Electronic Journal of Differential Equations, Vol. 2010(2010), No. 145, pp. 1–20.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

TRANSPORT EQUATION IN CELL POPULATION DYNAMICS II

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Abstract. In this work, we study the cellular profile in a cell proliferating model presented in [4]. Each cell is characterized by its degree of maturity and its maturation velocity. The boundary conditions generalizes the known biological rules. We study also the degenerate case corresponding to infinite maturation velocity, and describe mathematically the cellular profile.

1. Introduction

In this work, we continue the work in [4] in which we studied the transport equation

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial \mu} = -\sigma f + \int_a^b r(\mu, v, v')f(t, \mu, v')dv' \tag{1.1}$$

describing a cell proliferating. Here $f = f(t, \mu, v)$ is the cell density at time $t \geq 0$, $r(\mu, v, v')$ is the *transition rate* at which cells change their velocities from v to v' and

$$\sigma(\mu, v) = \int_a^b r(\mu, v', v)dv' \tag{1.2}$$

is the rate of cell mortality or cell loss due to causes other than division.

Each cell is distinguished by its degree of maturity $\mu \in (0, 1)$ and its maturation velocity v $(0 \leq a < v < b \leq \infty)$. During each cell mitotic, the degree of maturity of a mother cell is $\mu = 1$ and those of its daughter cells are $\mu = 0$.

We equip (1.1) with the *general biological rule* mathematically described by the boundary condition

$$f(t, 0, v) = [Kf(t, 1, \cdot)](v) \tag{1.3}$$

where, K is a linear operator into suitable spaces (see section 3).

This model was proposed, in [11], for the *transition biological rule* mathematically described by the operator

$$K\psi(v) = \frac{\beta}{v}\int_a^b k(v, v')\psi(v')v'\, dv' \tag{1.4}$$

2000 *Mathematics Subject Classification.* 92C37, 82D75.
Key words and phrases. Semigroups; operators; boundary value problem;
cell population dynamic; general boundary condition.

Submitted May 23, 2010. Published October 12, 2010.
Supported by LMCM-RMI.

where, $\beta \geq 0$ is the average number of daughter cells viable per mitotic and k expresses the correlation between the maturation velocity of a mother cell v' and that of its daughter v.

In [11], only a numerical study has been made for the transition biological rule (1.4) and since then, the model has been rarely studied because there are no methods or technics to study such a model.

When $0 < a < b < \infty$, we have proved, in [3], that the model (1.1)-(1.4) is governed by a strongly continuous semigroup and we have described its asymptotic behavior for the biological interesting case (i.e., $\beta > 1$).

When $0 < a < b = \infty$, then maturation velocities are obviously not bounded and so all announced results in [3] do not hold. Then, we have recently proved, in [4], that the general model (1.1)-(1.3) is governed by a strongly continuous semigroup and we have given its explicit expression which is very useful in the sequel. At the end of [4], we have set the following natural question: What happens when the cell density is increasing?

In this work, we are concerned with the question above which has no answers in the mathematical literature of the model (1.1)-(1.3). We organize this work as follows: Mathematical preliminaries; setting of the problem; positivity, irreducibility and domination; spectral properties; asymptotic behavior.

In Section 3, we recall some proved results about the model (1.1)-(1.3). In Sections 4 and 5, we study the positivity and the irreducibility of the semigroup solving the model (1.1)-(1.3) and we give its spectral properties. We end this work by describing the asymptotic behavior of this semigroup in the uniform topology as follows

Lemma 1.1 ([6, Theorem 9.10 and 9.11])**.** *Let $(T(t))_{t\geq 0}$ be a positive and irreducible strongly continuous semigroup on the Banach lattice X satisfying the inequality $\omega_{\rm ess}(T(t)) < \omega_0(T(t))$. Then, there exist a rank one projector $\mathbb{P}$ into X and positive constants M and δ such that*

$$\|e^{-s(A)t}T(t) - \mathbb{P}\|_{\mathcal{L}(X)} \leq Me^{-\delta t}, \quad t \geq 0.$$

A strongly continuous semigroup $(T(t))_{t\geq 0}$ satisfying Lemma above possesses an *asynchronous exponential growth* with *intrinsic growth constant* $s(T)$. The result above describes the cellular profile whose privileged direction is given by the projector $\mathbb{P}$. This is what the biologist observes in his laboratory. Finally, some of these results were announced in [5] and here we explicitly state the detailed conditions and outline all the proofs. For all theoretical results used here, we refer the reader to [9].

2. Mathematical preliminaries

Let X be a Banach space and let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup whose generator is A. The *type* $\omega(T(t))$ and the *essential type* $\omega_{\rm ess}(T(t))$ of the semigroup $(T(t))_{t\geq 0}$ are characterized by

$$\omega(T(t)) = \lim_{t\to\infty} \frac{\ln \|T(t)\|_{\mathcal{L}(X)}}{t}, \tag{2.1}$$

$$\omega_{\rm ess}(T(t)) = \lim_{t\to\infty} \frac{\ln \|T(t)\|_{\rm ess}}{t}. \tag{2.2}$$

Note that $\|C\|_{\text{ess}} = 0$ if and only if C is a compact operator. The spectral bound $s(A)$ of the generator A is

$$s(A) = \begin{cases} \sup\{\mathrm{Re}(\lambda),\ \lambda \in \sigma(A)\} & \text{if } \sigma(A) \neq \emptyset \\ -\infty & \text{if } \sigma(A) = \emptyset. \end{cases}$$

When X is an L_p space then

$$\omega(T(t)) = s(A); \tag{2.3}$$

see [13]. We need also the following results.

Lemma 2.1 ([6, Proposition 7.1 and 7.6]). *Let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on a Banach lattice space X whose generator is A.*

1. *$(T(t))_{t\geq 0}$ is a positive semigroup if and only if $(\lambda - A)^{-1}$ is a positive operator for some great λ.*
2. *Suppose that $(T(t))_{t\geq 0}$ is a positive semigroup. Then $(T(t))_{t\geq 0}$ is an irreducible semigroup if and only if $(\lambda - A)^{-1}$ is an irreducible operator for some large λ.*

Lemma 2.2 ([8]). *Let (Ω, Σ, μ) be a positive measure space and S, T be bounded linear operators on $L^1(\Omega, \mu)$.*

1. *The set of all weakly compact operators is norm-closed subset.*
2. *If T is weakly compact and $0 \leq S \leq T$, then S is weakly compact.*
3. *If S and T are weakly compact, then ST is compact.*

Lemma 2.3 ([12]). *Let (Ω, Σ, μ) be a positive measure space. Let A and $A + B$ be the generators, on $L^1(\Omega)$, of strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(U(t))_{t\geq 0}$ where, B is a linear bounded operator from $L^1(\Omega)$ into itself. Assume that $BT(t_1)BT(t_2)\cdots BT(t_n)$ is compact for some $n \in \mathbb{N}^*$ and for every $t_1, \cdots, t_n > 0$. Then, $\omega_{\text{ess}}(U(t)) = \omega_{\text{ess}}(T(t))$.*

3. Setting of the problem

In this section, we recall some facts about the model (1.1)-(1.3) already studied in [4]. Before we start, we suppose that the useful condition

$$a > 0 \tag{3.1}$$

holds in this work. So, let us consider the framework $L^1(\Omega)$ with norm

$$\|\varphi\|_1 = \int_\Omega |\varphi(\mu, v)|\, d\mu\, dv \tag{3.2}$$

where, $\Omega = (0,1) \times (a, \infty) := I \times J$ and let $W(\Omega)$ be the Sobolev space

$$W(\Omega) = \{\varphi \in L^1(\Omega),\ v\frac{\partial\varphi}{\partial\mu} \in L^1(\Omega) \text{ and } v\varphi \in L^1(\Omega)\}$$

whose norm is

$$\|\varphi\|_{W(\Omega)} = \|v\varphi\|_1 + \|v\frac{\partial\varphi}{\partial\mu}\|_1.$$

