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EXISTENCE OF POSITIVE SOLUTIONS FOR SOME NONLINEAR PARABOLIC EQUATIONS IN THE HALF SPACE

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Abstract. We prove the existence of positive solutions to the nonlinear parabolic equation

$$\Delta u - \frac{\partial u}{\partial t} = p(x,t)f(u)$$

in the half space $\mathbb{R}^n_+$, $n \geq 2$, subject to Dirichlet boundary conditions. The function f is nonnegative continuous non-increasing, and the potential p is nonnegative and satisfies some hypotheses related to the parabolic Kato class. We use potential theory arguments to prove our main result.

1. Introduction

In this article, we study the existence and asymptotic behaviour of continuous positive solution, in the sense of distributions, for the nonlinear parabolic equation

$$\begin{gathered}\Delta u - \frac{\partial u}{\partial t} = p(x,t)f(u) \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n_+ \\ u(z,t) = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0,\infty),\end{gathered} \tag{1.1}$$

where u_0 is a nonnegative measurable function in $\mathbb{R}^n_+$, the function $f : (0,\infty) \to [0,\infty)$ is non-increasing and continuous and the potential $p : \mathbb{R}^n_+ \times (0,\infty) \to [0,\infty)$ is measurable and satisfies some hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$ studied in [11, 13].

In this article, we denote $\mathbb{R}^n_+ = \{x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \geq 2$, we denote by $\partial\mathbb{R}^n_+$ the boundary of $\mathbb{R}^n_+$ and by $C(\mathbb{R}^n_+ \times (0,\infty))$ the set of continuous functions in $\mathbb{R}^n_+ \times (0,\infty)$. Note that $x \to \partial\mathbb{R}^n_+$ means that $x = (x', x_n)$ tends to a point $(\xi, 0)$ of $\partial\mathbb{R}^n_+$.

For each nonnegative measurable function f on $\mathbb{R}^n_+$, we denoted

$$P_t f(x) = Pf(x,t) = \int_{\mathbb{R}^n_+} \Gamma(x,t,y,0) f(y)\,dy, \quad t > 0,\ x \in \mathbb{R}^n_+,$$

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where $\Gamma(x,t,y,s)$ is the heat kernel in $\mathbb{R}^n_+ \times (0,\infty)$ with Dirichlet boundary conditions $u=0$ on $\partial\mathbb{R}^n_+ \times (0,\infty)$ given by

$$\Gamma(x,t,y,s) = (4\pi)^{-n/2}\frac{1}{(t-s)^{n/2}}\exp\big(-\frac{|x-y|}{4(t-s)}\big)(1-\exp\big(-\frac{x_n y_n}{t-s}\big))$$

for $t>s$, $x,y \in \mathbb{R}^n_+$.

We note that the family of kernels $(P_t)_{t>0}$ is sub-Markov semi-group, that is $P_{t+s} = P_t P_s$ for all $s,t>0$ and $P_t 1 \le 1$. We mention that for each nonnegative f on $\mathbb{R}^n_+$, the map $(x,t) \to P_t f(x)$ is lower semi-continuous on $\mathbb{R}^n_+$ and becomes continuous if f is further bounded. Moreover, let w be a nonnegative superharmonic function on $\mathbb{R}^n_+$, then for every $t>0$, $P_t w \le w$ and consequently the mapping $t \to P_t w$ is non-increasing.

The motivation for our study are the results presented in [6, 7, 8, 9, 10, 11, 12, 13, 15] and their references. Zhang [15] gave an existence result of the parabolic problem

$$\begin{gathered} \Delta u - \frac{\partial u}{\partial t} + q(x,t)u^p = 0, \quad \text{in } D\times(0,\infty) \\ u(x,0) = u_0(x), \quad x \in D, \end{gathered} \tag{1.2}$$

where $D=\mathbb{R}^n (n\ge 3)$, u_0 is a bounded function of class $C^2(\mathbb{R}^n)$ and $q(x,t)$ is in the parabolic Kato class $P^\infty(\mathbb{R}^n)$ which was introduced in [16].

