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FREDHOLM TYPE INTEGRODIFFERENTIAL EQUATION ON TIME SCALES

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ABSTRACT. The aim of this article is to study some basic qualitative properties of solutions to Fredholm type integrodifferential equations on time scales. A new integral inequality with explicit estimate on time scales is obtained and used to establish the results.

1. INTRODUCTION

The theory of time scales was introduced by Stefan Hilger [4] in 1988 which unifies continuous and discrete analysis. Since then many authors have studied various aspects of dynamic integral equations on time scales by using various techniques [5, 7, 8, 9, 10]. In this paper we consider the integrodifferential equation

$$x^{\Delta}(t) = f(t, x(t), x^{\Delta}(t), Hx(t)), \quad x(\alpha) = x_0, \tag{1.1}$$

for $t \in [\alpha, \beta] \subset I_{\mathbb{T}}$, where

$$Hx(t) = \int_{\alpha}^{\beta} h(t, \tau, x(\tau), x^{\Delta}(\tau))\Delta\tau, \tag{1.2}$$

f, h are given functions and x is unknown function to be found, and Δ denotes the delta derivative. We assume that $h : I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $f : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous functions, t is from a time scale $\mathbb{T}$, which is nonempty closed subset of $\mathbb{R}$, the set of real numbers, $\tau \leq t$ and $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty)$ the given subset of $\mathbb{R}$, $\mathbb{R}^n$ the real n-dimensional Euclidean space with appropriate norm defined by $|\cdot|$. The integral sign represents the delta integral. Recently in [1, 2, 5, 7, 8, 9, 10] the authors have studied the existence, uniqueness and other qualitative properties of solutions of various dynamic equations on time scales by using different techniques. In fact the study of equations of the form (1.1) is a challenging task, because of the occurrence of the x^{Δ} on the right hand side in (1.1). One can formulate existence and uniqueness result for (1.1) by using the idea recently employed in [8, Theorem 3.1]. Motivated by the results obtained by the present author in [8], in this paper we study some fundamental qualitative properties of solutions of equation (1.1). Time scale analogue of a variant of a

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certain integral inequality with explicit estimate is obtained and used to establish the results.

2. Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. We define the jump operators σ, ρ on $\mathbb{T}$ by the two mapping $\sigma, \rho : \mathbb{T} \to \mathbb{R}$ satisfying conditions

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

The jump operators classify the points of time scale $\mathbb{T}$ as left dense, left scattered, right dense and right scattered according to whether $\rho(t) = t$ or $\rho(t) < t$, $\sigma(t) = t$ and $\sigma(t) > t$ respectively for $t \in \mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right dense point in $\mathbb{T}$. The set of all rd-continuous functions is denoted by C_{rd}, If $\mathbb{T}$ has left scattered maximum m, then

$$\mathbb{T}^k = \begin{cases} \mathbb{T} - m & \text{if } \sup \mathbb{T} < \infty \\ \mathbb{T} & \text{if } \sup \mathbb{T} = \infty \end{cases} \tag{2.1}$$

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta} = f(t)$ holds for all $t \in I_{\mathbb{T}}$. We define the integral of f by

$$\int_s^t f(t)\Delta\tau = F(t) - F(s), \tag{2.2}$$

where $s, t \in \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to \mathbb{R}_+ = [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$. The function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in I_{\mathbb{T}}$. We denote by $\Re$ the set of all regressive and rd-continuous functions and define the set of all regressive functions by

$$\Re^+ = \{p \in \mathbb{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}. \tag{2.3}$$

For $p \in \Re^+$ we define (see [1]) the exponential function $e_p(., t_0)$ on time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^{\Delta}(t) = p(t)x(t), \quad x(t_0) = 1. \tag{2.4}$$

