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MULTIPLE SIGN-CHANGING SOLUTIONS FOR SUB-LINEAR IMPULSIVE THREE-POINT BOUNDARY-VALUE PROBLEMS

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Abstract. In this article, we study the existence of sign-changing solutions for some second-order impulsive boundary-value problem with a sub-linear condition at infinity. To obtain the results we use the Leray-Schauder degree and the upper and lower solution method.

1. Introduction

This article concerns the impulsive differential equation

$$\begin{gathered} y''(t) + f(t, y(t), y'(t)) = 0, \quad t \in J,\ t \neq t_k, \\ \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k)), \quad k = 1, 2, \dots, m, \\ y(0) = 0, \quad y(1) = \alpha y(\eta), \end{gathered} \tag{1.1}$$

where $J = [0, 1]$, $f \in C[J \times \mathbb{R}^2, \mathbb{R}^1]$, $\bar{I}_k \in C[\mathbb{R}^1, \mathbb{R}^1]$, $k = 1, 2, \dots, m$, $0 \le \alpha < 1$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < \eta < t_{m+1} = 1$.

The theory of impulsive differential equations describes processes which experience a sudden change of their state at certain moments. Processes with such a character arise naturally and often, for example, phenomena studied in physics, chemical technology, population dynamics, biotechnology and economics. For an introduction of the basic theory of impulsive differential equation, we refer the reader to [5].

In recent years, there have been many papers studying the existence of sign-changing solutions to some boundary-value problems, see [2, 6, 8, 9, 10, 15] and the references therein. However, to the authors best knowledge, there are few papers that considered the sign-changing solutions for the impulsive boundary-value problems. Usually, to show the existence of sign-changing solutions one employs the variational method and the Leray-Schauder degree method. However, a suitable variational structure for impulsive boundary-value problems is yet unknown. In [7, 12, 13], authors computed the algebraic multiplicities of the linear problems corresponding to the discussed boundary-value problems, but we know that the

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algebraic multiplicities of impulsive boundary-value problem are not easy to compute. Thus, there are many difficulties in studying the sign-changing solutions for the impulsive boundary-value problem (1.1) by the method mentioned above.

In this paper, we consider the sign-changing solutions for the impulsive three-point boundary-value problem (1.1) by the Leray-Schauder degree and strict upper and lower solution method. We assume a sub-linear condition at infinity, and we construct another pair of strict upper and lower solutions by conditions of f and $\bar{I}_k$. We will show a result of at least four sign-changing solutions, two positive solutions and two negative solutions for (1.1). Moreover, we will give a description of the exact locations of them.

2. Preliminaries

Let $J' = J \setminus \{t_1, t_2, \dots, t_m\}$, $PC^1[J, \mathbb{R}^1] = \{x : J \to \mathbb{R}^1,\ x'$ is continuous at $t \neq t_k$, x' is left continuous at $t = t_k$, $x'(t_k^+)$ exists$\}$. For $x \in PC^1[J, \mathbb{R}^1]$, let

$$\|x\|_{PC^1} = \max\{\|x\|, \|x'\|\},$$

where $\|x\| = \sup_{t\in J} |x(t)|$ and $\|x'\| = \sup_{t\in J} |x'(t)|$. Then $PC^1[J, \mathbb{R}^1]$ is a real Banach space with norm $\|\cdot\|_{PC^1}$. Let $x, y \in C[J, \mathbb{R}^1]$. Define $\prec$ as follows

$$x \prec y \text{ if } x(t) < y(t) \text{ for all } t \in J.$$

Definition 2.1. A function $u \in PC^1[J, \mathbb{R}^1] \cap C^2[J', \mathbb{R}^1]$ is called a strict lower solution of (1.1), if

$$\begin{gathered} u''(t) + f(t, u(t), u'(t)) > 0, \quad t \neq t_k, \\ \Delta u'|_{t=t_k} > \bar{I}_k(u(t_k)), \quad k = 1, 2, \dots, m, \\ u(0) < 0, \quad u(1) - \alpha u(\eta) < 0. \end{gathered} \tag{2.1}$$

