

if $x\in[-l,l]$, we obtain

$$
\begin{aligned}
J'*|u|(x,s) &\le \|J'\|_{L^2}\Big(\int_{l-1}^{l+1}|u(y,s)|^2dy\Big)^{1/2}\\
&\le \|J'\|_{L^2}\Big(\int_{\mathbb{R}}|u(y,s)|^2\chi_{l+1}\rho(y)\frac{1}{\rho_l}dy\Big)^{1/2}
\end{aligned}
$$

where χ_l is the characteristic function of the interval $[-l, l]$ and $\rho_l = \inf\{|\rho(x)| : x \in [-l-1, l+1]\}$. Then if $u_0 \in B(0,R)$, then

$$J' * |u|(x,s) \le \frac{R\|J'\|_{L^2}}{\sqrt{\rho_l}}. \tag{3.3}$$

Furthermore, differentiating w with respect to x, for $t \ge 0$, we have

$$\frac{\partial w}{\partial x}(x,t) = \int_0^t e^{-(t-s)} \left(J' * (f \circ u)\right)(x,s) ds.$$

Thus

$$\begin{aligned} \Big|\frac{\partial w(x,t)}{\partial x}\Big| &\le \int_0^t e^{-(t-s)} |J' * (f \circ u)(x,s)|_{L^2(\mathbb{R},\rho)} ds \\ &\le \int_0^t e^{-(t-s)} [k_1 J' * |u(x,s)| + k_2 \|J'\|_{L^1}] ds. \end{aligned}$$

But, proceeding as in the proof of (2.1), we obtain

$$|J' * (f \circ u)(x,s)| \le k_1 (J'|u|)(x,s) + k_2 \|J'\|_{L^1}.$$

Hence, using (3.3), results

$$\Big|\frac{\partial w(x,t)}{\partial x}\Big| \le k_1 \frac{R}{\sqrt{\rho_l}} \|J'\|_{L^2} + k_2 \|J'\|_{L^2}. \tag{3.4}$$

From (3.2) and (3.4) follows that the restriction of $w(\cdot,t)$ to the interval $[-l, l]$ is bounded in $H^1([-l, l])$ (by a constant independent of $u_0 \in B(0,R)$ and of t), and therefore the set $\{\chi_l w(\cdot,t)\}$ with $w(\cdot,0) \in B(0,R)$ is relatively compact subset of $L^2(\mathbb{R},\rho)$ for any $t > 0$ and, hence, it can be covered by a finite number of balls with radius smaller than $\frac{\eta}{4}$.

Now, from Lemma 3.2, follows that, for all $t \ge 0$ and any $u_0 \in B(0,R)$,

$$\|w(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \le 2R. \tag{3.5}$$

Then, let $l > 0$ be such that

$$2R(Ck_1R + k_2\|J\|_{L^1} + h)\Big(\int_{\mathbb{R}} (1 - \chi_l(x))^4 \rho(x) dx\Big)^{1/2} \le \frac{\eta}{4}. \tag{3.6}$$

Hence, using (3.2), (3.5) and (3.6), we obtain

$$\begin{aligned} &\|(1-\chi_l)w(\cdot,t)\|^2_{L^2(\mathbb{R},\rho)} \\ &= \int_{\mathbb{R}} \Big[w(x,t)\rho(x)^{1/2}(1-\chi_l)^2(x) w(x,t)\rho(x)^{1/2}\Big] dx \\ &\le \Big(\int_{\mathbb{R}} |w(x,t)|^2 \rho(x) dx\Big)^{1/2} \Big(\int_{\mathbb{R}} (1-\chi_l)^4(x) |w(x,t)|^2 \rho(x) dx\Big)^{1/2} \\ &\le \|w(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \Big((Ck_1R + k_2\|J\|_{L^1} + h)^2 \int_{\mathbb{R}} (1-\chi_l)^4(x)\rho(x) dx\Big)^{1/2} \\ &\le 2R(Ck_1R + k_2\|J\|_{L^1} + h) \left(\int_{\mathbb{R}} (1-\chi_l)^4(x)\rho(x) dx\right)^{1/2} \le \frac{\eta}{4}. \end{aligned}$$

Therefore, since

$$u(\cdot,t) = v(\cdot,t) + \chi_l w(\cdot,t) + (1-\chi_l) w(\cdot,t),$$

it follows that $S(t_\eta)B(0,R)$ has a finite covering by balls of $L^2(\mathbb{R},\rho)$ with radius smaller than η because

$$\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} = \|v(\cdot,t)\|_{L^2(\mathbb{R},\rho)} + \|\chi_l w(\cdot,t)\|_{L^2(\mathbb{R},\rho)} + \|(1-\chi_l)w(\cdot,t)\|_{L^2(\mathbb{R},\rho)}.$$

□

We denote by $\omega(D)$ the ω-limit of a set D.