If $p \in \Re^+$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$. The exponential function $e_p(., t_0)$ is given by

$$e_p(t, t_0) = \begin{cases} \exp\big(\int_{t_0}^t p(s)\Delta s\big) & \text{for } t \in \mathbb{T},\ \mu = 0; \\ \exp\big(\int_{t_0}^t \frac{\log(1+\mu(s)p(s))}{\mu(s)}\Delta s\big) & \text{for } t \in \mathbb{T},\ \mu > 0; \end{cases} \tag{2.5}$$

where log is a principle logarithm function. To allow a comparison of the results in the paper with the continuous case, we note that, if $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t, s) = \exp\Big(\int_s^t p(\tau)d\tau\Big), \quad e_\alpha(t, s) = \exp(\alpha(t - s)), \quad e_\alpha(t, 0) = \exp(\alpha t), \tag{2.6}$$

for $s, t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. To compare with the discrete case, if $\mathbb{T} = \mathbb{Z}$ (the set of integers), the exponential function is given by

$$e_p(t, s) = \prod_{\tau=s}^{t-1} [1 + p(\tau)], \quad e_\alpha(t, s) = (1 + \alpha)^{t-s}, \quad e_\alpha(t, 0) = (1 + \alpha)^t, \tag{2.7}$$

for $s, t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$. We use the following fundamental result proved in Bohner and Peterson [1] (see also [3]).

Lemma 2.1. *Suppose $u, b \in C_{rd}$ and $a \in \Re^+$. Then*

$$u^{\Delta}(t) \leq a(t)u(t) + b(t) \tag{2.8}$$

for all $t \in \mathbb{T}$, implies

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^{t} e_a(t, \sigma(\tau))b(\tau)\Delta\tau, \tag{2.9}$$

for all $t \in \mathbb{T}$.

3. Basic integral inequality on time scale

In this section we establish the time scale analogue of the variant of the integral inequality given in [6,Theorem 1.3.1 part(a2), p.41].

Theorem 3.1. *Let $u, a, b, c, d, f, g \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$ and suppose that*

$$\begin{aligned} u(t) \leq a(t) + b(t) &\int_{\alpha}^{t} f(t)\Big[u(s) + d(s)\int_{\alpha}^{\beta} g(\tau)u(\tau)\Delta\tau\Big]\Delta s \\ &+ c(t)\int_{\alpha}^{\beta} g(\tau)u(\tau)\Delta\tau, \end{aligned} \tag{3.1}$$

for $t \in I_{\mathbb{T}}$. If

$$k = \int_{\alpha}^{\beta} g(\xi)K_2(\xi)\Delta\xi < 1, \tag{3.2}$$

then

$$u(t) \leq K_1(t) + MK_2(t), \tag{3.3}$$

for $t \in I_{\mathbb{T}}$, where

$$K_1(t) = a(t) + b(t)\int_{\alpha}^{t} f(\tau)a(\tau)e_{fb}(t, \sigma(\tau))\Delta\tau, \tag{3.4}$$

$$K_2(t) = c(t) + b(t)\int_{\alpha}^{t} f(\tau)\{c(\tau) + d(\tau)\}e_{fb}(t, \sigma(\tau))\Delta\tau, \tag{3.5}$$

$$M = \frac{1}{1-k}\int_{\alpha}^{\beta} g(\xi)K_1(\xi)\Delta\xi. \tag{3.6}$$

Proof. Let

$$\lambda = \int_{\alpha}^{\beta} g(\tau)u(\tau)\Delta\tau, \tag{3.7}$$

and

$$z(t) = \int_{\alpha}^{t} f(s)[u(s) + d(s)\int_{\alpha}^{\beta} g(\tau)u(\tau)\Delta\tau]\Delta s = \int_{\alpha}^{t} f(s)[u(s) + d(s)\lambda]\Delta s, \tag{3.8}$$

then $z(\alpha) = 0$ and (3.1) can be restated as

$$u(t) \leq a(t) + b(t)z(t) + c(t)\lambda. \tag{3.9}$$

From (3.8) and (3.9), we have

$$\begin{aligned} z^{\Delta}(t) &\leq f(t)[a(t) + b(t)z(t) + c(t)\lambda + d(t)\lambda] \\ &= f(t)b(t)z(t) + f(t)[a(t) + \{c(t) + d(t)\}\lambda]. \end{aligned} \tag{3.10}$$