A function $v \in PC^1[J, \mathbb{R}^1] \cap C^2[J', \mathbb{R}^1]$ is called a strict upper solution of (1.1), if

$$\begin{gathered} v''(t) + f(t, v(t), v'(t)) < 0, \quad t \neq t_k, \\ \Delta v'|_{t=t_k} < \bar{I}_k(v(t_k)), \quad k = 1, 2, \dots, m, \\ v(0) > 0, \quad v(1) - \alpha v(\eta) > 0. \end{gathered} \tag{2.2}$$

Let us introduce the following constants:

$$\begin{gathered} \beta = \limsup_{|x|+|y|\to\infty} \max_{t\in J} \frac{|f(t, x, y)|}{|x| + |y|}, \\ \bar{\beta}_k = \limsup_{|x|\to\infty} \frac{|\bar{I}_k(x)|}{|x|}, \quad k = 1, 2, \dots m, \\ \gamma = \frac{4}{1 - \alpha\eta}(2\beta + \sum_{k=1}^{m} \bar{\beta}_k). \end{gathered} \tag{2.3}$$

To state the main results in this paper we need the following assumptions:

(H1) For each $k \in \{1, 2, \dots, m\}$, $\bar{I}_k(0) = 0$ and

$$\lim_{x\to 0} \frac{\bar{I}_k(x)}{x} = d_0 > 0.$$

(H2) $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^1$ is continuous, $f(t,0,0) = 0$ and
$$\lim_{x\to 0} \frac{f(t,x,y)}{x} = d_1 < 0,$$
uniformly for $t \in [0,1]$.

From [3, Lemma 5.4.1], we have the following result.

Lemma 2.2. *$H \subset PC^1[J,\mathbb{R}^1]$ is a relatively compact set if and only if for any $x \in H$, $x(t)$ and $x'(t)$ are uniformly bounded on J and equicontinuous at any $J_k (k = 1,2,\dots,m)$, where $J_1 = [0,t_1]$, $J_i = (t_{i-1}, t_i]$, $i = 2,3,\dots,m+1$.*

Now we define the operator $A : PC^1[J,\mathbb{R}^1] \to PC^1[J,\mathbb{R}^1]$ as follows:
$$\begin{aligned}
&(Ax)(t)\\
&= \frac{t}{1-\alpha\eta}\int_0^1 (1-s)f(s,x(s),x'(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta-s)f(s,x(s),x'(s))ds\\
&\quad - \int_0^t (t-s)f(s,x(s),x'(s))ds + \sum_{0<t_k<t}[\bar I_k(x(t_k))(t-t_k)]\\
&\quad - \frac{t}{1-\alpha\eta}\sum_{k=1}^m\{[1-t_k-\alpha(\eta-t_k)]\bar I_k(x(t_k))\}, \quad x \in PC^1[J,\mathbb{R}^1].
\end{aligned}$$
From Lemma 2.2, we know $A : PC^1[J,\mathbb{R}^1] \to PC^1[J,\mathbb{R}^1]$ is a completely continuous operator. The following Lemma can be easily obtained.

Lemma 2.3. *$y \in PC^1[J,\mathbb{R}^1]$ is a solution of (1.1) if and only if $y(t) = Ay(t)$ for $t \in [0,1]$*

Theorem 2.4. *Assume that u_1 and u_2 are two strict lower solutions of (1.1), $0 \le \gamma < 1$, then there exists $R_0 > 0$ large enough such that*
$$\deg(I - A, \Omega, \theta) = 1,$$
where $\Omega = \{x \in B(\theta, R_0) : \sigma_1 \prec x\}$, $\sigma_1(t) = \sup_{t\in J}\{u_1(t), u_2(t)\}$.

Proof. If we let $I_k = 0$ in the proof of [11, Theorem 2.1], we can easily get this theorem by slight modification. But for the completeness of this paper we will give details of the proof of this theorem. For $0 \le \gamma < 1$, we take $\beta' > \beta$, $\bar\beta'_k > \bar\beta_k$, $(k = 1,2,\dots,m)$ with
$$\gamma' := \frac{4}{1-\alpha\eta}(2\beta' + \sum_{k=1}^m \bar\beta'_k) < 1. \tag{2.4}$$
From the definition of β, there exists $N > 0$, such that
$$|f(t,x,y)| < \beta'(|x| + |y|), \quad \forall t \in J, |x| + |y| \ge N,$$
and so
$$|f(t,x,y)| \le \beta'(|x| + |y|) + M, \quad \forall t \in J,\ x,y \in \mathbb{R}^1, \tag{2.5}$$
where $M = \sup_{(t,x,y)\in J\times\mathbb{R}^2,\,|x|+|y|\le N} |f(t,x,y)|$. Similarly, we have
$$|\bar I_k(x)| \le \bar\beta'_k|x| + \overline{M}_k, \quad \forall x \in \mathbb{R}^1, \tag{2.6}$$
where $\overline{M}_k$ is a positive constant. Take
$$R_0 > \max\{\|u_1\|_{PC^1}, \|u_2\|_{PC^1}, \frac{1}{1-\gamma'}\frac{4}{1-\alpha\eta}(M + \sum_{k=1}^m \overline{M}_k)\}. \tag{2.7}$$