Theorem 3.5. *Assume the hypotheses in Lemma 3.4. Then $\mathcal{A} = \omega(B(0,R))$, is a global attractor for the flow $S(t)$ generated by (1.1) in $L^2(\mathbb{R},\rho)$ which is contained in the ball of radius R.*

Proof. From Lemma 3.2, it follows that $\mathcal{A}$ is contained in the ball of radius R and center in the origin of $L^2(\mathbb{R},\rho)$. Now, being $\mathcal{A}$ invariant by flow $S(t)$, it follows that $\mathcal{A} \subset S(t)B(0,R)$, for any $t \geq 0$ and then, from Lemma 3.4, it results that the measure of noncompactness of $\mathcal{A}$ is zero. Hence $\mathcal{A}$ is relatively compact and, since $\mathcal{A}$ is closed, follows that $\mathcal{A}$ is also compact. Finally, if D is bounded set in $L^2(\mathbb{R},\rho)$ then $S(t_0)D \subset B(0,R)$ for t_0 big enough and, therefore, $\omega(D) \subset \omega(B(0,R))$. □

4. Boundedness results

In this section we prove uniform estimates for the attractor whose existence was proved in the Theorem 3.5.

Theorem 4.1. *Assume the same hypotheses from Theorem 3.5, and $J \in C^r(\mathbb{R})$, for some integer $r > 0$. Then the attractor $\mathcal{A}$ is bounded in $C^r_\rho(\mathbb{R})$.*

Proof. Let $u(x,t)$ be a solution of (1.1) in $\mathcal{A}$. Then, by the variation of constants formula

$$u(x,t) = e^{-(t-t_0)}u(x,t_0) + \int_{t_0}^t e^{-(t-s)}[J*(f\circ u)(x,s)+h]ds.$$

From Theorem 3.5 follows that $\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq R$, where $R = \frac{2(k_2\|J\|_{L^1}+h)}{1-k_1\sqrt{K}\|J\|_{L^1}}$. Since $\|u(\cdot,t_0)\|_{L^2(\mathbb{R},\rho)} \leq R$, letting $t_0 \to -\infty$, we obtain

$$u(x,t) = \int_{-\infty}^t e^{-(t-s)}[J*(f\circ u)(x,s)+h]ds, \tag{4.1}$$

where the equality in (4.1) is in the sense of $L^2(\mathbb{R},\rho)$.

Using that $J \in C^1(\mathbb{R})$ follows, from (4.1), that $u(x,t)$ is differentiable with respect to x and

$$\frac{\partial u(x,t)}{\partial x} = \int_{-\infty}^t e^{-(t-s)}J'*(f\circ u)(x,s)ds. \tag{4.2}$$

Now, using that $J' \in C^1(\mathbb{R})$ follows, from (4.2), that $\frac{\partial u(x,t)}{\partial x}$ is differentiable with respect to x and

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \int_{-\infty}^t e^{-(t-s)}J''*(f\circ u)(x,s)ds.$$

Following this idea, using that $J^{(r-1)} \in C^1(\mathbb{R})$, we have that $\frac{\partial^{r-1}u(x,t)}{\partial x^{r-1}}$ is differentiable with respect to x and

$$\frac{\partial^r u(x,t)}{\partial x^r} = \int_{-\infty}^t e^{-(t-s)}J^r*(f\circ u)(x,s)ds. \tag{4.3}$$

Now, since J is bounded and compact supported, it also follows that $J^{(r)}$ is bounded and compact supported. Thus $J^{(r)} * v$ is well defined for $v \in L^1_{\rm loc}(\mathbb{R})$. Hence, proceeding as in the Lemma 2.2, obtain

$$\|J^{(r)} * v\|_{L^2(\mathbb{R},\rho)} \leq \sqrt{K}\|J^{(r)}\|_{L^1}\|v\|_{L^2(\mathbb{R},\rho)}.$$

Thus,

$$\|J^{(r)} * (f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq \sqrt{K}\|J^{(r)}\|_{L^1}\|(f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)}.$$

Using (1.5), we have

$$\|f(u(\cdot,s))\|_{L^2(\mathbb{R},\rho)} \leq k_1\|u(\cdot,s)\|_{L^2(\mathbb{R},\rho)} + k_2. \tag{4.4}$$

Since the ball $B(0,R)$ is invariant, $\|u(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq R$, from (4.4) results

$$\|(f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq k_1R + k_2.$$

Hence

$$\|J^{(r)} * (f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)} \leq \sqrt{K}\|J^{(r)}\|_{L^1}(k_1R + k_2). \tag{4.5}$$

