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ASYMPTOTIC AND NUMERICAL DESCRIPTION OF THE KINK/ANTIKINK INTERACTION

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Abstract. We consider a class of semi-linear wave equations with a small parameter and nonlinearities which provide the equations having exact kink-type solutions. We declare sufficient conditions for the nonlinearities under which the kink-kink and kink-antikink collisions occur, in the asymptotic sense, without changing the shape of the waves and with only some shifts of the solitary wave trajectories. Furthermore, we create an absolutely stable finite differences scheme to simulate the solution of the Cauchy problem and obtain some numerical results for two-wave interaction. We present also some unexpected results about three-wave interaction.

1. Introduction

We consider the semilinear wave equation

$$\varepsilon^2(u_{tt}-u_{xx})+F'(u)=0,\quad x\in\mathbb{R}^1,\ t>0 \tag{1.1}$$

with some smooth nonlinearities $F(u)$ and the parameter $\varepsilon\to 0$. It is well known [1] that the unique completely integrable representative of the family (1.1) is the sine-Gordon equation (see e.g. [2])

$$\varepsilon^2(u_{tt}-u_{xx})+\sin(u)=0. \tag{1.2}$$

At the seme time, there are many nonlinearities $F(u)$ such that the equation (1.1) admits exact travelling wave solutions of the kink/antikink type:

$$\begin{gathered} u(x,t,\varepsilon)=\omega\Big(\pm\beta\frac{x-Vt}{\varepsilon}\Big),\quad \beta=(1-V^2)^{-1/2},\quad \omega(\eta)\in\mathcal{C}^\infty(\mathbb{R}),\\ \omega(\eta)\to 0\quad\text{for } \eta\to-\infty,\quad \omega(\eta)\to 1\quad\text{for }\eta\to+\infty. \end{gathered}\tag{1.3}$$

It is easy to check that the conditions

(A) $F(z)\in\mathcal{C}^\infty(\mathbb{R})$, $F(z)>0$ for $z\in(0,1)$,
(B) $F^{(i)}(z_0)=0$, $i=0,1,\dots,k$, $F^{(k+1)}(z_0)>0$, where $z_0=0$ and $z_0=1$, and $k=1$ or $k=3$,

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are sufficient for the existence of kink/antikink solutions such that

$$|\omega(\eta)| \leq c_1\eta^{-2} \quad \text{as } \eta \to -\infty, \quad |\omega(\eta) - 1| \leq c_2\eta^{-2} \quad \text{as } \eta \to \infty.$$

Moreover, under the periodicity condition

(C) $F(z+1) = F(z)$

any combination of kink-antikink waves

$$u_\Sigma = \sum_{i=1}^{N} \omega\Big(\pm \beta_i \frac{x - V_i t - x_i^0}{\varepsilon}\Big), \quad x_{i+1}^0 - x_i^0 > 1, \quad 0 < t \ll 1 \tag{1.4}$$

will approximate sufficiently well the exact solution of the corresponding Cauchy problem. This bring up the question about the character of interaction between the entities (1.3).

There are some known asymptotic results about the interaction character for the equation (1.1) with a small parameter ε. Namely, there are constructed asymptotic in the weak sense solutions of the equation (1.1) to describe the interaction of two kinks [3] and the kink–antikink pair [4]. As a result, in the cited articles are found sufficient conditions for $F(u)$ and V_i, $i = 1, 2$, under which the interaction of two solitary waves (1.3) preserves the sine-Gordon scenario (see Sec. 2). This means that the interaction occurs without changing the shape of the waves and with shifts of the trajectories. The main tool there was the weak asymptotic method (see e.g. [5, 6, 3, 4] and references therein). The main advantage of this approach is the possibility to reduce the problem of describing nonlinear waves interaction to a qualitative analysis of some ordinary differential equations (instead of partial differential equations). This method takes into account the fact that kinks (as well as solitons [7, 6]) which are smooth for $\varepsilon > 0$ become non-smooth in the limit as $\varepsilon \to 0$. So it is possible to treat such solutions as a mapping $\mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\mathbb{R}_x^1))$ for $\varepsilon > 0$ and only as $\mathcal{C}(0, T; \mathcal{D}'(\mathbb{R}_x^1))$ uniformly in $\varepsilon \geq 0$. Accordingly, the remainder should be small in the weak sense. This sufficiently trivial observation allowed to reach a progress for some old problems about nonlinear wave interaction for nonintegrable equations.