Inspired by the papers by Zhang [15] and Zhang and Zhao [14], Maatoug and Riahi introduced for the case of the half space a parabolic Kato class $P^\infty(\mathbb{R}^n_+)$ and gave an existence result for (1.2) where $D=\mathbb{R}^n_+$.

Maagli et al [11] studied the problem

$$\begin{gathered} \Delta u - u\varphi(.,u) - \frac{\partial u}{\partial t} = 0 \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n_+ \\ u = 0 \quad \text{in } \partial\mathbb{R}^n_+ \times (0,\infty), \end{gathered} \tag{1.3}$$

where u_0 is a nonnegative measurable function defined on $\mathbb{R}^n_+$ and satisfies some properties which allows u_0 to be not bounded, the perturbed nonlinear term $u\varphi(.,u)$ satisfies some hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$.

Under some conditions imposed on the initial value u_0 and the nonlinear term φ, the authors proved in [11] the following result.

Theorem 1.1. *Problem* (1.3) *has a positive continuous solution u in $\mathbb{R}^n_+ \times (0,\infty)$ satisfying*

$$cP_t u_0(x) \le u(x,t) \le P_t u_0(x),$$

for each $t>0$ and $x\in\mathbb{R}^n_+$, where $c\in(0,1)$.

The elliptic counterpart of the problem (1.1) was studied in [2]. There the author proved existence and nonexistence results for the semilinear elliptic equation

$$\begin{gathered} \Delta u = g(u) \quad \text{in } D \\ u = \varphi \quad \text{on } \partial D, \end{gathered} \tag{1.4}$$

where D is a simply connected bounded C^2-domain in $\mathbb{R}^d$ $(d\ge 3)$, g is a continuous function on $(0,\infty)$ such that $0 \le g(u) \le \max(1, u^{-\alpha})$, for $0<\alpha<1$ and φ is a nontrivial nonnegative continuous function on ∂D. More precisely, Athreya [2] proved the following result.

Theorem 1.2. *There exists $0 < c_1 < \infty$ such that if $u \in C(\partial D)$ and $\varphi(x) \geq (1+c_1)h_0(x)$, then there exists a solution u of* (1.4) *such that $u \geq h_0$, where h_0 is a fixed positive harmonic function.*

Hence it is interesting to discuss the parabolic problem (1.1) by adopting similar techniques as in [11] based on potential theory tools.

For the study of (1.1), a basic assumption on the function p requires to fix a nonnegative superharmonic function ω on $\mathbb{R}^n_+$ satisfying condition (H0) defined as follows.

Definition 1.3. We say that a nonnegative superharmonic function w satisfies condition (H0) if w is locally bounded in $\mathbb{R}^n_+$ such that the map $(x,t) \mapsto Pw(x,t)$ is continuous in $\mathbb{R}^n_+ \times (0,\infty)$ and $\lim_{x\to\partial\mathbb{R}^n_+} P_t w(x) = 0$, for every $t > 0$.

To illustrate the above definition, we consider the following examples of functions satisfying (H0); see [11].

- Every bounded nonnegative superharmonic function ω in $\mathbb{R}^n_+$ satisfies (H0).
- $\omega(x) = x_n^\beta$, $0 < \beta \leq 1$. Indeed, $\Delta\omega = \beta(1-\beta)\omega^{\frac{\beta-2}{\beta}}$, then ω is a superharmonic function. Moreover, by a simple calculation we obtain

$$\omega(x) - P_t\omega(x) = \int_0^t P_s \omega^{\frac{\beta-2}{\beta}}(x)ds, (x,t) \in \mathbb{R}^n_+ \times (0,\infty).$$

Hence, $P\omega \leq \omega$ and so $\lim_{x\to\partial\mathbb{R}^n_+} P_t\omega(x) = 0$. Furthermore, the function $(x,t) \to \omega(x) - P_t\omega(x)$ is upper semicontinuous, which ensures the continuity of the function $(x,t) \to P_t\omega(x)$.