Applying lemma 2.1 to (3.10) yields

$$z(t) \le \int_\alpha^t f(\tau)[a(\tau) + \lambda\{c(\tau) + d(\tau)\}]e_{fb}(t, \sigma(\tau))\Delta\tau. \tag{3.11}$$

Using (3.11) in (3.9), we obtain

$$\begin{aligned} u(t) &\le a(t) + b(t)\Big\{\int_\alpha^t f(\tau)[a(\tau) + \lambda\{c(\tau) + d(\tau)\}]e_{fb}(t, \sigma(\tau))\Delta\tau\Big\} + c(t)\lambda \\ &= a(t) + b(t)\int_\alpha^t f(\tau)a(\tau)e_{fb}(t, \sigma(\tau))\Delta\tau \\ &\quad + \lambda\Big\{c(t) + b(t)\int_\alpha^t f(\tau)\{c(\tau) + d(\tau)\}e_{fb}(t, \sigma(t))\Delta\tau\Big\} \\ &= K_1(t) + \lambda K_2(t). \end{aligned} \tag{3.12}$$

From this inequality and (3.7), it is easy to observe that $\lambda \le M$. Using this inequality in (3.12), we obtain (3.3). □

4. Estimates on the solutions

In this section we obtain estimates on the solutions of equation (1.1) by applying Theorem 3.1, under some suitable conditions on the functions involved therein.

First, we shall give the following theorem concerning the estimate on the solution of equation (1.1).

Theorem 4.1. *Suppose that the functions f, h in (1.1) satisfy the conditions*

$$|f(t, u, v, w)| \le \gamma[|u| + |v| + |w|], \tag{4.1}$$

$$|h(t, u, v, w)| \le q(t)r(\tau)[|u| + |v|], \tag{4.2}$$

where $0 \le \gamma < 1$ is a constant and $q, r \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$. Let

$$L_1(t) = \frac{|x_0|}{1-\gamma} + \frac{1}{1-\gamma}\Big[\int_\alpha^t \frac{\gamma}{1-\gamma}|x_0|e_{\frac{\gamma}{1-\gamma}}(t, \tau)\Delta\tau\Big], \tag{4.3}$$

$$L_2(t) = \frac{\gamma}{1-\gamma}q(t) + \frac{1}{1-\gamma}\Big[\int_\alpha^t \gamma[\frac{\gamma}{1-\gamma} + 1]q(t)e_{\frac{\gamma}{1-\gamma}}(t, \tau)\Delta\tau\Big], \tag{4.4}$$

for $t \in I_{\mathbb{T}}$ and

$$\lambda = \int_\alpha^\beta r(\xi)L_2(\xi)\Delta\xi < 1, \quad Q = \frac{1}{1-\lambda}\int_\alpha^\beta r(\xi)L_1(\xi)\Delta\xi. \tag{4.5}$$

If $x(t)$ is a solution of (1.1) on $I_{\mathbb{T}}$, then

$$|x(t)| + |x^\Delta(t)| \le L_1(t) + QL_2(t), \tag{4.6}$$

for $t \in I_{\mathbb{T}}$.