Finally, let $Y_1 := L^1(J,\ vdv)$ be the trace space whose norm is

$$\|\psi\|_{Y_1} = \int_a^\infty |\psi(v)| v dv.$$

Lemma 3.1 ([2]). *The trace mapping $\gamma_0\varphi = \varphi(0,\cdot)$ and $\gamma_1\varphi = \varphi(1,\cdot)$ are continuous from $W(\Omega)$ into Y_1.*

In this context, we introduce a boundary operator K from Y_1 into itself allowing us to define the operator A_K by

$$\begin{gathered} A_K\varphi = -v\frac{\partial\varphi}{\partial\mu} \text{ on the domain} \\ D(A_K) = \{\varphi \in W(\Omega),\ \text{satisfying } \gamma_0\varphi = K\gamma_1\varphi\}. \end{gathered} \tag{3.3}$$

When $K = 0$, it is easy to check that the corresponding operator A_0 has some properties summarized as follows

Lemma 3.2. *Let A_0 be the unbounded operator*

$$A_0\varphi = -v\frac{\partial\varphi}{\partial\mu} \text{ on the domain } D(A_0) = \{\varphi \in W(\Omega),\ \gamma_0\varphi = 0\}. \tag{3.4}$$

Then

(1) *A_0 generates, on $L^1(\Omega)$, a strongly continuous semigroup $(U_0(t))_{t\geq 0}$, given by*
$$U_0(t)\varphi(\mu, v) = \chi(\mu, v, t)\varphi(\mu - tv, v) \tag{3.5}$$
where
$$\chi(\mu, v, t) = \begin{cases} 1 & \text{if} \quad \mu \geq tv; \\ 0 & \text{if} \quad \mu < tv. \end{cases} \tag{3.6}$$
(2) *$U_0(t) = 0$ (and so compact) for all $t > 1/a$.*
(3) *For all $\lambda > 0$, the operator $(\lambda - A_0)^{-1}$ (resp. $\gamma_1(\lambda - A_0)^{-1}$) is strictly positive from $L^1(\Omega)$ into $L^1(\Omega)$ (resp. Y_1).*

In the general case, we set the following definition.

Definition 3.3. *Let K be a linear operator from Y_1 into itself. Then, K is said to be an* admissible *operator if (K is bounded and $\|K\|_{\mathcal{L}(Y_1)} < 1$) or ($K$ is compact and $\|K\|_{\mathcal{L}(Y_1)} \geq 1$).*

Lemma 3.4 ([4]). *Let K be an admissible operator and let K_λ be the operator*

$$K_\lambda := \theta_\lambda K \quad \textit{where} \quad \theta_\lambda(v) = e^{-\frac{\lambda}{v}}. \tag{3.7}$$

Then, there exists a real constant $\omega_0 := \omega_0(K) \geq 0$ such that

$$\omega_0 \begin{cases} = 0, & \text{if } K \text{ is bounded and } \|K\|_{\mathcal{L}(Y_1)} < 1; \\ \geq 0, & \text{if } K \text{ is compact and } \|K\|_{\mathcal{L}(Y_1)} \geq 1. \end{cases} \tag{3.8}$$

satisfying $\|K_{\omega_0}\| \leq 1$ and

$$\lambda > \omega_0 \Longrightarrow \|K_\lambda\| < 1. \tag{3.9}$$

The number $\omega_0 := \omega_0(K)$ is called the *abscissa* of the admissible K. In this context, the unbounded operator defined by (3.3) satisfies the following result.

Lemma 3.5 ([4]). *Let K be an admissible operator whose abscissa is ω_0.*

(1) *For all $\lambda > \omega_0$, the resolvent operator of (3.3) is given by*
$$(\lambda - A_K)^{-1}g = \varepsilon_\lambda K(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}g + (\lambda - A_0)^{-1}g \tag{3.10}$$
for all $g \in L^1(\Omega)$, where $\varepsilon_\lambda(\mu, v) = e^{-\lambda\frac{\mu}{v}}$.

(2) *The operator defined by* (3.3) *generates, on* $L^1(\Omega)$*, a strongly continuous semigroup* $(U_K(t))_{t\geq 0}$ *satisfying*
$$\|U_K(t)\varphi\|_1 \leq e^{\frac{\omega_0}{a}} e^{t\omega_0}\|\varphi\|_1, \quad t \geq 0, \tag{3.11}$$
for all $\varphi \in L^1(\Omega)$.

(3) *The operator* $U_K(t)$ *is given by*
$$U_K(t) = U_0(t) + B_K(t), \quad t \geq 0, \tag{3.12}$$
where
$$B_K(t)\varphi(\mu, v) = \xi(\mu, v, t)\big[K\gamma_1 U_K\big(t - \frac{\mu}{v}\big)\varphi\big](v) \tag{3.13}$$
for almost all $(\mu, v) \in \Omega$*, with*
$$\xi(\mu, v, t) = \begin{cases} 1 & \text{if} \quad \mu < tv; \\ 0 & \text{if} \quad \mu \geq tv. \end{cases} \tag{3.14}$$

(4) *For all* $\varphi \in L^1(\Omega)$*, we have*
$$\int_a^\infty \int_0^t |\gamma_1\left(U_K(x)\varphi\right)(v)| v\, dx\, dv \leq \frac{e^{t\omega}\|\varphi\|_1}{1 - \|K_\omega\|_{\mathcal{L}(Y_1)}}, \quad t \geq 0. \tag{3.15}$$

Note that a rank one operator is compact and therefore its admissibility holds. In this case we have the following useful result.

Lemma 3.6 ([4])**.** *Let* K *be a rank one operator in* Y_1*; i.e.,*
$$K\psi = h \int_a^\infty k(v')\psi(v')v'\, dv', \quad h \in Y_1,\ k \in L^\infty(J).$$
Then, for all $\varphi \in L^1(\Omega)$*, we have*
$$U_K(t)\varphi = \sum_{m=0}^{\infty} U_m(t)\varphi, \quad t \geq 0, \tag{3.16}$$
where, $U_0(t)$ *is given by* (3.5) *and*
$$\begin{aligned} &U_1(t)\varphi(\mu, v) \\ &= \xi(\mu, v, t)h(v) \int_a^\infty k(v_1)\chi\left(1, v_1, t - \frac{\mu}{v}\right)\varphi\left(1 - \big(t - \frac{\mu}{v}\big)v_1, v_1\right) v_1 dv_1 \end{aligned}$$
and, for $m \geq 2$*, by*
$$\begin{aligned} U_m(t)\varphi(\mu, v) = \xi(\mu, v, t)h(v) &\underbrace{\int_a^\infty \cdots \int_a^\infty}_{m \text{ times}} \prod_{j=1}^{m-1} h(v_j) \prod_{j=1}^{m} k(v_j) \\ &\times \xi\Big(1, v_{m-1}, t - \frac{\mu}{v} - \sum_{i=1}^{(m-2)} \frac{1}{v_i}\Big)\chi\Big(1, v_m, t - \frac{\mu}{v} - \sum_{i=1}^{(m-1)} \frac{1}{v_i}\Big) \\ &\times \varphi\Big(1 - \Big(t - \frac{\mu}{v} - \sum_{i=1}^{m-1} \frac{1}{v_i}\Big)v_m, v_m\Big) v_1 v_2 \cdots v_m\, dv_1 \cdots dv_m. \end{aligned}$$
Furthermore, for all $t \geq 0$ *we have*
$$\lim_{N\to\infty} \|U_K(t) - \sum_{m=0}^{N} U_m(t)\|_{\mathcal{L}(L^1(\Omega))} = 0. \tag{3.17}$$

A stability result about the semigroup $(U_K(t))_{t\geq 0}$ is given as follows.

Lemma 3.7 ([4]). *Let K and K' be two compact operators. Then*
$$\|U_K(t)-U_{K'}(t)\|_{\mathcal{L}(L^1(\Omega))}\leq 4e^{\omega(\frac{1}{a}+t)}\|K-K'\|_{\mathcal{L}(Y_1)},\quad t\geq 0, \tag{3.18}$$
for all ω big enough.