Let $\sigma_1(t) = \sup_{t\in J}\{u_1(t), u_2(t)\}$ for all $t \in J$. Then $\sigma_1 \in PC[J, \mathbb{R}^1]$. Now we define $h_1 : J \times \mathbb{R}^2 \to \mathbb{R}^1$, $\bar{J}_{k,1} : \mathbb{R}^1 \to \mathbb{R}^1$, $(k = 1, 2, \dots, m)$ as follows:

$$h_1(t, x, y) = \begin{cases} f(t, \sigma_1(t), y), & x < \sigma_1(t), \\ f(t, x, y), & x \ge \sigma_1(t), \end{cases} \tag{2.8}$$

$$\bar{J}_{k,1}(x) = \begin{cases} \bar{I}_k(\sigma_1(t_k)), & x < \sigma_1(t_k), \\ \bar{I}_k(x), & x \ge \sigma_1(t_k). \end{cases} \tag{2.9}$$

Define the nonlinear operator $A_1 : PC^1[J, \mathbb{R}^1] \to PC^1[J, \mathbb{R}^1]$ as follows:

$$\begin{aligned}
&(A_1x)(t) \\
&= \frac{t}{1-\alpha\eta}\int_0^1 (1-s)h_1(s, x(s), x'(s))ds - \frac{\alpha t}{1-\alpha\eta}\int_0^\eta (\eta - s)h_1(s, x(s), x'(s))ds \\
&\quad - \int_0^t (t-s)h_1(s, x(s), x'(s))ds + \sum_{0<t_k<t}[\bar{J}_{k,1}(x(t_k))(t-t_k)] \\
&\quad - \frac{t}{1-\alpha\eta}\sum_{k=1}^m \{[1 - t_k - \alpha(\eta - t_k)]\bar{J}_{k,1}(x(t_k))\}, \quad \forall t \in J.
\end{aligned}$$

Clearly, $A_1 : PC^1[J, \mathbb{R}^1] \to PC^1[J, \mathbb{R}^1]$ is a completely continuous operator. Let

$$B(\theta, R_0) = \{x \in PC^1[J, \mathbb{R}^1] : \|x\|_{PC^1} < R_0\}.$$

For any $x \in \overline{B}(\theta, R_0)$, by (2.5)-(2.9), we have for all $t \in J$,

$$|h_1(t, x(t), x'(t))| \le \beta' \sup_{t\in J}\{|x(t)|, |u_1(t)|, |u_2(t)|\} + \beta'|x'(t)| + M \le 2\beta' R_0 + M,$$

and for $k = 1, 2, \dots, m$,

$$|\bar{J}_{k,1}(x(t_k))| \le \bar{\beta}'_k \max\{|x(t_k)|, |u_1(t_k)|, |u_2(t_k)|\} + \overline{M}_k \le \bar{\beta}'_k R_0 + \overline{M}_k.$$

Then

$$\begin{aligned}
&|A_1x(t)| \\
&\le [\frac{1}{1-\alpha\eta}\int_0^1 (1-s)ds + \frac{\alpha}{1-\alpha\eta}\int_0^\eta (\eta - s)ds + \int_0^1 (1-s)ds](2\beta' R_0 + M) \\
&\quad + \frac{1}{1-\alpha\eta}\sum_{k=1}^m (\bar{\beta}'_k R_0 + \overline{M}_k) + \sum_{k=1}^m (\bar{\beta}'_k R_0 + \overline{M}_k) \\
&\le \frac{2}{1-\alpha\eta}(2\beta' R_0 + M) + \sum_{k=1}^m (\frac{1}{1-\alpha\eta} + 1)\bar{\beta}'_k R_0 + \sum_{k=1}^m (\frac{1}{1-\alpha\eta} + 1)\overline{M}_k \\
&\le \frac{2}{1-\alpha\eta}(2\beta' + \sum_{k=1}^m \bar{\beta}'_k)R_0 + \frac{2}{1-\alpha\eta}(M + \sum_{k=1}^m \overline{M}_k).
\end{aligned} \tag{2.10}$$