Proof. Let $m(t) = |x(t)| + |x^\Delta(t)|$, $t \in I_{\mathbb{T}}$. Using the fact that $x(t)$ is a solution of (1.1) and the hypotheses, we have

$$\begin{aligned} m(t) &= \Big|x_0 + \int_\alpha^t f(s, x(s), x^\Delta(s), Hx(s))\Delta s\Big| + |f(t, x(t), x^\Delta(t), Hx(t))| \\ &\le |x_0| + \int_\alpha^t \gamma\Big[m(s) + \int_\alpha^\beta q(s)r(\tau)m(\tau)\Delta\tau\Big]\Delta s \end{aligned}$$

$$+\gamma\Big[m(t)+\int_\alpha^\beta q(t)r(\tau)m(\tau)\Delta\tau\Big].$$

From this inequality, we have

$$\begin{aligned}m(t)\le\frac{|x_0|}{1-\gamma}+\frac{1}{1-\gamma}\int_\alpha^t\gamma\Big[m(s)+q(s)\int_\alpha^\beta r(\tau)m(\tau)\Delta\tau\Big]\Delta s\\
+\frac{\gamma}{1-\gamma}q(t)\int_\alpha^\beta r(\tau)m(\tau)\Delta\tau.\end{aligned}$$

Now an application of theorem 3.1 to the above inequality, we have (4.6). □

Remark 4.2. The estimate obtained in (4.6) yields bounds on the solution $x(t)$ and its delta derivative. If the estimate in (4.6) is bounded, then the solution $x(t)$ of (1.1) and its delta derivative are also bounded on $I_{\mathbb{T}}$.

Consider the IVP (1.1) with the IVP

$$z^\Delta(t)=g(t,z(t),z^\Delta(t),Hz(t)),\quad z(\alpha)=z_0, \tag{4.7}$$

for $t\in I_{\mathbb{T}}$, where H is given by (1.2) and $g\in C_{rd}(I_{\mathbb{T}}\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n,\mathbb{R}^n)$.

The next result deals with the closeness of solutions of (1.1) and (4.7).

Theorem 4.3. *Suppose that the functions f,h in* (1.1) *satisfy the conditions*

$$|f(t,u,v,w)-f(t,\overline{u},\overline{v},\overline{w})|\le\gamma[|u-\overline{u}|+|v-\overline{v}|+|w-\overline{w}|], \tag{4.8}$$

$$|h(t,\tau,u,v)-h(t,\tau,\overline{u},\overline{v})|\le q(t)r(\tau)[|u-\overline{u}|+|v-\overline{v}|], \tag{4.9}$$

where $0\le\gamma<1$ is a constant and $q,r\in C_{rd}(I_{\mathbb{T}},\mathbb{R}_+)$ and

$$|f(t,u,v,w)-g(t,u,v,w)|\le\epsilon, \tag{4.10}$$

$$|x_0-z_0|\le\delta, \tag{4.11}$$

where f,x_0 and g,z_0 are as in (1.1) *and* (4.7). *Let*

$$w(t)=\delta+\epsilon[1+t-\alpha], \tag{4.12}$$

$\lambda,L_2(t)$ be as in (4.4), (4.5) *and*

$$Q_0=\frac{1}{1-\lambda}\int_\alpha^\beta r(\xi)A_0(\xi)\Delta\xi, \tag{4.13}$$

in which

$$A_0(t)=\frac{w(t)}{1-\gamma}+\frac{1}{1-\gamma}\Big[\int_\alpha^t\frac{\gamma}{1-\gamma}w(\tau)e_{\frac{\gamma}{1-\gamma}}(t,\tau)\Delta\tau\Big]. \tag{4.14}$$

Let $y(t)$ and $z(t)$ be respectively, solutions of (1.1) *and* (4.7) *on $I_{\mathbb{T}}$, then*

$$|x(t)-z(t)|+|x^\Delta(t)-z^\Delta(t)|\le A_0(t)+Q_0L_2(t), \tag{4.15}$$

for $t\in I_{\mathbb{T}}$.