Now, let us define the following two perturbation operators
$$R\varphi(\mu,v)=\int_a^\infty r(\mu,v,v')\varphi(\mu,v')dv',$$
$$S\varphi(\mu,v)=-\sigma(\mu,v)\varphi(\mu,v)$$
where σ is given by (1.2). Let us impose the following hypothesis

(H1) r is measurable positive, and $\sigma\in L^\infty(\Omega)$.

Denoting
$$\underline{\sigma}:=\operatorname{ess\,inf}_{(\mu,v)\in\Omega}\sigma(\mu,v)\quad\text{and}\quad\overline{\sigma}:=\operatorname{ess\,sup}_{(\mu,v)\in\Omega}\sigma(\mu,v),$$
by [4, Lemma 4.1], the operators S and R are bounded from $L^1(\Omega)$ into itself and $S+R$ is a dissipative operator. Furthermore, the following two perturbed operators
$$\begin{gathered}L_K:=A_K+S,\\ D(L_K)=D(A_K)\end{gathered}\tag{3.19}$$
and
$$\begin{gathered}T_K:=L_K+R=A_K+S+R,\\ D(T_K)=D(A_K)\end{gathered}\tag{3.20}$$
are infinitesimal generators as follows.

Lemma 3.8 ([4]). *Suppose that* (H1) *holds and let K be an admissible operator whose abscissa is ω_0. Then*

(1) *The operator defined by* (3.19) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(V_K(t))_{t\geq 0}$ satisfying*
$$\|V_K(t)\varphi\|_1\leq e^{\frac{\omega_0}{a}}e^{t(\omega_0-\underline{\sigma})}\|\varphi\|_1\quad t\geq 0, \tag{3.21}$$
for all $\varphi\in L^1(\Omega)$.

(2) *The operator defined by* (3.20) *generates, on $L^1(\Omega)$, a strongly continuous semigroup $(W_K(t))_{t\geq 0}$. Furthermore, if $\|K\|_{\mathcal{L}(Y_1)}<1$ then*
$$\|W_K(t)\varphi\|_1\leq\|\varphi\|_1\quad t\geq 0, \tag{3.22}$$
for all $\varphi\in L^1(\Omega)$.

Remark 3.9. The corresponding case to $\|K\|_{\mathcal{L}(Y_1)}<1$ is biologically uninteresting because the cell density is decreasing. Indeed, for all $t\geq 0$ and $s\geq 0$ such that $t>s$, (3.22) leads to
$$\|W_K(t)\varphi\|_1=\|W_K(t-s)W_K(s)\varphi\|_1\leq\|W_K(s)\varphi\|_1$$
for all initial data $\varphi\in L^1(\Omega)$ $(p\geq 1)$.

4. Positivity, irreducibility and domination

In this section, we are concerned with the positivity and the irreducibility of the generated semigroup $(W_K(t))_{t\geq 0}$. We end this section by a domination result.

Lemma 4.1. *Let K be an admissible operator whose abscissa is ω_0. Then*

1. *If K is positive, then the semigroup $(U_K(t))_{t\geq 0}$ is positive too.*
2. *Suppose that K is positive. If K is irreducible, then the positive semigroup $(U_K(t))_{t\geq 0}$ is irreducible.*

Proof. (1) Let $\lambda > \omega_0$ and $g \in (L^1(\Omega))_+$. First, as K is a positive operator, then K_λ $(\lambda \geq 0)$ (given by (3.7)) is a positive operator because of

$$K_\lambda \geq e^{-\lambda/a} K. \tag{4.1}$$

Next, by (3.9) and (3.10) we are led to

$$\begin{aligned}
(\lambda - A_K)^{-1} g &= \varepsilon_\lambda K (I - K_\lambda)^{-1} \gamma_1 (\lambda - A_0)^{-1} g + (\lambda - A_0)^{-1} g \\
&= \varepsilon_\lambda K \sum_{n\geq 0} K_\lambda^n \gamma_1 (\lambda - A_0)^{-1} g + (\lambda - A_0)^{-1} g
\end{aligned}$$

and therefore

$$(\lambda - A_K)^{-1} g \geq \varepsilon_\lambda K \sum_{n\geq 0} K_\lambda^n \gamma_1 (\lambda - A_0)^{-1} g \tag{4.2}$$

because of the third point of Lemma 3.2 and hence, $(\lambda - A_K)^{-1}$ is a positive operator. Now, the positivity of the semigroup $(U_K(t))_{t\geq 0}$ follows from the first point of Lemma 2.1.

(2) Let $\lambda > \omega_0$ and $g \in (L^1(\Omega))_+$ with $g \neq 0$. First, from the third point of Lemma 3.2, $\gamma_1(\lambda - A_0)^{-1} g$ is a strictly positive function. Hence, by the irreducibility of the positive operator K, there exists an integer $m > 0$ such that

$$K^m \gamma_1 (\lambda - A_0)^{-1} g(v) > 0 \quad \text{for almost all} \quad v \in J. \tag{4.3}$$

Next, (4.2) leads to

$$(\lambda - A_K)^{-1} g \geq \varepsilon_\lambda K K_\lambda^{m-1} \gamma_1 (\lambda - A_0)^{-1} g$$

which implies, by (4.1), that

$$(\lambda - A_K)^{-1} g \geq \varepsilon_\lambda e^{-\frac{\lambda(m-1)}{a}} K^m \gamma_1 (\lambda - A_0)^{-1} g$$

and therefore

$$(\lambda - A_K)^{-1} g(\mu, v) > 0 \quad \text{a.e. } (\mu, v) \in \Omega$$

because of (4.3). Now, the second point of Lemma 2.1 completes the proof. $\square$

The positivity property of the semigroup $(V_K(t))_{t\geq 0}$ is given by the following theorem.

Theorem 4.2. *Suppose that* (H1) *holds and let K be an admissible operator whose abscissa is ω_0.*

1. *If K is positive, then the semigroups $(V_K(t))_{t\geq 0}$ and $(W_K(t))_{t\geq 0}$ are positive. Furthermore, we have*
$$e^{-t\overline{\sigma}} U_K(t) \leq V_K(t) \quad t \geq 0 \tag{4.4}$$
and
$$V_K(t) \leq U_K(t) \quad t \geq 0 \tag{4.5}$$

and

$$V_K(t) \leq W_K(t) \quad t \geq 0. \tag{4.6}$$

(2) *Suppose K is positive. If K is irreducible, then the positive semigroup $(W_K(t))_{t\geq 0}$ is irreducible.*

Proof. (1). Let $t \geq 0$ and $\varphi \in (L^1(\Omega))_+$. Thanks to Lemma above we get the positivity of the semigroup $(U_K(t))_{t\geq 0}$ and therefore

$$e^{-t\overline{\sigma}} U_K(t)\varphi \leq \Big[e^{-t\sigma/n} U_K(\frac{t}{n})\Big]^n \varphi \leq e^{-t\underline{\sigma}} U_K(t)\varphi \leq U_K(t)\varphi$$

for all integers $n \geq 1$. Passing to the limit $n \to \infty$ in the relation above together with Trotter's formula

$$V_K(t)\varphi = \lim_{n\to\infty} \Big[e^{-t\sigma/n} U_K(t/n)\Big]^n \varphi$$

we obtain (4.4) and (4.5) hold. Furthermore, the positivity of the semigroup $(V_K(t))_{t\geq 0}$ obviously follows from that of the semigroup $(U_K(t))_{t\geq 0}$ and (4.4). Next, by (H1) we obtain R is a positive operator and therefore, for all integers $n \geq 1$, we have

$$\big[e^{\frac{t}{n}R} V_K(\frac{t}{n})\big]^n \varphi = \Big[\Big(\sum_{p\geq 0} \frac{\big(\frac{t}{n}R\big)^p}{p!}\Big) V_K(\frac{t}{n})\Big]^n \varphi \geq [I.V_K(\frac{t}{n})]^n \varphi = V_K(t)\varphi$$

because of the positivity of the semigroup $(V_K(t))_{t\geq 0}$. Passing to the limit $n \to \infty$ in the relation above together with Trotter's formula

$$W_K(t)\varphi = \lim_{n\to\infty} \big[e^{\frac{t}{n}R} V_K(\frac{t}{n})\big]^n \varphi$$

we obtain that (4.6) holds and therefore the positivity of the semigroup $(W_K(t))_{t\geq 0}$ follows.