Also we have

$$
\begin{aligned}
|(A_1x)'(t)| &\le [\frac{1}{1-\alpha\eta}\int_0^1(1-s)ds+\frac{\alpha}{1-\alpha\eta}\int_0^\eta(\eta-s)ds+1](2\beta' R_0+M)\\
&\quad+\frac{1}{1-\alpha\eta}\sum_{k=1}^m(\bar\beta_k' R_0+\overline{M}_k)+\sum_{k=1}^m(\bar\beta_k' R_0+\overline{M}_k)\\
&\le \frac{2}{1-\alpha\eta}(2\beta'+\sum_{k=1}^m\bar\beta_k')R_0+\frac{2}{1-\alpha\eta}(M+\sum_{k=1}^m\overline{M}_k).
\end{aligned}
\tag{2.11}
$$

Thus

$$
\|A_1x\|_{PC^1}\le\frac{4}{1-\alpha\eta}(2\beta'+\sum_{k=1}^m\bar\beta_k')R_0+\frac{4}{1-\alpha\eta}(M+\sum_{k=1}^m\overline{M}_k)<R_0.
$$

Then $A_1(\overline{B}(\theta,R_0))\subset B(\theta,R_0)$. Hence

$$
\deg(I-A_1,B(\theta,R_0),\theta)=1. \tag{2.12}
$$

Now we prove that $x_0\in\Omega$ whenever $x_0\in\overline{B}(\theta,R_0)$ with $x_0=A_1x_0$. By Lemma 2.3, we have

$$
\begin{gathered}
x_0''(t)+h_1(t,x_0(t),x_0'(t))=0,\quad t\in J,\ t\ne t_k,\\
\Delta x_0'|_{t=t_k}=\bar J_{k,1}(x_0(t_k)),\quad k=1,2,\dots,m,\\
x_0(0)=0,\quad x_0(1)-\alpha x_0(\eta)=0,
\end{gathered}
\tag{2.13}
$$

for any $x_0\in\overline{B}(\theta,R_0)$ with $x_0=A_1x_0$. We need to prove

$$
\sigma_1\prec x_0. \tag{2.14}
$$

Let $\omega(t)=\sigma_1(t)-x_0(t)$ for all $t\in J$. Then $\omega\in PC[J,\mathbb{R}^1]$. If (2.14) is not true, then $\sup_{t\in J}\omega(t)\ge 0$. We have several cases to consider.

(1) $\omega(0)=\sup_{t\in J}\omega(t)\ge 0$. In this case,

$$
0\le\omega(0)=\sigma_1(0)-x_0(0)=\sigma_1(0)=\max\{u_1(0),u_2(0)\}<0,
$$

which is a contradiction.

(2) $\omega(1)=\sup_{t\in J}\omega(t)\ge 0$. Assume without loss of generality that $\sigma_1(1)=u_1(1)$. Then

$$
0\le\omega(1)=u_1(1)-x_0(1)<\alpha u_1(\eta)-\alpha x_0(\eta)\le\alpha\omega(\eta)\le\alpha\omega(1),
$$

which is a contradiction.

(3) There exists $k_0\in\{1,2,\dots,m,m+1\}$ and $\tau_0\in(t_{k_0-1},t_{k_0})$ such that $\omega(\tau_0)=\sup_{t\in J}\omega(t)\ge 0$. We may assume $\sigma_1(\tau_0)=u_1(\tau_0)$. We have two subcases: (3A) $u_2(\tau_0)<u_1(\tau_0)$, and (3B) $u_2(\tau_0)=u_1(\tau_0)$.