- $\omega(x) = K\nu(x)$, where ν is a nonnegative measure on $\partial\mathbb{R}^n_+$ satisfying for $0 < \alpha \leq n/2$

$$\sup_{x\in\mathbb{R}^n_+} \int_{\partial\mathbb{R}^n_+} \frac{x_n}{|x-z|^{n-2\alpha}} \nu(dz) < \infty.$$

- $\omega(x) = \Sigma_{p=1}^\infty \min(p, \alpha_p \mathcal{G}(x, e_p))$, where $\mathcal{G}$ is the Green's function of Δ in $\mathbb{R}^n_+$ with zero boundary condition, $e_p = (0,\dots,0,p)$ and $\alpha_p > 0$ is chosen such that $\alpha_p\mathcal{G}(x,e_p) \leq 2^{-p}$ for $x \in B^c(e_p, \frac{1}{2}) \cap \mathbb{R}^n_+$. This last example is studied in [11], where the authors proved that the function ω is an unbounded potential satisfying condition (H0).

For the rest of this article, we fix a nonnegative superhahmonic function ω satisfying the condition (H0), and we assume the following hypotheses:

(H1) The function $f : (0,\infty) \to [0,\infty)$ is nonincreasing and continuous.

(H2) For all $x \in \mathbb{R}^n_+$, we have $\lim_{t\to 0} P_t u_0(x) = u_0(x)$ and

$$Pu_0 \in C(\mathbb{R}^n_+ \times (0,\infty)) \text{ and } \lim_{x\to\xi\in\partial\mathbb{R}^n_+} P_t u_0(x) = 0. \tag{1.5}$$

We note that if there exists $c > 0$ such that $0 \leq u_0 \leq c\omega$, then (1.5) is satisfied.

(H3) $p : \mathbb{R}^n_+ \times (0,\infty) \to [0,\infty)$ is measurable such that the function

$$\widetilde{p} := \frac{pf(P\omega)}{P\omega}$$

belongs to the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$.

Before stating our main result, we give an example where (H3) is satisfied.

Example 1.4. Let f be a non-increasing continuous function such that there exists $\eta > 0$ satisfying
$$0 \le f(t) \le \eta(t+1) \quad \forall t > 0.$$
Let $\omega(x) = x_n$, $x \in \mathbb{R}^n_+$ and let p be a nonnegative function such that
$$p \le \frac{\omega}{1+\omega} q$$
for some $q \in P^\infty(\mathbb{R}^n_+)$. Then we have
$$\widetilde{p} = \frac{pf(P\omega)}{P\omega} = \frac{pf(\omega)}{\omega} \le \eta \frac{1+\omega}{1+\omega} q = \eta q$$
which belongs to $P^\infty(\mathbb{R}^n_+)$.

More examples where (H3) is satisfied will be developed in section 4. Now, we give our main result.

Theorem 1.5. *Under the assumptions* (H1)–(H3)*, there exist a constant* $c > 1$ *such that if* $u_0 \ge c\omega$ *on* $\mathbb{R}^n_+$*, then* (1.1) *has a positive continuous solution* u *satisfying, for each* $x \in \mathbb{R}^n_+$ *and* $t > 0$,
$$P_t\omega(x) \le u(x,t) \le P_t u_0(x).$$

The outline of this article is as follows. In section 2, we give some notations and we recall some properties of the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$. Section 3 concerns the proof of Theorem 1.5 by using a potential theory approach. The last section is reserved for examples.

2. Preliminary results

In this section we collect some useful results concerning the parabolic Kato class $P^\infty(\mathbb{R}^n_+)$, which is stated in [11] and [13].

Definition 2.1 ([11])**.** A Borel measurable function q in $\mathbb{R}^n_+ \times \mathbb{R}$ belongs to the class $P^\infty(\mathbb{R}^n_+)$ if for all $c > 0$,
$$\lim_{h\to 0} \sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} \int_{t-h}^{t+h} \int_{B(x,\sqrt{h})\cap\mathbb{R}^n_+} \min(1, \frac{y_n^2}{|t-s|}) G_c(x,|t-s|,y,0)|q(y,s)|dyds = 0$$
and
$$\sup_{(x,t)\in\mathbb{R}^n_+\times\mathbb{R}} \int_{-\infty}^{+\infty} \int_{\mathbb{R}^n_+} \min(1, \frac{y_n^2}{|t-s|}) G_c(x,|t-s|,y,0)|q(y,s)|dyds < \infty,$$
where
$$G_c(x,t,y,s) := \frac{1}{(t-s)^{n/2}} exp(-c\frac{|x-y|^2}{t-s}), \quad t > s, x, y \in \mathbb{R}^n_+.$$

Remark 2.2. The parabolic Kato class $P^\infty(\mathbb{R}^n_+)$ is quite rich. In particular, it contains the time independent Kato class $K^\infty(\mathbb{R}^n_+)$ used in the study of elliptic equations (See [3, 4] for definition and properties).