(2) Clearly (4.4) and (4.6) lead to

$$W_K(t) \geq V_K(t) \geq e^{-t\overline{\sigma}} U_K(t), \quad t \geq 0,$$

and therefore the irreducibility of the semigroup $(U_K(t))_{t\geq 0}$ obviously implies that of the semigroup $(W_K(t))_{t\geq 0}$. The proof is now complete. $\square$

We end this section with a domination result.

Theorem 4.3. *Let K be an admissible operator whose abscissa is ω_0 and let K' be a positive admissible operator whose abscissa is ω_0' such that*

$$|K\psi| \leq K'|\psi|$$

for all $\psi \in Y_1$. Then

$$|U_K(t)\varphi| \leq U_{K'}(t)|\varphi|, \quad t \geq 0, \tag{4.7}$$

for all $\varphi \in L^1(\Omega)$.

Proof. First, note that (4.7) is obvious for $t = 0$. So, let $t > 0$ and $\lambda > \max\{\omega_0,\ \omega_0'\}$. For all $\varphi \in L^1(\Omega)$, (3.10) infers that

$$\begin{aligned} |(\lambda - A_K)^{-1}\varphi| &\leq |\varepsilon_\lambda K(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}\varphi| + |(\lambda - A_0)^{-1}\varphi| \\ &\leq \varepsilon_\lambda K'|(I - K_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}\varphi| + |(\lambda - A_0)^{-1}\varphi| \end{aligned}$$

which leads, by (3.9), to

$$\begin{aligned}|(\lambda - A_K)^{-1}\varphi| &\le \varepsilon_\lambda K' \Big|\sum_{n\ge 0} K_\lambda^n \gamma_1(\lambda - A_0)^{-1}\varphi\Big| + |(\lambda - A_0)^{-1}\varphi| \\ &\le \varepsilon_\lambda K' \sum_{n\ge 0} |K_\lambda^n \gamma_1(\lambda - A_0)^{-1}\varphi| + |(\lambda - A_0)^{-1}\varphi| \\ &\le \varepsilon_\lambda K' \sum_{n\ge 0} {K'}_\lambda^n |\gamma_1(\lambda - A_0)^{-1}\varphi| + |(\lambda - A_0)^{-1}\varphi|\end{aligned}$$

and therefore

$$\begin{aligned}|(\lambda - A_K)^{-1}\varphi| &\le \varepsilon_\lambda K' \sum_{n\ge 0} {K'}_\lambda^n \gamma_1(\lambda - A_0)^{-1}|\varphi| + (\lambda - A_0)^{-1}|\varphi| \\ &= \varepsilon_\lambda K'(I - K'_\lambda)^{-1}\gamma_1(\lambda - A_0)^{-1}|\varphi| + (\lambda - A_0)^{-1}|\varphi|\end{aligned}$$

because of the third point of Lemma 3.2. Hence,

$$|(\lambda - A_K)^{-1}\varphi| \le (\lambda - T_{K'})^{-1}|\varphi|$$

and by iteration we are led to

$$|\left[\lambda(\lambda - A_K)^{-1}\right]^n \varphi| \le \left[\lambda(\lambda - T_{K'})^{-1}\right]^n |\varphi|$$

for all integers $n > 0$. Putting $\lambda = \frac{n}{t}$ we obtain

$$|\left[\frac{n}{t}\left(\frac{n}{t} - A_K\right)^{-1}\right]^n |\varphi|| \le \left[\frac{n}{t}\left(\frac{n}{t} - T_{K'}\right)^{-1}\right]^n |\varphi|.$$

Now, Trotter's formula completes the proof. $\square$

5. Spectral properties

In this section, we estimate the type $\omega(W_K(t))$ of the semigroup $(W_K(t))_{t\ge 0}$. This is obtained by the characterization of the spectrum of the generator A_K. Before we start, let us note that a compact operator K is admissible and therefore all semigroups of this work exist. So, let us commence by characterizing all elements of $\sigma_p(A_K)$ belonging to $\mathbb{C}_+ := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \ge 0\}$.

Lemma 5.1. *Let K be a compact operator in Y_1 and let $\lambda \in \mathbb{C}_+$. Then*

$$\lambda \in \sigma(A_K) \Longrightarrow 1 \in \sigma_p(K_\lambda). \tag{5.1}$$

Proof. Let $\lambda \in \mathbb{C}_+$. If $1 \in \rho(K_\lambda)$, then for all $g \in L^1(\Omega)$, the equation

$$h = K_\lambda h + \gamma_1(\lambda - A_0)^{-1} g \tag{5.2}$$

has a unique solution $h \in Y_1$. Let φ be the function

$$\varphi = \varepsilon_\lambda K h + (\lambda - A_0)^{-1} g \in L^1(\Omega). \tag{5.3}$$

So, it is easy to check that

$$\lambda\varphi + v\frac{\partial\varphi}{\partial\mu} = g$$

and as we have

$$\gamma_1\varphi = \gamma_1\left(\varepsilon_\lambda K h + (\lambda - A_0)^{-1} g\right) = K_\lambda h + \gamma_1(\lambda - A_0)^{-1} g = h$$

then

$$\gamma_0\varphi = K h = K\gamma_1\varphi$$

and therefore $\varphi \in D(A_K)$. Hence $(\lambda - A_K)$ is invertible operator, which leads to $\lambda \in \rho(A_K)$. The proof is complete. $\square$

Let us finish this section with the following main result.

Theorem 5.2. *Suppose that* (H1) *holds and let K be a positive, irreducible and compact operator in Y_1 with $r(K) > 1$. Then,*

$$\omega(W_K(t)) > -\infty. \tag{5.4}$$

Proof. We divide the proof in several steps.

Step one. Let $\lambda \geq 0$. As K is a positive and compact operator then K_λ, given by (3.7), is a positive and compact operator too. Furthermore its irreducibility follows from that of the operator K because of $K_\lambda \geq e^{-\lambda/a}K$. Now, by [10], we obtain $r(K_\lambda) > 0$ and there exist a quasi-interior vector ψ_λ of $(Y_1)_+$ and a strictly positive functional $\psi_\eta^* \in (Y_q)_+$ such that

$$\begin{aligned} K_\lambda \psi_\lambda &= r(K_\lambda)\psi_\lambda \quad \text{with} \quad \|\psi_\lambda\|_{Y_1} = 1 \\ K_\lambda^* \psi_\lambda^* &= r(K_\lambda)\psi_\lambda^* \quad \text{with} \quad \|\psi_\lambda^*\|_{Y_q} = 1 \end{aligned} \tag{5.5}$$

where K_λ^* is the adjoint operator of K_λ and $p^{-1} + q^{-1} = 1$. So, in the next step, we prove that the mapping

$$\lambda \longrightarrow r(K_\lambda), \tag{5.6}$$

is continuous and strictly decreasing.