Proof. Let $u(t)=|x(t)-z(t)|+|x^\Delta(t)-z^\Delta(t)|$, $t\in I_{\mathbb{T}}$. Using the hypotheses, we have

$$\begin{aligned}u(t)\le|x_0-z_0|+\int_\alpha^t|f(s,x(s),x^\Delta(s),Hx(s))-f(s,z(s),z^\Delta(s),Hz(s))|\Delta s\\
+\int_\alpha^t|f(s,z(s),z^\Delta(s),Hz(s))-g(s,z(s),z^\Delta(s),Hz(s))|\Delta s\\
+|f(t,x(t),x^\Delta(t),Hx(t))-f(t,z(t),z^\Delta(t),Hz(t))|\end{aligned}$$

$$
\begin{aligned}
&+|f(t,z(t),z^{\Delta}(t),Hz(t))-g(t,z(t),z^{\Delta}(t),Hz(t))| \\
&\le \delta+\int_{\alpha}^{t}\gamma\Big[u(s)+q(s)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau\Big]\Delta s \\
&\quad+\int_{\alpha}^{t}\epsilon\Delta s+\gamma[u(t)+g(t)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau]+\epsilon \\
&= w(t)+\int_{\alpha}^{t}\gamma[u(s)+q(s)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau]\Delta s \\
&\quad+\gamma\Big[u(t)+q(t)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau\Big].
\end{aligned}
$$

Then we obtain

$$
\begin{aligned}
u(t)\le \frac{w(t)}{1-\gamma}+\frac{1}{1-\gamma}\int_{\alpha}^{t}\gamma\Big[u(s)+q(s)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau\Big]\Delta s \\
+\frac{\gamma}{1-\gamma}q(t)\int_{\alpha}^{\beta}r(\tau)u(\tau)\Delta\tau.
\end{aligned}
$$

Now an application of Theorem 3.1 yields (4.15). □

Remark 4.4. The result given in theorem 4.2 relates the solutions of (1.1) and (4.7) in the sense that if f is close to g and x_0 is close to z_0, then the solutions of (1.1) and (3.10) are also close to each other.

5. Continuous dependence of Solutions

In this section we study continuous dependence of solutions of (1.1) and its variants. The following theorem deals with the continuous dependence of solution of (1.1) on given initial values.

Theorem 5.1. *Suppose that f, h in (1.1) satisfy (4.8), (4.9). Let $x_i(t)$, $(i = 1, 2)$ be respectively solutions of equation*

$$x^{\Delta}(t)=f(t,x(t),x^{\Delta}(t),Hx(t)), \tag{5.1}$$

with the given initial conditions

$$x_i(\alpha)=c_i, \tag{5.2}$$

on $I_{\mathbb{T}}$, where f, H are as in (1.1) and c_i are given constants. Let λ and $L_2(t)$ be as in (4.4) and (4.5) and

$$Q_1=\frac{1}{1-\lambda}\int_{\alpha}^{\beta}r(\xi)A_1(\xi)\Delta\xi, \tag{5.3}$$

where $A_1(t)$ is defined by the right hand side of (4.14) by replacing $w(t)$ with the expression $|c_1 - c_2|$. Then

$$|x_1(t)-x_2(t)|+|x_1^{\Delta}(t)-x_2^{\Delta}(t)|\le A_1(t)+Q_1L_2(t), \tag{5.4}$$

for $t \in I_{\mathbb{T}}$.

Proof. Let $v(t) = |x_1(t) - x_2(t)| + |x_1^{\Delta}(t) - x_2^{\Delta}(t)|$, $t \in I_{\mathbb{T}}$. From the hypotheses, we have

$$v(t)\le |c_1-c_2|+\int_{\alpha}^{t}|f(s,x_1(s),x_1^{\Delta}(s),Hx_1(s))-f(s,y_2(s),y_2^{\Delta}(s),Hy_2(s))|\Delta s$$