For case (3A), we take $\delta_0>0$ small enough such that $[\tau_0-\delta_0,\tau_0+\delta_0]\subset(t_{k_0-1},t_{k_0})$ and $\sigma_1(t)=u_1(t)$ for all $t\in[\tau_0-\delta_0,\tau_0+\delta_0]$. Then $\omega(t)=u_1(t)-x_0(t)$ for all $t\in[\tau_0-\delta_0,\tau_0+\delta_0]$. Thus, $\omega\in C^2[\tau_0-\delta_0,\tau_0+\delta_0]$ and $\omega(\tau_0)$ is a local maximum of ω in $[\tau_0-\delta_0,\tau_0+\delta_0]$. Therefore $\omega'(\tau_0)=0$, $\omega''(\tau_0)\le 0$ and so

$$
\begin{aligned}
0\ge\omega''(\tau_0)&=u_1''(\tau_0)-x_0''(\tau_0)\\
&=u_1''(\tau_0)+h_1(\tau_0,x_0(\tau_0),x_0'(\tau_0))\\
&=u_1''(\tau_0)+f(\tau_0,u_1(\tau_0),u_1'(\tau_0))>0,
\end{aligned}
$$

which is a contradiction.

For case (3B), let $\omega_1(t) = u_2(t) - x_0(t)$ for all $t \in (t_{k_0-1}, t_{k_0})$. For $t' \in (t_{k_0-1}, t_{k_0})$, we have

$$\begin{aligned} \omega_1(\tau_0) &= u_2(\tau_0) - x_0(\tau_0) \\ &= \sigma_1(\tau_0) - x_0(\tau_0) = \omega(\tau_0) \\ &\geq \omega(t') = \sigma_1(t') - x_0(t') \\ &\geq u_2(t') - x_0(t') = \omega_1(t'). \end{aligned}$$

Then $\omega_1(\tau_0)$ is a local maximum of ω_1 in (t_{k_0-1}, t_{k_0}). Thus $\omega_1'(\tau_0) = 0$, $\omega_1''(\tau_0) \leq 0$. Therefore

$$\begin{aligned} 0 \geq \omega_1''(\tau_0) &= u_2''(\tau_0) - x_0''(\tau_0) \\ &= u_2''(\tau_0) + h_1(\tau_0, x_0(\tau_0), x_0'(\tau_0)) \\ &= u_2''(\tau_0) + f(\tau_0, u_2(\tau_0), u_2'(\tau_0)) > 0, \end{aligned}$$

which is a contradiction.

(4) There exists $k_0 \in \{1, 2, \dots, m\}$ such that $\omega(t_{k_0}) = \sup_{t \in J} \omega(t) \geq 0$. We take $\delta_0 > 0$ small enough such that $\omega(t_{k_0})$ is a local maximum of $\omega(t)$ in $[t_{k_0} - \delta_0, t_{k_0} + \delta_0]$, then we have $\omega'(t_{k_0}) \geq 0$ and $\omega'(t_{k_0}^+) \leq 0$. Thus,

$$\begin{aligned} 0 \geq \omega'(t_{k_0}^+) &= u_1'(t_{k_0}^+) - x_0'(t_{k_0}^+) \\ &> [u_1'(t_{k_0}) + \bar{I}_{k_0}(u_1(t_{k_0}))] - [x_0'(t_{k_0}) + \bar{J}_{k_0,1}(x_0(t_{k_0}))] \\ &= u_1'(t_{k_0}) - x_0'(t_{k_0}) \\ &= \omega'(t_{k_0}) \geq 0, \end{aligned}$$

which is a contradiction.

From the discussion of cases (1)-(4), we see that (2.14) holds. Since $\Omega = \{x \in B(\theta, R_0) | \sigma_1 \prec x\}$, it follows that $\Omega \subset PC^1[J, \mathbb{R}^1]$ is an open set. We see from (2.12) (2.14) and the properties of topological degree that

$$\deg(I - A_1, \Omega, \theta) = 1.$$

Notice that $A_1 x = Ax$ for all $x \in \overline{\Omega}$, and so we have

$$\deg(I - A, \Omega, \theta) = 1.$$

This completes the proof. $\square$

Corollary 2.5. *Assume that u_1 is a strict lower solution of* (1.1), $0 \leq \gamma < 1$, *then there exists $R_0 > 0$ large enough such that*

$$\deg(I - A, \Omega, \theta) = 1,$$

where $\Omega = \{x \in B(\theta, R_0) : u_1 \prec x\}$.

Also we have the following Theorems.