Step two. Let $\lambda \geq 0$ and $\eta \geq 0$. First, writing (5.5) for η, it follows that

$$K_\lambda^* \psi_\eta^* = r(K_\eta)\psi_\eta^* \quad \text{with} \quad \|\psi_\eta^*\|_{Y_q} = 1 \tag{5.7}$$

where, ψ_η is a quasi-interior vector of $(Y_1)_+$ and $\psi_\eta^* \in (Y_q)_+$ is a strictly positive functional and K_η^* is the adjoint operator of K_η. Now, by (5.5) and (5.7) we obtain

$$\begin{aligned} r(K_\eta) &= \frac{\langle K_\eta^* \psi_\eta^*, \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} = \frac{\langle \psi_\eta^*, K_\eta \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} \\ &= \frac{\langle \psi_\eta^*, K_\lambda \psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} + \frac{\langle \psi_\eta^*, (K_\eta - K_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} \\ &= r(K_\lambda) + \frac{\langle \psi_\eta^*, (K_\eta - K_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} \end{aligned}$$

which implies

$$r(K_\eta) - r(K_\lambda) = \frac{\langle \psi_\eta^*, (K_\eta - K_\lambda)\psi_\lambda\rangle}{\langle \psi_\eta^*, \psi_\lambda\rangle} \tag{5.8}$$

and therefore

$$\begin{aligned} |r(K_\eta) - r(K_\lambda)| &\leq \frac{\|\psi_\eta^*\|_{Y_q}}{\langle \psi_\eta^*, \psi_\lambda\rangle} \left\|(K_\eta - K_\lambda)\psi_\lambda\right\|_{Y_1} \\ &= \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\psi \in B} \|K_\eta \psi - K_\lambda \psi\|_{Y_1} \\ &= \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\varphi \in K(B)} \|\theta_\eta \varphi - \theta_\lambda \varphi\|_{Y_1} \\ &\leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda\rangle} \sup_{\varphi \in \overline{K(B)}} \|\theta_\eta \varphi - \theta_\lambda \varphi\|_{Y_1} \end{aligned}$$

where B is the unit ball in Y_1. Thanks to the compactness of $\overline{K(B)}$, there exists $\varphi_0 \in \overline{K(B)}$ such that

$$|r(K_\eta) - r(K_\lambda)| \leq \frac{1}{\langle \psi_\eta^*, \psi_\lambda \rangle} \|\theta_\eta \varphi_0 - \theta_\lambda \varphi_0\|_{Y_1}$$

which leads to

$$\lim_{\eta \to \lambda} |r(K_\eta) - r(K_\lambda)| \leq \lim_{\eta \to \lambda} \frac{1}{\langle \psi_\eta^*, \psi_\lambda \rangle} \|\theta_\eta \varphi_0 - \theta_\lambda \varphi_0\|_{Y_1} = 0$$

because of the dominated convergence Theorem and hence, (5.6) is a continuous mapping. Now, let us prove that (5.6) is a strictly decreasing mapping. So, let $\lambda > \eta \geq 0$, then we have

$$K_\lambda \psi_\lambda = \theta_\lambda K \psi_\lambda = \theta_{\lambda-\eta} \theta_\eta K \psi_\lambda = \theta_{\lambda-\eta} K_\eta \psi_\lambda < K_\eta \psi_\lambda$$

which leads to $K_\eta \psi_\lambda - K_\lambda \psi_\lambda > 0$ and therefore $\langle \psi_\eta^*, (K_\eta - K_\lambda)\psi_\lambda \rangle > 0$ because ψ_η^* is a strictly positive functional of $(Y_q)_+$. Now, thanks to (5.8) we can say that (5.6) is a strictly decreasing mapping.

Step three. First, let us note that $r(K_0) = r(K) > 1$. Next, let B be the unit ball in Y_1. Thanks to the compactness of $\overline{K(B)}$, there exists $\varphi_0 \in \overline{K(B)}$ such that

$$\begin{aligned} \|K_\lambda\|_{\mathcal{L}(Y_1)} &= \sup_{\psi \in B} \|K_\lambda \psi\|_{Y_1} = \sup_{\varphi \in K(B)} \|\theta_\lambda \varphi\|_{Y_1} \\ &\leq \sup_{\varphi \in \overline{K(B)}} \|\theta_\lambda \varphi\|_{Y_1} = \|\theta_\lambda \varphi_0\|_{Y_1} \end{aligned}$$

and therefore

$$\lim_{\lambda \to \infty} r(K_\lambda) \leq \lim_{\lambda \to \infty} \|K_\lambda\|_{\mathcal{L}(Y_1)} \leq \lim_{\lambda \to \infty} \|\theta_\lambda \varphi_0\|_{Y_1} = 0$$

because of the dominated convergence Theorem. So, there exists a unique λ_0 such that

$$\lambda_0 > 0 \quad \text{and} \quad r(K_{\lambda_0}) = 1. \tag{5.9}$$

Step four. Let us prove that $s(A_K) = \lambda_0$. Let $\lambda \in \sigma(A_K) \cap \mathbb{C}_+$. By (5.1), there exists ψ such that $K_\lambda \psi = \psi$ which implies

$$\begin{aligned} |\psi| &= |K_\lambda \psi| = |\theta_\lambda K \psi| \\ &\leq |\theta_\lambda| K |\psi| \leq \theta_{\mathrm{Re}(\lambda)} K |\psi| \\ &= K_{\mathrm{Re}(\lambda)} |\psi|; \end{aligned}$$

therefore $(K_{\mathrm{Re}(\lambda)})^n |\psi| \geq |\psi|$ for all integers n. Clearly $r\left(K_{\mathrm{Re}(\lambda)}\right) \geq 1$ and hence, $\mathrm{Re}\,(\lambda) \leq \lambda_0$ because the mapping (5.6) is strictly decreasing. Whence

$$s(A_K) \leq \lambda_0. \tag{5.10}$$

Conversely, by (5.5) and (5.9) we obtain $K_{\lambda_0} \psi_{\lambda_0} = \psi_{\lambda_0}$. If we set $\varphi = \varepsilon_{\lambda_0} K \psi_{\lambda_0}$ then it is easy to check that

$$-v \frac{\partial \varphi}{\partial \mu} = \lambda_0 \varphi$$

and

$$K \gamma_1 \varphi = K\left[\theta_{\lambda_0} K \psi_{\lambda_0}\right] = K[K_{\lambda_0} \psi_{\lambda_0}] = K \psi_{\lambda_0} = \gamma_0 \varphi$$

which implies that $A_K \varphi = \lambda_0 \varphi$ and therefore $\lambda_0 \in \sigma_1(A_K) \subset \sigma(A_K)$. Whence

$$\lambda_0 \leq s(A_K). \tag{5.11}$$

Thanks to (5.10) and (5.11) and (5.9) we obtain $s(A_K) = \lambda_0 > 0$, and by (2.3) we finally are led to

$$\omega(U_K(t)) > 0. \tag{5.12}$$

Step five. Thanks to (4.4) and (4.6), we can write

$$W_K(t) \geq V_K(t) \geq e^{-t\overline{\sigma}} U_K(t)$$

and therefore,

$$\|W_K(t)\|_{\mathcal{L}(L^1(\Omega))} \geq e^{-t\overline{\sigma}} \|U_K(t)\|_{\mathcal{L}(L^1(\Omega))}.$$

Finally, (2.1) and (5.12) obviously lead to

$$\omega(W_K(t)) = \lim_{t\to\infty} \frac{\ln \|W_K(t)\|_{\mathcal{L}(L^1(\Omega))}}{t} \geq \omega(U_K(t)) - \overline{\sigma} > -\overline{\sigma}.$$

Now, the hypothesis (H1) completes the proof. $\square$

6. Asymptotic Behavior

In this section we are going to give a mathematical description of the cellular profile of the model (1.1)-(1.3). This can be obtained as the asymptotic behavior of the semigroup $(W_K(t))_{t\geq 0}$. To this end, we use the precious assumption (3.1) that is

$$a > 0$$

and we firstly prove the compactness of the semigroup $(U_K(t))_{t\geq 0}$ for $t > \frac{2}{a}$. Actually, when $a > 0$, then after a transitory phase, all cells will be divided or dead. This explain the eventual compactness property which we are going to prove. Before we start, let us recall that a finite rank operator K is compact and therefore its admissibility holds. Hence, all semigroups of this work exist. So, let us commence by the following useful result

Lemma 6.1. *Let K be a compact operator from Y_1 into itself. Then, for all $t > \frac{2}{a}$, $U_K(t)$ is a weakly compact operator in $L^1(\Omega)$.*

Proof. Let $t > 2/a$ and $\varphi \in (L^1(\Omega))_+$. In the sequel, we are going to divide the proof in several steps.