$$
\begin{aligned}
&+ |f(t, x_1(t), x_1^\Delta(t), Hx_1(t)) - f(t, x_2(t), x_2^\Delta(t), Hx_2(t))| \\
&\le |c_1 - c_2| + \int_\alpha^t \gamma\Big[v(s) + \int_\alpha^\beta q(s)r(\tau)v(\tau)\Delta\tau\Big]\Delta s \\
&+ \gamma\Big[v(t) + \int_\alpha^\beta q(t)r(\tau)v(\tau)\Delta\tau\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
v(t) \le \frac{|c_1 - c_2|}{1-\gamma} &+ \frac{1}{1-\gamma}\int_\alpha^t \gamma\Big[v(s) + q(s)\int_\alpha^\beta r(\tau)v(\tau)\Delta\tau\Big]\Delta s \\
&+ \frac{\gamma}{1-\gamma} q(t)\int_\alpha^\beta r(\tau)v(\tau)\Delta\tau.
\end{aligned}
$$

Now applying theorem 3.1 gives (5.4), which shows the dependency of solution of (5.1) on given initial values. □

Remark 5.2. If we put $c_1 = c_2 = 0$, then we have $A_1(t) = 0$, $Q_1 = 0$ and the uniqueness of solutions of equation (5.1) follows.

Now we consider the integrodifferential equations on time scales

$$
z^\Delta(t) = f(t, z(t), z^\Delta(t), Hz(t), \mu), z(\alpha) = z_0, \tag{5.5}
$$

$$
z^\Delta(t) = f(t, z(t), z^\Delta(t), Hz(t), \mu_0), z(\alpha) = z_0, \tag{5.6}
$$

for $t \in I_{\mathbb{T}}$, where H is given as in (1.2), $f \in C_{rd}(I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and μ, μ_0 are parameters.

Our next theorem deals with the dependency of solutions of (5.5) and (5.6) on parameters.

Theorem 5.3. *Suppose that the functions h and f in* (5.5) *and* (5.6) *satisfy respectively the conditions* (4.9) *and*

$$
|f(t, u, v, w, \mu) - f(t, \overline{u}, \overline{v}, \overline{w}, \mu)| \le \gamma[|u - \overline{u}| + |v - \overline{v}| + |w - \overline{w}|], \tag{5.7}
$$

$$
|f(t, u, v, w, \mu) - f(t, u, v, w, \mu_0)| \le m(t)|\mu - \mu_0|, \tag{5.8}
$$

where $0 \le \gamma \le 1$ is a constant and $m \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}_+)$. Let

$$
\overline{m}(t) = m(t) + \int_\alpha^\beta m(s)\Delta s, \tag{5.9}
$$

$\lambda, L_2(t)$ be as in (4.4), (4.5) *and*

$$
Q_2 = \frac{1}{1-\lambda}\int_\alpha^\beta r(\xi)A_2(\xi)\Delta\xi, \tag{5.10}
$$

where $A_2(t)$ is defined by the right hand side of (4.14) *by replacing $w(t)$ with the expression $|\mu - \mu_0|\overline{m}(t)$.*

Let $z_1(t)$ and $z_2(t)$ be respectively, the solutions of (5.5) *and* (5.6) *on $I_{\mathbb{T}}$. Then*

$$
|z_1(t) - z_2(t)| + |z_1^\Delta(t) - z_2^\Delta(t)| \le A_2(t) + Q_2 L_2(t), \tag{5.11}
$$

for $t \in I_{\mathbb{T}}$.

Proof. Let $z(t) = |z_1(t) - z_2(t)| + |z_1^\Delta(t) - z_2^\Delta(t)|$, $t \in I_{\mathbb{T}}$. Using that $z_1(t)$ and $z_2(t)$ are respectively, the solutions of (5.5) and (5.6) and hypotheses, we have