Theorem 2.6. *Assume that v_1 and v_2 are two strict upper solutions of* (1.1), $0 \leq \gamma < 1$, *then there exists $R_0 > 0$ large enough such that*

$$\deg(I - A, \Omega, \theta) = 1,$$

where $\Omega = \{x \in B(\theta, R_0) : x \prec \sigma_2\}$, $\sigma_2(t) = \inf_{t \in J}\{v_1(t), v_2(t)\}$.

Corollary 2.7. *Assume that v_1 is a strict upper solution of* (1.1), $0 \le \gamma < 1$, *then there exists $R_0 > 0$ large enough such that*

$$\deg(I - A, \Omega, \theta) = 1,$$

where $\Omega = \{x \in B(\theta, R_0) : x \prec v_1\}$.

Theorem 2.8. *Assume that u_1 is a strict lower solution and v_1 is a strict upper solution of* (1.1), $0 \le \gamma < 1$, *then there exist $R_0 > 0$ large enough such that*

$$\deg(I - A, \Omega, \theta) = 1,$$

where $\Omega = \{x \in B(\theta, R_0) : u_1 \prec x \prec v_1\}$.

3. Main Results

Theorem 3.1. *Assume that* (H1), (H2) *are satisfied,* $0 \le \gamma < 1$ *and* (1.1) *has a strict lower solution u_1 and a strict upper solution v_1, such that $u_1 \prec v_1$ and u_1, v_1 are sign-changing on* $[0, 1]$. *Then* (1.1) *has at least four sign-changing solutions, two positive solutions and two negative solutions.*

Proof. From (H2), there exists $0 < \varepsilon_0 < R_0$ such that

$$f(t, -\varepsilon, 0) > 0, \quad f(t, \varepsilon, 0) < 0, \quad \forall t \in [0, 1], \ \forall \varepsilon \in (0, \varepsilon_0).$$

Let $u_{1,i}(t) = -1/i$, $v_{1,j}(t) = \frac{1}{j}$, $i, j = 1, 2, \dots$. Then there exists a natural number $n_0 > \frac{1}{\varepsilon_0}$ such that

$$u_{1,i} \not\preceq v_1, \quad u_1 \not\preceq v_{1,j},$$

for each $i, j \ge n_0$. Since $u_{1,i}(t) = -\frac{1}{i} < 0$, it follows that $u_{1,i}(t_k) = -1/i < 0$, $k = 1, 2, 3, \dots, m$. By (H1) and (H2), we can easily show that

$$\begin{gathered}
u''_{1,i}(t) + f(t, u_{1,i}(t), u'_{1,i}(t)) > 0, \quad t \ne t_k, \\
\Delta u'_{1,i}|_{t=t_k} > \bar{I}_k(u_{1,i}(t_k)), \quad k = 1, 2, \dots, m, \\
u_{1,i}(0) < 0, \quad u_{1,i}(1) - \alpha u_{1,i}(\eta) < 0.
\end{gathered}$$

So, $u_{1,i}(t)$ is a strict lower solution of (1.1). Similarly, we know $v_{1,j}$ is a strict upper solution of (1.1).

Take u_{1,n_0} and v_{1,n_0}, let

$$\begin{gathered}
O_1 = \{x \in B(\theta, R_0) | u_1 \prec x\}, \quad O_2 = \{x \in B(\theta, R_0) | x \prec v_1\}, \\
O_3 = \{x \in B(\theta, R_0) | u_{1,n_0} \prec x\}, \quad O_4 = \{x \in B(\theta, R_0) | x \prec v_{1,n_0}\}, \\
\Omega_1 = O_1 \setminus (\overline{O_1 \cap O_2}) \cup (\overline{O_1 \cap O_3}), \quad \Omega_2 = O_2 \setminus (\overline{O_1 \cap O_2}) \cup (\overline{O_2 \cap O_4}), \\
\Omega_3 = O_3 \setminus (\overline{O_1 \cap O_3}) \cup (\overline{O_3 \cap O_4}), \quad \Omega_4 = B(\theta, R_0) \setminus (\overline{O}_1 \cup \overline{O}_4 \cup \overline{\Omega}_2 \cup \overline{\Omega}_3).
\end{gathered}$$