Step one. Let K be the operator

$$K\psi = h \int_a^\infty k(v')\psi(v')v'\,dv', \quad h \in C_c(J),\ k \in L^\infty(J). \tag{6.1}$$

So, by Lemma 3.6, the operator $U_K(t)$ can be written as

$$U_K(t) = \sum_{m=0}^\infty U_m(t) \tag{6.2}$$

where $U_0(t)$ is given by (3.5) and

$$U_1(t)\varphi(\mu, v) = \xi(\mu, v, t)h(v) \int_a^\infty k(v_1)\chi\left(1, v_1, t - \frac{\mu}{v}\right)\varphi\left(1 - (t - \frac{\mu}{v})v_1, v_1\right) v_1\,dv_1$$

and, for $m \geq 2$, by

$$
\begin{aligned}
U_m(t)\varphi(\mu, v) = \xi(\mu, v, t)h(v) & \underbrace{\int_a^\infty \cdots \int_a^\infty}_{m \text{ times}} \prod_{j=1}^{m-1} h(v_j) \prod_{j=1}^{m} k(v_j) \\
& \times \xi\Big(1, v_{m-1}, t - \frac{\mu}{v} - \sum_{i=1}^{(m-2)} \frac{1}{v_i}\Big)\chi\Big(1, v_m, t - \frac{\mu}{v} - \sum_{i=1}^{(m-1)} \frac{1}{v_i}\Big) \\
& \times \varphi\Big(1 - \Big(t - \frac{\mu}{v} - \sum_{i=1}^{m-1} \frac{1}{v_i}\Big)v_m, v_m\Big) v_1 v_2 \cdots v_m \, dv_1 \cdots dv_m.
\end{aligned}
$$

First. As $t > 2/a$, then on the one hand we have

$$
\mu - tv < 1 - \frac{2}{a}a = -1 < 0
$$

for all $(\mu, v) \in \Omega$. This implies that $\chi(\mu, v, t) = 0$ and therefore $U_0(t) = 0$ because of (3.5). On the other hand we have

$$
1 - \big(t - \frac{\mu}{v}\big)v_1 < 1 - \big(\frac{2}{a} - \frac{1}{a}\big)v_1 < 1 - \big(\frac{2}{a} - \frac{1}{a}\big)a = 0
$$

for all $(\mu, v, v_1) \in \Omega \times J$. This leads to

$$
\chi\Big(1, v_1, \big(t - \frac{\mu}{v}\big)\Big) = 0
$$

and therefore $U_1(t) = 0$. Whence, (6.2) becomes

$$
U_K(t) = \sum_{m=2}^{\infty} U_m(t). \tag{6.3}
$$

Next. As $h \in C_c(J)$, there exists a finite real number b ($a < b < \infty$) such that $\operatorname{supp} h \subset (a, b)$. Let us denotes $m = [tb] + 2$ where $[tb]$ is the integer part of tb. So, as we clearly have $m - 1 \leq tb + 1 < m$, for all $v_i \in (a, b)$, $i = 1 \cdots (m-1)$,

$$
\begin{aligned}
\Big(t - \frac{\mu}{v} - \sum_{i=1}^{m-2} \frac{1}{v_i}\Big)v_{m-1} &\leq \Big(t - \frac{(m-2)}{b}\Big)v_{m-1} \\
&\leq \Big(t - \frac{(m-2)}{b}\Big)b < 1.
\end{aligned}
$$

This implies, by (3.14), that

$$
\xi\Big(1, v_{m-1}, t - \frac{\mu}{v} - \sum_{i=1}^{(m-2)} \frac{1}{v_i}\Big) = 0
$$

and therefore, $U_m(t) = 0$ for all $m > bt + 1$. So, (6.2) becomes the finite sum

$$
U_K(t) = \sum_{m=2}^{[bt]+1} U_m(t). \tag{6.4}
$$

Now, let us prove that $U_m(t)$ is a weakly compact operator in $L^1(\Omega)$ for all $2 \leq m \leq [bt] + 1$. So, the change of variables

$$x = 1 - \Big(t - \frac{\mu}{v} - \sum_{i=1}^{(m-1)} \frac{1}{v_i}\Big) v_m$$
$$v_{m-1}^2 dx = -v_m dv_{m-1}$$

together with some simplifications infer that

$$\begin{aligned} |U_m(t)\varphi(\mu, v)| &\leq \frac{m^3}{t^3} \|h\|_\infty \|h\|_{L^1(J)}^{m-2} \|k\|_\infty^m \xi(\mu, v, t)|h(v)| \int_\Omega \varphi(x, v_m)\, dx\, dv_m \\ &:= C_m(t)\mathbb{I} \otimes \mathbb{I}\varphi(\mu, v), \end{aligned} \tag{6.5}$$

where, $\mathbb{I} \otimes \mathbb{I}$ is the operator

$$\mathbb{I} \otimes \mathbb{I}\varphi(\mu, v) = \xi(\mu, v, t)|h(v)| \int_\Omega \varphi(x, v_m)\, dx\, dv_m$$

and $C_m(t)$ is the constant

$$C_m(t) = \frac{m^3}{t^3} \|h\|_\infty \|h\|_{L^1(J)}^{m-2} \|k\|_\infty^m .$$

As, we clearly have

$$\int_\Omega \xi(\mu, v, t)|h(v)|\, d\mu\, dv = t\|h\|_{Y_1} < \infty,$$

then $\xi(\cdot, \cdot, t)h \in L^1(\Omega)$ and therefore $\mathbb{I} \otimes \mathbb{I}$ is rank one operator in $L^1(\Omega)$. Hence $C_m(t)\mathbb{I} \otimes \mathbb{I}$ is a compact operator in $L^1(\Omega)$. Using (6.5) we obtain

$$0 \leq U_m(t) + C_m(t)\mathbb{I} \otimes \mathbb{I} \leq 2C_m(t)\mathbb{I} \otimes \mathbb{I}$$

which , by Lemma 2.2, implies $U_m(t) + C_m(t)\mathbb{I} \otimes \mathbb{I}$ is a weakly compact operator in $L^1(\Omega)$ and therefore

$$U_m(t) = (U_m(t) + C_m(t)\mathbb{I} \otimes \mathbb{I}) - C_m(t)\mathbb{I} \otimes \mathbb{I}$$

is a weakly compact operator in $L^1(\Omega)$. Finally, thanks to (6.4), we can say that $U_K(t)$, like a finite sum, is a weakly compact operator in $L^1(\Omega)$.

Step two. Let K be the rank one operator

$$K\psi = h \int_a^\infty k(v')\psi(v')v'\, dv', \quad h \in Y_1,\ k \in L^\infty(J). \tag{6.6}$$

As $h \in Y_1$, there exists a sequence $(h_n)_n$ of $C_c(J)$ converging to h in Y_1. Let us define the operator

$$K_n\psi = h_n \int_a^\infty k(v')\psi(v')v'\, dv'$$

which has the form (6.1). By the third step, we obtain $U_{K_n}(t)$ is a weakly compact operator in $L^1(\Omega)$. On the other hand, it follows that

$$|(K_n - K)\psi| \leq |h_n - h| \int_a^\infty |k(v')\psi(v')|v'\, dv'$$

which leads to

$$|(K_n - K)\psi| \leq |h - h_n| \|k\|_{L^\infty(J)} \|\psi\|_{Y_1}$$

and therefore

$$\|K_n - K\|_{\mathcal{L}(Y_1)} \leq \|h_n - h\|_{Y_1} \|k\|_{L^\infty(J)}.$$

Hence,

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0. \tag{6.7}$$

Now, (3.18) obviously implies that

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0,$$

and therefore we can say : $U_K(t)$ is a weakly compact operator in $L^1(\Omega)$.