Therefore, from (4.3) and (4.5), follows that

$$\begin{aligned}
\|\frac{\partial^r u(x,t)}{\partial x^r}\|_{L^2(\mathbb{R},\rho)} &\leq \int_{-\infty}^t e^{-(t-s)}\|J^{(r)} * (f\circ u)(\cdot,t)\|_{L^2(\mathbb{R},\rho)}ds \\
&\leq \sqrt{K}\|J^{(r)}\|_{L^1}(k_1R+k_2)\int_{-\infty}^t e^{-(t-s)}ds \\
&= \sqrt{K}\|J^{(r)}\|_{L^1}(k_1R+k_2).
\end{aligned}$$

Therefore, we can obtain boundedness for the derivatives of u of any order, in terms only of J and of the derivatives of J, concluding the proof. $\square$

Theorem 4.2. *Assume the same hypotheses from Theorem 3.5. Then the attractor $\mathcal{A}$ belongs to the ball $\|\cdot\|_\infty \leq a$, where $a = Ck_1R + k_2\|J\|_{L^1} + h$.*

Proof. Let $u(x,t)$ be a solution of (1.1) in $\mathcal{A}$. Then as we see in (4.1)

$$u(x,t) = \int_{-\infty}^t e^{-(t-s)}[J * (f\circ u)(x,s) + h]ds,$$

where the equality above is in the sense of $L^2(\mathbb{R},\rho)$. Thus, using (3.1), obtain

$$\begin{aligned}
|u(x,t)| &\leq \int_{-\infty}^t e^{-(t-s)}[|J * (f\circ u)(x,s)| + h]ds \\
&\leq \int_{-\infty}^t (Ck_1R + k_2\|J\|_{L^1} + h)e^{-(t-s)}ds \\
&= \int_{-\infty}^t ae^{-(t-s)}ds = a.
\end{aligned}$$

$\square$

5. Upper semicontinuity of attractors with respect to J

A natural question to examine is the dependence of this attractors on the function J present in (1.1). We denote by $\mathcal{A}_J$ the global attractor whose existence was proved in the Theorem 3.5

Let us recall that a family of subsets $\{\mathcal{A}_J\}$, is upper semicontinuous at J_0 if

$$\operatorname{dist}(\mathcal{A}_J, \mathcal{A}_{J_0}) \to 0, \quad \text{as } J \to J_0,$$

where

$$\operatorname{dist}(\mathcal{A}_J, \mathcal{A}_{J_0}) = \sup_{x\in\mathcal{A}_J} \operatorname{dist}(x, \mathcal{A}_{J_0}) = \sup_{x\in\mathcal{A}_J} \inf_{y\in\mathcal{A}_{J_0}} \|x-y\|_{L^2(\mathbb{R},\rho)}.$$

In this section, we prove that the family of attractors is upper semicontinuous, in $L^2(\mathbb{R},\rho)$, with respect to function J at J_0 with $J \in C^1(\mathbb{R})$ non negative even and supported in the interval $[-1,1]$ and $J(x) \le C\rho(x)$, $\forall\, x \in [-1,1]$, where C is the constant given in the Lemma 3.4.

Lemma 5.1. *Assume* (H1), (H2), (H3) *hold. Then the flow $S_J(t)$ is continuous with respect to variations of J, in the L^1-norm, at J_0, uniformly for $t \in [0,b]$ with $b < \infty$ and u in bounded sets.*

Proof. As shown above the solutions of (1.1) satisfy the variations of constants formula,

$$S_J(t)u = e^{-t}u + \int_0^t e^{-(t-s)}[J*(f\circ S_J(s)u + h]ds.$$

Let $J_0 \in C^1(\mathbb{R})$ be a non negative even function supported in the interval $[-1,1]$, $b > 0$ and D a bounded set in $L^2(\mathbb{R},\rho)$, for example the ball $B(0,R)$ (Although R depends on J, it can be uniformly chosen in a neighborhood of J_0) . Given $\varepsilon > 0$, we want to find $\delta > 0$ such that $\|J - J_0\|_{L^1} < \delta$ implies

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} < \varepsilon,$$

for $t \in [0,b]$ and $u \in D$. Note that

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \le \int_0^t e^{-(t-s)}\|J*(f\circ S_J(s)u) - J_0*(f\circ S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}ds.$$

Subtracting and summing the term $J_0*(f\circ S_J(s)u)$ and using Lemma 2.2, for any $t > 0$, we obtain

$$\begin{aligned}
\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} &\le \int_0^t e^{-(t-s)}[\|(J-J_0)*(f\circ S_J(s)u)\|_{L^2(\mathbb{R},\rho)} \\
&\quad + \|J_0*[f\circ S_J(s)u - f\circ S_{J_0}(s)u]\|_{L^2(\mathbb{R},\rho)}]ds \\
&\le \int_0^t e^{-(t-s)}[\sqrt{K}\|J-J_0\|_{L^1}\|f\circ S_J(s)u\|_{L^2(\mathbb{R},\rho)} \\
&\quad + \sqrt{K}\|J_0\|_{L^1}\|f\circ S_J(s)u - f\circ S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}]ds.
\end{aligned}$$