$$\begin{aligned}
z(t) &\le \int_\alpha^t |f(s, z_1(s), z_1^\Delta(s), Hz_1(s), \mu) - f(s, z_2(s), z_2^\Delta(s), Hz_2(s), \mu)|\Delta s \\
&\quad + \int_\alpha^t |f(s, z_2(s), z_2^\Delta(s), Hz_2(s), \mu) - f(s, z_2(s), z_2^\Delta(s), Hz_2(s), \mu_0)|\Delta s \\
&\quad + |f(t, z_1(t), z_1^\Delta(t), Hz_1(t), \mu) - f(t, z_2(t), z_2^\Delta(t), Hz_2(t), \mu)| \\
&\quad + |f(t, z_2(t), z_2^\Delta(t), Hz_2(t), \mu) - f(t, z_2(t), z_2^\Delta(t), Hz_2(t), \mu_0)| \\
&\le \int_\alpha^t \gamma[z(s) + \int_\alpha^\beta q(s)r(\tau)z(\tau)\Delta\tau]\Delta s + \int_\alpha^t m(s)|\mu - \mu_0|\Delta s \\
&\quad + \gamma[z(t) + \int_\alpha^\beta q(t)r(\tau)z(\tau)\Delta\tau] + m(t)|\mu - \mu_0| \\
&= |\mu - \mu_0|\overline{m}(t) + \int_\alpha^t \gamma[z(s) + q(s)\int_\alpha^\beta r(\tau)z(\tau)\Delta\tau]\Delta s \\
&\quad + \gamma[z(t) + q(t)\int_\alpha^\beta r(\tau)z(\tau)\Delta\tau].
\end{aligned}$$

Then we have

$$\begin{aligned}
z(t) &\le \frac{|\mu - \mu_0|\overline{m}(t)}{1-\gamma} + \frac{1}{1-\gamma}\int_\alpha^t \gamma\Big[z(s) + q(s)\int_\alpha^\beta r(\tau)z(\tau)\Delta\tau\Big]\Delta s \\
&\quad + \frac{\gamma}{1-\gamma} q(t)\int_\alpha^\beta r(\tau)z(\tau)\Delta\tau.
\end{aligned}$$

Now applying Theorem 3.1 yields (5.11), which shows the dependency of solutions of (5.5) and (5.6) on parameters. □

Application. It is often difficult to obtain explicitly the solutions to the equations of the form (1.1) and thus need a new insight for handling the qualitative properties of its solutions. The method of integral inequalities with explicit estimates provides a powerful analytic tool in the study of various dynamic equations. It enable us to obtain valuable information about solutions without the need to know in advance the solution explicitly. To illustrate this fact and the main ideas, we consider the following special version of equation (1.1).

$$x^\Delta(t) = F(t, x(t), x^\Delta(t)), \quad x(\alpha) = x_0, \tag{5.12}$$

for $t \in I_{\mathbb{T}}$, where $F : I_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is rd-continuous function and $x(t)$ is unknown function.

Let $y \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ be a function such that $y^\Delta(t)$ exists for $t \in I_{\mathbb{T}}$ and satisfies the inequality

$$|y^\Delta(t) - F(t, y(t), y^\Delta(t))| \le \epsilon, \tag{5.13}$$

for a given $\epsilon > 0$, where it is supposed that the initial condition $y(\alpha) = x_0$ is fulfilled. Then we call $y(t)$ an ϵ-approximate solution with respect to problem (5.12).

The relation between an ϵ-approximate solution of (5.12) and a solution of (5.12) is shown in the following example.

Example. Suppose that the function F in (5.12) satisfies the condition

$$|F(t,u,v) - F(t,\overline{u},\overline{v})| \le \gamma[|u-\overline{u}| + |v-\overline{v}|], \tag{5.14}$$

where $0 \le \gamma < 1$ is a constant. Let $x(t), y(t) \in C_{rd}(I_{\mathbb{T}}, \mathbb{R}^n)$ are respectively a solution of (5.12) and an ϵ-approximate solution of (5.12). Then from (5.12) and (5.13), we have

$$x(t) = x_0 + \int_\alpha^t F(s, x(s), x^\Delta(s))\Delta s, \tag{5.15}$$

and

$$\begin{aligned}
\epsilon(t-\alpha) &\ge \int_\alpha^t |y^\Delta(s) - F(s, y(s), y^\Delta(s))|\Delta s \\
&\ge \Big|\int_\alpha^t \{y^\Delta(s) - F(s, y(s), y^\Delta(s))\}\Delta s\Big| \\
&= \Big|y(t) - x_0 - \int_\alpha^t F(s, y(s), y^\Delta(s))\Big|.
\end{aligned} \tag{5.16}$$