Step three. Let K be the finite rank operator

$$K\psi = \sum_{i=1}^{M_K} h_i \int_a^\infty k_i(v')\psi(v')v'\,dv', \quad h_i \in Y_1,\ k_i \in L^\infty(J),\ i = 1\cdots M_K$$

where $M_K < \infty$, and let K' be the positive rank-one operator

$$K'\psi = h\int_a^\infty k(v')\psi(v')v'\,dv',$$

where

$$h = \sum_{i=1}^{M_K} |h_i| \in (Y_1)_+ \quad \text{and} \quad k = \sum_{i=1}^{M_K} |k_i| \in (L^\infty(J))_+.$$

As K' has the form (6.6), then thanks to the step above it follows that $U_{K'}(t)$ is a weakly compact operator in $L^1(\Omega)$. Furthermore, for all $\psi \in Y_1$, we have

$$\begin{aligned}
|K\psi| &\le \sum_{i=1}^{M_K} |h_i| \int_a^\infty |k_i(v')||\psi(v')|v'\,dv' \\
&\le \Big[\sum_{i=1}^{M_K} |h_i|\Big] \int_a^\infty \Big[\sum_{i=1}^{M_K} |k_i(v')|\Big] |\psi(v')|v'\,dv' = K'|\psi|
\end{aligned}$$

which leads, by Theorem 4.3, to

$$|U_K(t)\varphi| \le U_{K'}(t)|\varphi|$$

for all $\varphi \in L^1(\Omega)$. This clearly implies

$$0 \le U_K(t) + U_{K'}(t) \le 2U_{K'}(t)$$

and therefore, the operator $U_K(t) + U_{K'}(t)$ is weakly compact in $L^1(\Omega)$ by Lemma 2.2. Now, we can say that

$$U_K(t) = (U_K(t) + U_{K'}(t)) - U_{K'}(t)$$

is a weakly compact operator in $L^1(\Omega)$.

Step four. Let K be a compact operator in Y_1. So, by [7, Corollary 5.3, pp.276], there exists a sequence $(K_n)_n$ of finite rank operators converging to K in $\mathcal{L}(Y_1)$; i.e.,

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0.$$

On the one hand, the step above leads to the weak compactness of the operator $U_{K_n}(t)$ in $L^1(\Omega)$, and on the other hand (3.18) implies

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0$$

which leads to the weak compactness of the operator $U_K(t)$. The proof is complete. $\square$

Let us consider the hypothesis

(H2) There exist $\overline{r} \in (L^1(J))_+ \cap (L^1(J))_+$ and $n_r, m_r \geq 0$ such that
$$r(\mu, x, y) \leq \frac{\mu^{n_r+1}}{y^{m_r+2}} \overline{r}(x),$$
for almost all $(\mu, x, y) \in I \times J^2$.

Note that when (H2) holds, the σ given by (1.2) satisfies
$$\sigma(\mu, v) \leq \frac{\mu^{n_r+1}}{v^{m_r+2}} \int_a^\infty \overline{r}(v')dv' \leq \frac{1}{a^{m+2}} \|\overline{r}\|_{L^1(J)} < \infty$$
for almost all $(\mu, v) \in \Omega$ and therefore (H1) holds too. Accordingly we have the following result.

Lemma 6.2. *Suppose that* (H2) *holds and let K be a compact operator from Y_1 into itself. Then, for all $t > 0$, $RU_K(t)R$ is a weakly compact operator in $L^1(\Omega)$.*

Proof. Let $t > 0$ and let $\omega > \omega_0$ be a given real where, ω_0 is the abscissa of the operator K. In the sequel, we divide the proof in several steps.

Step one. Let K be the operator
$$K\psi = h \int_a^\infty k(v')\psi(v')v'\,dv', \quad h \in C_c(J),\ k \in L^\infty(J) \tag{6.8}$$
and let $\varphi \in (L^1(\Omega))_+$. Then (3.12) implies
$$RU_K(t)R\varphi = RU_0(t)R\varphi + RB_K(t)R\varphi. \tag{6.9}$$
First, by (3.13) and (6.8), a simple calculation implies
$$\begin{aligned} &RB_K(t)R\varphi(\mu, v)\\ &= \int_a^\infty \int_a^\infty r(\mu, v, v'')\xi(\mu, v'', t)h(v'')k(v')\gamma_1\Big(U_K\big(t - \frac{\mu}{v''}\big)R\varphi\Big)(v')v'\,dv'dv'' \end{aligned}$$
for almost all $(\mu, v) \in \Omega$; therefore,
$$\begin{aligned} &|RB_K(t)R\varphi(\mu, v)|\\ &\leq \|h\|_\infty \|k\|_\infty \overline{r}(v) \int_a^\infty \int_a^\infty \frac{\mu^{n_r+1}}{(v'')^{m_r+2}} \xi(\mu, v'', t)|\gamma_1\big(U_K\big(t - \frac{\mu}{v''}\big)R\varphi\big)(v')|v'\,dv'dv''\\ &\leq \frac{\|h\|_\infty \|k\|_\infty}{a^{m_r}} \overline{r}(v) \int_a^\infty \int_a^\infty \frac{\mu}{(v'')^2} \xi(\mu, v'', t)|\gamma_1\big(U_K\big(t - \frac{\mu}{v''}\big)R\varphi\big)(v')|v'\,dv'dv'' \end{aligned}$$
because of hypothesis (H2). Now, a suitable change of variable in the above integral leads to
$$|RB_K(t)R\varphi(\mu, v)| \leq \frac{\|h\|_\infty \|k\|_\infty}{a^{m_r}} \overline{r}(v) \int_a^\infty \int_0^t |\gamma_1\left(U_K\left(x\right)R\varphi\right)(v')|v'\,dx\,dv'$$
which, by (3.15), implies
$$|RB_K(t)R\varphi(\mu, v)| \leq \frac{\|h\|_\infty \|k\|_\infty e^{t\omega}}{a^{m_r}\left(1 - \|K_\omega\|_{\mathcal{L}(Y_1)}\right)} \overline{r}(v) \int_\Omega (R\varphi)(\mu, v)\,d\mu\,dv;$$
therefore,
$$|RB_K(t)R\varphi| \leq (\alpha_t \overline{r}\mathbb{I})\mathbb{I} \otimes \mathbb{I}\varphi, \tag{6.10}$$
where
$$\mathbb{I} \otimes \mathbb{I}\varphi = \int_\Omega \varphi(\mu, v)d\mu\,dv, \quad \alpha_t = \frac{\|h\|_\infty \|k\|_\infty \|R\|_{\mathcal{L}(L^1(\Omega))} e^{t\omega}}{a^{m_r}(1 - \|K_\omega\|_{\mathcal{L}(Y_1)})}$$

and $\mathbb{I}(\mu, v) = 1$ for all $(\mu, v) \in \Omega$.

Next, thanks to (3.5), a simple calculation gives us

$$\begin{aligned} &RU_0(t)R\varphi(\mu, v) \\ &= \int_a^\infty \int_0^\infty r(\mu, v, v')r(\mu - tv', v', v'')\chi(\mu, v', t)\varphi(\mu - tv', v'')dv'\, dv'' \end{aligned}$$

which, by (H2), implies

$$\begin{aligned} &|RU_0(t)R\varphi(\mu, v)| \\ &\le \int_a^\infty \int_0^\infty \overline{r}(v)\frac{\mu^{n_r+1}}{(v')^{m_r+2}}\overline{r}(v')\frac{(\mu - tv')^{n_r+1}}{(v'')^{m_r+2}}\chi(\mu, v', t)\varphi(\mu - tv', v'')dv'\, dv'' \\ &\le \frac{\|\overline{r}\|_\infty}{a^{2m_r+4}}\overline{r}(v)\int_a^\infty \int_0^\infty \chi(\mu, v', t)\varphi(\mu - tv', v'')dv'\, dv''. \end{aligned}$$

A suitable change in last integral easily leads to

$$|RU_0(t)R\varphi(\mu, v)| \le \frac{1}{t}\frac{\|\overline{r}\|_\infty}{a^{2m_r+4}}\overline{r}(v)\int_\Omega \varphi(x, v'')\, dx\, dv''$$

and thus

$$|RU_0(t)R\varphi| \le (\beta_t\overline{r}\mathbb{I})\mathbb{I}\otimes\mathbb{I}\varphi, \tag{6.11}$$

where $\beta_t = \|\overline{r}\|_\infty t^{-1}a^{-2m_r+4}$. Finally, thanks to (6.9), (6.10) and (6.11), we infer that

$$|RU_K(t)R\varphi| \le (\gamma_t\overline{r}\mathbb{I})\, \mathbb{I}\otimes\mathbb{I}\varphi$$

where, $\gamma_t = \alpha_t + \beta_t$, and therefore,

$$0 \le RU_K(t)R + (\gamma_t\overline{r}\mathbb{I})\, \mathbb{I}\otimes\mathbb{I} \le 2(\gamma_t\overline{r}\mathbb{I})\mathbb{I}\otimes\mathbb{I}.$$

By (H2), we obtain $\overline{r}\mathbb{I} \in L^1(\Omega)$ which implies that the right-hand side of the relation above is clearly rank one operator in $L^1(\Omega)$ and therefore weakly compact. Then, by the second point of Lemma 2.2, we infer the weak compactness, in $L^1(\Omega)$, of the operator $RU_K(t)R+(\gamma_t\overline{r}\mathbb{I})\mathbb{I}\otimes\mathbb{I}$ and therefore the weak compactness of the operator

$$RU_K(t)R = \Big(RU_K(t)R + (\gamma_t\overline{r}\mathbb{I})\mathbb{I}\otimes\mathbb{I}\Big) - (\gamma_t\overline{r}\mathbb{I})\mathbb{I}\otimes\mathbb{I}$$

follows obviously.