Other examples of functions belonging to $P^\infty(\mathbb{R}^n_+)$ are given by the following proposition.

Proposition 2.3 ([11])**.** (i) $L^\infty(\mathbb{R}^n_+) \otimes L^1(\mathbb{R}) \subset P^\infty(\mathbb{R}^n_+)$.
(ii) $K^\infty(\mathbb{R}^n_+) \otimes L^\infty(\mathbb{R}) \subset P^\infty(\mathbb{R}^n_+)$.

(iii) *For $1 < p < +\infty$ and $q \geq 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then for $s > \frac{np}{2}$ and $\delta < \frac{2}{p} - \frac{n}{s} < \nu$ we have*
$$\frac{L^s(\mathbb{R}^n_+)}{\theta(.)^\delta(1+|.|)^{\nu-\delta}} \otimes L^q(\mathbb{R}^n_+) \subset P^\infty(\mathbb{R}^n_+),$$
where θ is defined on $\mathbb{R}^n_+$ by $\theta(x) = x_n$.

We state now an elementary inclusion of the class $P^\infty(\mathbb{R}^n_+)$ as follows.

Proposition 2.4 ([11]). *Let $q \in P^\infty(\mathbb{R}^n_+)$, then the function $(y,s) \mapsto y_n^2 q(y,s)$ is in $L^1_{\rm loc}(\overline{\mathbb{R}^n_+} \times \mathbb{R})$. In particular, we have $P^\infty(\mathbb{R}^n_+) \subset L^1_{\rm loc}(\mathbb{R}^n_+ \times \mathbb{R})$.*

For any nonnegative measurable function f in $\mathbb{R}^n_+ \times (0,\infty)$, we denote
$$Vf(x,t) := \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x,t,y,s) f(y,s)\,dy\,ds = \int_0^t P_{t-s}(f(.,s))(x)\,ds$$
and we give the following propositions that will be useful in proving the existence and continuity of solutions to (1.1).

Proposition 2.5 ([11]). *Let q be a nonnegative function in $P^\infty(\mathbb{R}^n_+)$ then there exists a positive constant α_q such that for each nonnegative superharmonic function v in $\mathbb{R}^n_+$,*
$$V(qPv)(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x,t,y,s) f(y,s) P_t v(y)\,dy\,ds \leq \alpha_q P_t v(x),$$
for every $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$.

Proposition 2.6 ([11]). *Let w be a nonnegative superharmonic function in $\mathbb{R}^n_+$ satisfying (H0) and q be a nonnegative function in $P^\infty(\mathbb{R}^n_+)$ then the family of functions*
$$\Big\{(x,t) \to Vf(x,t) = \int_0^t \int_{\mathbb{R}^n_+} \Gamma(x,t,y,s) f(y,s)\,dy\,ds,\ |f| \leq qPw\Big\}$$
is equicontinuous in $\mathbb{R}^n_+ \times (0,\infty)$. Moreover, for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, we have
$$\lim_{s\to 0} Vf(x,s) = \lim_{y \to \partial\mathbb{R}^n_+} Vf(y,t) = 0,$$
uniformly on f.

We will apply the following auxiliary result, several times in this article.