From (5.14)–(5.16), we observe that

$$\begin{aligned}
|y(t)-x(t)| &= \Big|y(t) - x_0 - \int_\alpha^t F(s, y(s), y^\Delta(s))\Delta s \\
&\quad + \int_\alpha^t \{F(s, y(s), y^\Delta(s)) - F(s, x(s), x^\Delta(s))\}\Delta s\Big| \\
&\le \Big|y(t) - x_0 - \int_\alpha^t F(s, y(s), y^\Delta(s))\Delta s\Big| \\
&\quad + \int_\alpha^t |F(s, y(s), y^\Delta(s)) - F(s, x(s), x^\Delta(s))| \\
&\le \epsilon(t-\alpha) + \int_\alpha^t \gamma[|y(s)-x(s)| + |y^\Delta(s) - x^\Delta(s)|]\Delta s.
\end{aligned} \tag{5.17}$$

Also, from (5.12)–(5.14), we observe that

$$\begin{aligned}
&|y^\Delta(t) - x^\Delta(t)| \\
&= \big|y^\Delta(t) - F(t, y(t), y^\Delta(t)) + F(t, y(t), y^\Delta(t)) - F(t, x(t), x^\Delta(t))\big| \\
&\le |y^\Delta(t) - F(t, y(t), y^\Delta(t))| + |F(t, y(t), y^\Delta(t)) - F(t, x(t), x^\Delta(t))| \\
&\le \epsilon + \gamma[|y(t)-x(t)| + |y^\Delta(t) - x^\Delta(t)|].
\end{aligned} \tag{5.18}$$

Let $z(t) = |y(t)-x(t)| + |y^\Delta(t) - x^\Delta(t)|$ for $t \in I_{\mathbb{T}}$. From (5.17) and (5.18), we have

$$z(t) \le \varepsilon(t-\alpha) + \int_\alpha^t \gamma z(s)\Delta s + \epsilon + \gamma z(t). \tag{5.19}$$

From (5.19), we observe that

$$z(t) \le \frac{\varepsilon[1+(t-\alpha)]}{1-\gamma} + \frac{1}{1-\gamma}\int_\alpha^t \gamma z(s)\Delta s. \tag{5.20}$$

Now a suitable application of Theorem 3.1 (when $a(t) = \frac{\varepsilon[1+(t-\alpha)]}{1-\gamma}$, $b(t) = \frac{1}{1-\gamma}$, $f(t) = \gamma, g(t) = 0$) to (5.20) yields

$$z(t) \le \frac{\epsilon[1+(t-\alpha)]}{1-\gamma} + \frac{1}{1-\gamma}\int_\alpha^t \gamma\Big\{\frac{\epsilon[1+(\tau-\alpha)]}{1-\gamma}\Big\}e_{\gamma(\frac{1}{1-\gamma})}(t,\sigma(\tau))\Delta\tau.$$

From where, we obtain

$$
\begin{aligned}
&|y(t)-x(t)| \\
&\le \frac{\epsilon}{1-\gamma}\Big[[1+(t-\alpha)]+\int_\alpha^t \frac{\gamma[1+(\tau-\alpha)]}{1-\gamma} e_{\gamma(\frac{1}{1-\gamma})}(t,\sigma(\tau))\Delta\tau\Big].
\end{aligned}
\tag{5.21}
$$

Clearly, this estimate provides the relationship between an ϵ-approximate solution of (5.12) and a solution of (5.12), without knowing in advance their explicit solutions. Moreover, from (5.21), it follows that if $\epsilon = 0$, then the uniqueness of solutions of equation (5.12) is established.

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