Step two. Let K be rank one operator into Y_1; i.e.,

$$K\psi = h\int_a^\infty k(v')\psi(v')v'\, dv', \quad h \in Y_1,\ k \in L^\infty(J). \tag{6.12}$$

Then, there exists a sequence $(h_n)_n$ of $C_c(J)$ converging to h in Y_1. This implies the operator

$$K_n\psi = h_n\int_a^\infty k(v')\psi(v')v'\, dv'$$

has the form (6.8) and therefore, $RU_{K_n}(t)R$ is a weakly compact operator in $L^1(\Omega)$ because of the preceding step. On the other hand, it is easy to check that

$$\lim_{n\to\infty}\|K_n - K\|_{\mathcal{L}(Y_1)} = 0$$

and therefore

$$\lim_{n\to\infty}\|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0 \tag{6.13}$$

because of (3.18). Now, the weak compactness of the operator $RU_K(t)R$ in $L^1(\Omega)$ follows from

$$\|RU_K(t)R - RU_{K_n}(t)R\|_{\mathcal{L}(L^1(\Omega))} \le \|R\|^2 \|U_K(t) - U_{K_n}(t)\|_{\mathcal{L}(L^1(\Omega))} \tag{6.14}$$

together with (6.13) and from the first point of Lemma 2.2.

Step three. Let K be the finite rank operator

$$K\psi = \sum_{i=1}^{M_K} h_i \int_a^\infty k_i(v')\psi(v')v'\,dv', \quad h_i \in Y_1,\ k_i \in L^\infty(J),\ i = 1\cdots M_K$$

where $M_K < \infty$. Setting

$$h = \sum_{i=1}^{M_K} |h_i| \in (Y_1)_+ \quad \text{and} \quad k = \sum_{i=1}^{M_K} |k_i| \in (L^\infty(J))_+$$

it follows that the positive operator

$$K'\psi = h \int_a^\infty k(v')\psi(v')v'\,dv'$$

has the form (6.12) and therefore, $RU_{K'}(t)R$ is a weakly compact operator in $L^1(\Omega)$ because of the preceding step. On the other hand,

$$|K\psi| \le \Big[\sum_{i=1}^{M_K} |h_i|\Big] \int_a^\infty \Big[\sum_{i=1}^{M_K} |k_i(v')|\Big] |\psi(v')| v'\,dv' = K'|\psi|$$

for all $\psi \in Y_1$. Then, thanks to the positivity of the operator R and Theorem 4.3, we obtain

$$|RU_K(t)R\varphi| \le RU_{K'}(t)R|\varphi|;$$

therefore

$$0 \le RU_K(t)R + RU_{K'}(t)R \le 2RU_{K'}(t)R.$$

Now, the second point of Lemma 2.2 implies the weak compactness, in $L^1(\Omega)$, of the operator $RU_K(t)R + RU_{K'}(t)R$ and hence, that of the operator

$$RU_K(t)R = \Big(RU_K(t)R + RU_{K'}(t)R\Big) - RU_{K'}(t)R$$

clearly follows.

Step four. Now, let K be a compact operator in Y_1. Thanks to [7, Corollary 5.3, pp.276], there exists a sequence $(K_n)_n$ of finite rank operators such that

$$\lim_{n\to\infty} \|K_n - K\|_{\mathcal{L}(Y_1)} = 0.$$

So, on one hand the weak compactness of the operator $RU_{K_n}(t)R$, in $L^1(\Omega)$, follows from the step above and on the other hand

$$\lim_{n\to\infty} \|U_{K_n}(t) - U_K(t)\|_{\mathcal{L}(L^1(\Omega))} = 0$$

because of (3.18). Finally, a relation like (6.14) together with the first point of Lemma 2.2 imply the weak compactness of the operator $RU_K(t)R$ in $L^1(\Omega)$. The proof is now complete. □

In the next result, we compute the essential type of the semigroup $(W_K(t))_{t\ge0}$ as follows

Theorem 6.3. *Suppose that* (H2) *holds and let K be a positive compact operator from Y_1 into itself. Then we have*

$$\omega_{\rm ess}(W_K(t)) = -\infty. \tag{6.15}$$

Proof. First, let $t > 4/a$. Thanks to Lemma 6.1, we obtain $U_K(t/2)$ is a weakly compact operator in $L^1(\Omega)$. As, (4.5) leads to

$$0 \le V_K(t/2) \le U_K(t/2).$$

Then, by Lemma 2.2, we obtain $V_K(\frac{t}{2})$ is a weakly compact operator in $L^1(\Omega)$. Once more Lemma 2.2 implies that $V_K(t) = \big(V_K(t/2)\big)^2$ is a compact operator in $L^1(\Omega)$ which leads, by (2.2), to

$$\omega_{\rm ess}\left(V_K(t)\right) = \lim_{t\longrightarrow\infty} \frac{\ln \|V_K(t)\|_{\rm ess}}{t} = -\infty. \tag{6.16}$$

Next, let $t > 0$. The positivity of the operators R and K together with Theorem 4.2 clearly imply

$$0 \le RV_K(t)R \le RU_K(t)R.$$

The relation above together with Lemma 6.2 and the second point of Lemma 2.2 imply the weak compactness of the operator $RV_K(t)R$. So, for all $t_1, t_2, t_3 > 0$, the operator

$$RV_K(t_1)RV_K(t_2)RV_K(t_3)R = \left(RV_K(t_1)R\right) V_K(t_2) \left(RV_K(t_3)R\right)$$

is compact in $L^1(\Omega)$ because of the third point of Lemma 2.2; therefore, Lemma 2.3 leads to

$$\omega_{\rm ess}\left(W_K(t)\right) = \omega_{\rm ess}\left(V_K(t)\right). \tag{6.17}$$

Finally, (6.16) and (6.17) complete the proof. □

Now, we are ready to give the main result of this work. Before we state it, let us point out that contrary to Remark 3.9, the case $\|K\|_{\mathcal{L}(Y_1)} > 1$ is the most observed and biologically interesting because the cell density is increasing during each mitotic. Now, we give the asymptotic behavior, in this case, for the semigroup $(W_K(t))_{t\ge 0}$, as follows.

Theorem 6.4. *Suppose that* (H2) *holds and let K be a positive, irreducible and compact operator with $r(K) > 1$. Then, there exist a rank one projector $\mathbb{P}$ into X and positive constants M and δ such that*

$$\|e^{-ts(T_K)}W_K(t) - \mathbb{P}\|_{\mathcal{L}(L^1(\Omega))} \le Me^{-\delta t}, \quad t \ge 0.$$

Proof. First, let us note that the admissibility of the operator K holds and therefore the semigroup $(W_K(t))_{t\ge 0}$ exists. Next, Theorem 4.2 implies that $(W_K(t))_{t\ge 0}$ is a positive and irreducible semigroup. Finally, (5.4) and (6.15) lead to $\omega_{\rm ess}(W_K(t)) < \omega(W_K(t))$. Since all conditions of Lemma 1.1 are satisfied, the proof is complete. □

Remark 6.5. Note that $a > 0$ has been used in many places of this work. So the open question is: What happens when $a = 0$?

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