Proposition 2.7. *Let ω be a nonnegative superharmonic function satisfying condition* (H0) *and φ be a nonnegative measurable function such that $\varphi \leq \omega$ on $\mathbb{R}^n_+$, then the function $(x,t) \to P_t\varphi(x)$ is continuous on $\mathbb{R}^n_+ \times (0,\infty)$ and $\lim_{x\to\partial\mathbb{R}^n_+} P_t\varphi(x) = 0$, for every $t > 0$.*

Proof. For each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, we write
$$P_t\omega(x) = P_t\varphi(x) + P_t(\omega - \varphi)(x).$$
So, from (H0) we have $(x,t) \to P_t\omega(x)$ is continuous in $\mathbb{R}^n_+ \times (0,\infty)$ and from the fact that $(x,t) \to P_t\varphi(x)$ and $(x,t) \to P_t(\omega-\varphi)(x)$ are lower semicontinuous in $\mathbb{R}^n_+ \times (0,\infty)$, we deduce that $(x,t) \to P_t\varphi(x)$ is continuous in $\mathbb{R}^n_+ \times (0,\infty)$. On the other hand since $0 \leq P_t\varphi \leq P_t\omega$ and $\lim_{x\to\partial\mathbb{R}^n_+} P_t\omega(x) = 0$, then we have $\lim_{x\to\partial\mathbb{R}^n_+} P_t\varphi(x) = 0$, for every $t > 0$. $\square$

3. Proof of theorem 1.5

Let $\widetilde{p}$ be the function given in hypothesis (H3) and let $\alpha_{\widetilde{p}}$ be the constant defined in Proposition 2.5. We put $c := 1 + \alpha_{\widetilde{p}}$ and we consider a nonnegative continuous function u_0 on $\mathbb{R}^n_+$ such that $u_0 \geq c\omega$. Let Λ be the non-empty closed convex set given by

$$\Lambda = \{v \,\text{measurable function in}\; \mathbb{R}^n_+ \times (0,\infty) : P\omega \leq v \leq Pu_0\}.$$

We define the integral operator T on Λ by

$$T(v) = Pu_0 - V(pf(v)).$$

We aim to prove that T has a fixed point u in Λ. First, we prove that T maps Λ into itself. Let $v \in \Lambda$, since $v \geq P\omega \geq 0$, we have

$$Tv \leq Pu_0.$$

Furthermore, by the monotonicity of the function f we have

$$\begin{aligned} Tv &= Pu_0 - V(pf(v)) \\ &\geq Pu_0 - V(\widetilde{p}P\omega) \\ &\geq c_1 P\omega - \alpha_{\widetilde{p}} P\omega \\ &\geq (c_1 - \alpha_{\widetilde{p}})P\omega \geq P\omega. \end{aligned}$$

Secondly, we claim that T is nondecreasing on Λ. Indeed, let $u, v \in \Lambda$ such that $u \leq v$. Then it follows from the monotonicity of the function f that

$$Tv - Tu = V(p(f(u) - f(v))) \geq 0.$$

Now, we define the sequence $(v_k)_{k\in\mathbb{N}}$ by

$$v_0 = P\omega \quad \text{and} \quad v_{k+1} = Tv_k, \quad \text{for } k \in \mathbb{N}.$$

Since $T\Lambda \subset \Lambda$, then from the monotonicity of T, we obtain for all $k \in \mathbb{N}$

$$P\omega \leq v_k \leq v_{k+1} \leq Pu_0.$$

So, the sequence $(v_k)_{k\in\mathbb{N}}$ converge to a function $u \in \Lambda$. Moreover, using hypothesis (H3) and the monotonicity of the function f we obtain for each $k \in \mathbb{N}$

$$pf(v_k) \leq pf(Pw) = \widetilde{p}P\omega.$$

So, by Proposition 2.5 and Lebesgue's theorem we deduce that $V(pf(v_k)$ converges to $V(pf(u))$ as k tends to infinity. Then, on $\mathbb{R}^n_+ \times (0,\infty)$, u satisfies

$$u = Pu_0 - V(pf(u)). \tag{3.1}$$

At the remainder of the proof, we aim to show that u is a desired solution of (1.1). It is obvious that

$$pf(u) \leq \widetilde{p}Pw. \tag{3.2}$$

So, from the hypothesis (H0) and Proposition 2.4, we deduce that

$$pf(u) \in L^1_{\rm loc}(\mathbb{R}^n_+ \times (0,\infty))$$

moreover, by (3.2) and Proposition 2.6, we obtain

$$V(pf(u)) \in C(\mathbb{R}^n_+ \times (0,\infty)) \subset L^1_{\rm loc}(\mathbb{R}^n_+ \times (0,\infty)).$$

In addition, using (1.5) and Proposition 2.7 we obtain

$$Pu_0 \in C(\mathbb{R}^n_+ \times (0,\infty)).$$

Thus, by (3.1) it follows that $u \in C(\mathbb{R}^n_+ \times (0,\infty))$.