From Theorems 2.4-2.8 and Corollaries 2.5-2.7, we have

$$\deg(I - A, O_1, \theta) = 1, \tag{3.1}$$

$$\deg(I - A, O_2, \theta) = 1, \tag{3.2}$$

$$\deg(I - A, O_1 \cap O_2, \theta) = 1, \tag{3.3}$$

$$\deg(I - A, O_1 \cap O_3, \theta) = 1, \tag{3.4}$$

$$\deg(I - A, O_2 \cap O_4, \theta) = 1. \tag{3.5}$$

Thus,

$$\deg(I-A,\Omega_1,\theta)=1-1-1=-1, \tag{3.6}$$
$$\deg(I-A,\Omega_2,\theta)=1-1-1=-1. \tag{3.7}$$

So, there exist $x_1\in O_1\cap O_2$, $x_2\in\Omega_1$, $x_3\in\Omega_2$, which are sign-changing solutions of (1.1). From Corollaries 2.5-2.7 and Theorem 2.8, we have

$$\deg(I-A,O_3,\theta)=1, \tag{3.8}$$
$$\deg(I-A,O_4,\theta)=1, \tag{3.9}$$
$$\deg(I-A,O_3\cap O_4,\theta)=1. \tag{3.10}$$

Thus, from (3.4), (3.7) and (3.9), we have

$$\deg(I-A,\Omega_3,\theta)=1-1-1=-1. \tag{3.11}$$

From the proof of (2.12), it is easy to get

$$\deg(I-A,B(\theta,R_0),\theta)=1. \tag{3.12}$$

Then we have from (3.1), (3.7), (3.9), (3.11) and (3.12) that

$$\deg(I-A,\Omega_4,\theta)=1-1-1-(-1)-(-1)=1.$$

So, there exists a fourth sign-changing solution $x_4\in\Omega_4$. By (3.4), we can get a solution $x_{5,i}\in O_1\cap O_3$ for $i\ge n_0$. From $\|x_{5,i}\|=\|Ax_{5,i}\|<R_0$, we know $\{x_{5,i}\}_{i=n_0}^{\infty}$ is a bounded set. Notice that A is a completely continuous operator, then $\{x_{5,i}\}_{i=n_0}^{\infty}$ is a relatively compact set. Without loss of generality, assume that $x_{5,i}\to x_5$ as $i\to\infty$. Then x_5 is a solution of (1.1). Since $u_{1,i}\to 0$ as $i\to\infty$, then x_5 is a positive solution of (1.1). Similarly, we can get x_6, x_7 and x_8 such that

$$\begin{gathered}
\theta\prec x_6\prec R_0,\quad u_1\not\prec x_6\not\prec v_{1,n_0}.\\
-R_0\prec x_7\prec v_1,\quad -R_0\prec x_7\prec\theta,\\
-R_0\prec x_8\prec\theta,\quad u_{1,n_0}\not\prec x_8\not\prec v_1.
\end{gathered}$$

It is easy to see that x_6 is a positive solution of (1.1), x_7 and x_8 are two negative solutions of (1.1). This completes the proof. $\square$

Remark 3.2. Obviously, we can replace the sub-linear condition $0\le\gamma<1$ with a pair of strict upper and lower solutions, but then we need to introduce a Nagumo condition for nonlinear item f.

In this paper, we give some existence results for sign-changing solutions. Up to now, there were few papers that considered the existence of sign-changing solutions for impulsive multi-point boundary-value problem. Moreover, we give the exact positions of them. Therefore, the result of this paper is new.

The method of this paper is of interest even if there exists a jump of $x(t)$ at $t=t_k$, $k=1,2,3,\dots,m$ at the same time.

Example 3.3. Let $R_0=100$ and

$$u_1(t)=\sin\frac{3}{2}\pi t-\frac{1}{2},\quad v_1(t)=\sin\frac{1}{2}\pi t+\frac{1}{2},\quad \forall t\in[0,1].$$

Obviously, $u_1(t)$ and $v_1(t)$ are sign-changing on $[0,1]$ and $u_1 \prec v_1$. Now let the sets D_1, D_2, D_3, and $\widetilde{D}_4$ be defined by

$$D_1 = \{(t, u_1(t), u_1'(t)) : t \in [0,1]\}, \quad D_2 = \{(t, v_1(t), v_1'(t)) : t \in [0,1]\},$$
$$D_3 = \{(t, 100, 0) : t \in [0,1]\}, \quad \widetilde{D}_4 = \{(t, 0, 0) : t \in [0,1]\}.$$