Using (4.4), we obtain

$$\|f\circ S_J(s)u\|_{L^2(\mathbb{R},\rho)} \le k_1\|u(\cdot,s)\|_{L^2(\mathbb{R},\rho)} + k_2 \le k_1R + k_2$$

and, using (H1), we obtain

$$\|f\circ S_J(s)u - f\circ S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)} \le k_1\|S_J(s)u - S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}.$$

Therefore,

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \leq (k_1R + k_2)\sqrt{K}\|J - J_0\|_{L^1} + \int_0^t e^{-(t-s)}\sqrt{K}\|J_0\|_{L^1}k_1\|S_J(s)u - S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}.$$

Hence

$$e^t\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \leq (k_1R + k_2)\sqrt{K}\|J - J_0\|_{L^1}e^t + \int_0^t e^s\sqrt{K}\|J_0\|_{L^1}k_1\|S_J(s)u - S_{J_0}(s)u\|_{L^2(\mathbb{R},\rho)}.$$

Therefore, by Gronwall's Lemma, it follows that

$$\|S_J(t)u - S_{J_0}(t)u\|_{L^2(\mathbb{R},\rho)} \leq (k_1R + k_2)\sqrt{K}\|J - J_0\|_{L^1}e^{(\sqrt{K}\|J_0\|_{L^1}k_1)t}.$$

From this, the results follows immediately. □

Theorem 5.2. *Assume the same hypotheses as in Lemma 5.1. Then the family of attractors $\mathcal{A}_J$ is upper semicontinuous with respect to J at J_0.*

Proof. From hypotheses of the theorem, it follows that, for every $J \in C^1(\mathbb{R})$, sufficiently close to J_0 in the L^1-norm, non negative even supported in $[-1, 1]$ and satisfying $J(x) \leq C\rho(x)$, for all $x \in [-1, 1]$, the attractor, $\mathcal{A}_J$, given by Theorem 3.5 is in the closed ball $B[0, R]$ in $L^2(\mathbb{R}, \rho)$. Therefore

$$\cup_J\mathcal{A}_J \subset B[0, R].$$

Since $\mathcal{A}_{J_0}$ is global attractor and $B[0, R]$ is a bounded set then, for every $\varepsilon > 0$, there exists $t^* > 0$ such that $S_{J_0}(t)B[0, R] \subset \mathcal{A}_{J_0}^{\varepsilon/2}$, for all $t \geq t^*$, where $\mathcal{A}_{J_0}^{\frac{\varepsilon}{2}}$ is $\frac{\varepsilon}{2}$-neighborhood of $\mathcal{A}_{J_0}$.

From Lemma 5.1, it follows that $S_J(t)$ is continuous at J_0, uniformly for u in a bounded set and t in compacts. Thus, there exists $\delta > 0$ such that

$$\|J - J_0\|_{L^1} < \delta \Rightarrow \|S_J(t^*)u - S_{J_0}(t^*)u\|_{L^2(\mathbb{R},\rho)} < \frac{\varepsilon}{2}, \quad \forall\, u \in B[0, R].$$

We will show that if $\|J - J_0\| < \delta$ then $\mathcal{A}_J \subset \mathcal{A}_{J_0}^{\varepsilon}$. In fact, let $u \in \mathcal{A}_J$. Since $\mathcal{A}_J$ is invariant, $v = S_J(-t^*)u \in \mathcal{A}_J \subset B[0, R]$. Therefore,

$$S_{J_0}(t^*)v \in \mathcal{A}_{J_0}^{\varepsilon/2}, \tag{5.1}$$

$$\|S_J(t^*)v - S_{J_0}(t^*)v\|_{L^2(\mathbb{R},\rho)} < \frac{\varepsilon}{2}. \tag{5.2}$$

From (5.1) and (5.2), it follows that

$$u = S_J(t^*)S_J(-t^*)u = S_J(t^*)v \in \mathcal{A}_{J_0}^{\varepsilon}$$

and the upper semicontinuity of $\mathcal{A}_J$ follows. □

Remark 5.3. Similar results can be obtained for the flow of (1.1) in

$$C_\rho(\mathbb{R}) \equiv \{f : \mathbb{R} \to \mathbb{R} \text{ continuous with the norm } \|\cdot\|_\rho\},$$

where

$$\|u\|_\rho = \sup_{x\in\mathbb{R}}\{|u(x)|\rho(x)\} < \infty,$$

being ρ a positive continuous function on $\mathbb{R}$.

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