Now, applying the heat operator $\Delta - \frac{\partial}{\partial t}$ in (3.1), we obtain clearly that u is a positive continuous solution (in the distributional sense) of

$$\Delta u - \frac{\partial u}{\partial t} = p(x,t)f(u) \text{ in } \mathbb{R}^n_+ \times (0,\infty).$$

Next, using (1.5) and hypothesis (H2), it follows that

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} P_t u_0(x) = u_0(x) \quad \text{and} \quad \lim_{x\to\xi\in\partial\mathbb{R}^n_+} P_t u_0(x) = 0.$$

Finally, from (3.2) and Proposition 2.6, we conclude that for each $x \in \mathbb{R}^n_+$ we have

$$\lim_{t\to 0} V(pf(u))(x,t) = 0.$$

Hence, u is a positive continuous solution in $\mathbb{R}^n_+ \times (0,\infty)$ of the problem (1.1). This completes the Proof.

4. Examples

In this section we give some examples. The first one concerns functions satisfying the hypothesis (H3), the second is an application of Theorem 1.5.

Example 4.1. Let f be a nonnegative bounded continuous function on $(0,\infty)$ and σ be a nonnegative measure on $\partial\mathbb{R}^n_+$ satisfying

$$\sup_{x\in\mathbb{R}^n_+} \int_{\partial\mathbb{R}^n_+} \frac{x_n}{|x-z|^{n-2\alpha}} \sigma(dz) < \infty,$$

for some $0 < \alpha \leq n/2$. Then, it was shown in [11], that the harmonic function defined on $\mathbb{R}^n_+$ by

$$K\sigma(x) := \Gamma(\frac{n}{2})\pi^{-n/2} \int_{\partial\mathbb{R}^n_+} \frac{x_n}{|x-z|^n} \sigma(dz)$$

satisfies condition (H0).

Now, let $\omega = K\sigma$ and let p be a nonnegative function such that $p \leq qP(\omega)$ for some $q \in P^\infty(\mathbb{R}^n_+)$, then

$$\widetilde{p} = \frac{pf(P\omega)}{P\omega} \leq ||f||_\infty q \in P^\infty(\mathbb{R}^n_+).$$

Hence, hypothesis (H3) is satisfied.

Example 4.2. Let $1 \leq s < \infty$ and $r \geq 1$ such that $\frac{1}{s} + \frac{1}{r} = 1$. Let $\sigma \geq \frac{ns}{2}$ and $\rho < \frac{2}{s} - \frac{n}{\sigma} < \mu$. For each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$, We put

$$p(x,t) = \frac{|g(x)|}{x_n^{\rho-(\gamma+1)}(1+|x|)^{\mu-\rho}} |h(t)|,$$

where $\gamma > 0$, $g \in L^\sigma(\mathbb{R}^n_+)$ and $h \in L^r(\mathbb{R})$.

Let u_0 be a nonnegative continuous function on $\mathbb{R}^n_+$ satisfying hypothesis (H2). Then, there exist a constant $c > 1$ such that if $u_0(x) \geq cx_n$, for all $x \in \mathbb{R}^n_+$, the problem

$$
\begin{gathered}
\Delta u - \frac{\partial u}{\partial t} = p(x,t)u^{-\gamma} \quad \text{in } \mathbb{R}^n_+ \times (0,\infty) \\
u(x,0) = u_0(x) \quad \text{in } \mathbb{R}^n_+ \\
u(z,t) = 0 \quad \text{on } \partial\mathbb{R}^n_+ \times (0,\infty),
\end{gathered}
$$

has a positive continuous solution u satisfying, for each $(x,t) \in \mathbb{R}^n_+ \times (0,\infty)$,

$$x_n \leq u(x,t) \leq P_t u_0(x).$$

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