Then D_1, D_2, D_3, and $\widetilde{D}_4$ are four disjoint closed sets of $\mathbb{R}^3$. Let

$$r_0 = \frac{1}{2}\min\{d(\widetilde{D}_4, D_1), d(\widetilde{D}_4, D_2), d(\widetilde{D}_4, D_3)\} > 0$$

and

$$D_4 = \{(t,x,y) \in \mathbb{R}^3 : d((t,x,y), \widetilde{D}_4) \le r_0\}.$$

Define the function $\widetilde{f}$ by

$$\widetilde{f}(t,x,y) = \begin{cases} 30, & (t,x,y) \in D_1, \\ -30, & (t,x,y) \in D_2, \\ 1, & (t,x,y) \in D_3, \\ \frac{1}{100}(-x+y), & (t,x,y) \in D_4. \end{cases}$$

From Dugundji's extension theorem, see [4], there exists a continuous function $f : [0,1] \times \mathbb{R}^2 \mapsto \mathbb{R}^1$ such that $f(t,x,y) = \widetilde{f}(t,x,y)$ while $(t,x,y) \in D_i$ for each $i = 1,2,3,4$, and $f([0,1] \times \mathbb{R}^2) \subset \widetilde{f}([0,1] \times \mathbb{R}^2) \subset B(\theta, 100)$. Consider the impulsive three-point boundary-value problem

$$\begin{gathered} y''(t) + f(t, y(t), y'(t)) = 0, \quad t \in J,\ t \ne t_k, \\ \Delta y'|_{t=t_k} = \bar{I}_k(y(t_k)), \quad k = 1,2, \\ y(0) = 0, \quad y(1) = \alpha y(\eta), \end{gathered} \tag{3.13}$$

where $t_1 = \frac{1}{10}$, $t_2 = \frac{2}{3}$, $\alpha = \frac{1}{2}$, $\eta = \frac{3}{4}$ and $\bar{I}_k(x) = \frac{1}{50k}x$, $k = 1,2$. From the definition of $u_1(t)$ and f we have

$$u_1''(t) + f(t, u_1(t), u_1'(t)) = -\frac{9}{4}\pi^2 \sin\frac{3}{2}\pi t + \widetilde{f}(t, u_1(t), u_1'(t)) > -\frac{9}{4}\pi^2 + 30 > 0,$$

for all $t \in [0,1]$,

$$\begin{gathered} \bar{I}_1(u_1(t_1)) = \frac{1}{50}(\sin\frac{3}{20}\pi - \frac{1}{2}) < 0 = \Delta u_1'|_{t=t_1}, \\ \bar{I}_2(u_1(t_2)) = \frac{1}{100}(\sin\pi - \frac{1}{2}) < 0 = \Delta u_1'|_{t=t_2}, \\ u_1(0) < 0, \quad \alpha u_1(\eta) = \frac{1}{2}(\sin\frac{9}{8}\pi - \frac{1}{2}) > -\frac{3}{2} = u_1(1). \end{gathered}$$

Then $u_1(t)$ is a strict lower solution of (3.12). Similarly, $v_1(t)$ is a strict upper solution of (3.12). From

$$\lim_{x\to 0} \frac{\bar{I}_k(x)}{x} = \frac{1}{50k} > 0, \quad \bar{I}_k(0) = 0, \quad k = 1,2,$$

we see that (H1) holds. Next note

$$\lim_{x\to 0} \frac{f(t,x,0)}{x} = \lim_{x\to 0} \frac{\widetilde{f}(t,x,0)}{x} = -\frac{1}{100} < 0, f(t,0,0) = 0,$$

uniformly for $t \in [0,1]$, then (H2) holds. Since

$$\beta = \limsup_{|x|+|y|\to\infty} \max_{t\in J} \frac{|f(t,x,y)|}{|x|+|y|} = 0,$$

$$\bar{\beta}_k = \limsup_{|x|\to\infty} \frac{|\bar{I}_k(x)|}{|x|} = \frac{1}{50k}, \quad k = 1,2,$$

it follows that

$$\gamma = \frac{4}{1-\alpha\eta}(2\beta + \bar{\beta}_1 + \bar{\beta}_2) = \frac{24}{125} < 1.$$

Now all conditions of Theorem 3.1 hold. Therefore, the impulsive boundary-value problem (3.2) has at least four sign-changing solutions, two positive solutions and two negative solutions.

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