However, the constructed asymptotics are formal only. Moreover, it is clear that the conditions [3, 4] are excessively restrictive. For this reason we created an absolutely stable finite differences scheme for the equation (1.1) (Sec. 3) and applied it to some nonintegrable versions of (1.1). The numerical results (Sec. 4) show that the kink–kink and kink–antikink pairs interact without changing the shape of the waves including the case when the conditions [3, 4] are violated.

At the same time it turns out that the multi–wave situation is more unexpected. The point is that there is a hypotheses (Vladimir Danilov et alii [8], Boris Dubrovin's, private communication) that there are sufficiently many equations with sine-Gordon scenario of two solitary waves interaction, but three waves can interact in the same manner for the completely integrable equations only. Apparently, we can not be such categorical: our numerical results show that three kinks can interact preserving the sine-Gordon scenario (see Conclusion).

2. Asymptotic solution

For essentially nonintegrable interaction problems it is impossible to construct either explicit solutions (classical or weak) or asymptotics in the classical sense. However, it is possible to construct an asymptotic solution in the weak sense [6].

It should be noted that there is an obstacle to apply the standard $\mathcal{D}'$ construction. Indeed, in the $\mathcal{D}'$ sense, the differential terms of the equation (1.1) are subordinated to the nonlinear term. Moreover, the left-hand side of (1.1) is of the value $O(\varepsilon^2)$ in the weak sense for any u of the form (1.4) and $t \ll 1$. Obviously, this prevents the construction of the correct asymptotics for the Cauchy problem. To overcome this obstacle, in [3] has been constructed a new definition of asymptotic solutions, which involves in the leading term the derivatives of u with arguments x/ε and t/ε:

Definition 2.1. A sequence $u(t, x, \varepsilon)$, belonging to $\mathcal{C}^\infty(0, T; \mathcal{C}^\infty(\mathbb{R}^1_x))$ for ε a positive constant and belonging to $\mathcal{C}(0, T; \mathcal{D}'(\mathbb{R}^1_x))$ uniformly in ε, is called a weak asymptotic mod $O_{\mathcal{D}'}(\varepsilon^2)$ solution of (1.1) if the relation

$$2\frac{d}{dt}\int_{-\infty}^{\infty} \varepsilon^2 u_t u_x \psi dx + \int_{-\infty}^{\infty} \{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\}\psi_x dx = O(\varepsilon^2) \tag{2.1}$$

holds for any test function $\psi = \psi(x) \in \mathcal{D}(\mathbb{R}^1)$.

Here the right-hand side is a $\mathcal{C}^\infty$-function for $\varepsilon = \text{const} > 0$ and a piecewise continuous function uniformly in $\varepsilon \geq 0$. The estimate is understood in the $\mathcal{C}(0, T)$ sense:

$$g(t, \varepsilon) = O(\varepsilon^k) \leftrightarrow \max_{t\in[0,T]} |g(t, \varepsilon)| \leq c\varepsilon^k.$$

The left-hand side of (2.1) is the result of multiplication of (1.1) by $\psi(x)u_x$ and integration by parts in the case of smooth u. Therefore, it is zero for any exact solution. On the other hand, the relation (2.1) is just the orthogonality condition which appears for single-phase asymptotics [9, 10]. This condition guarantees both the first correction existence and allows to find an equation for the distorted kink's front motion.

Definition 2.2. A function $v(t, x, \varepsilon)$ is said to be of the value $O_{\mathcal{D}'}(\varepsilon^k)$ if the relation

$$\int_{-\infty}^{\infty} v(t, x, \varepsilon)\psi(x)dx = O(\varepsilon^k)$$

holds for any test function $\psi \in \mathcal{D}(\mathbb{R}^1_x)$.

Let us consider the interaction of two kinks,

$$u|_{t=0} = \sum_{i=1}^{2} \omega(\beta_i \frac{x - x_i^0}{\varepsilon}), \quad \varepsilon\frac{\partial u}{\partial t}\Big|_{t=0} = -\sum_{i=1}^{2} \beta_i V_i \omega'(\beta_i \frac{x - x_i^0}{\varepsilon}), \tag{2.2}$$

where $\beta_i = 1/\sqrt{1 - V_i^2}$, $|V_i| \in (0, 1)$, and the initial front positions x_i^0 are such that $x_2^0 - x_1^0 > 1$. Obviously, it is assumed that the trajectories $x = V_i t + x_i^0$ have a joint point $x = x^*$ at a time instant $t = t^*$.

The asymptotic anzatz for the problem (1.1), (2.2) has the following form:

$$u = \sum_{i=1}^{2} \Big\{\omega\Big(\beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\varepsilon}\Big) + A_i(\tau) U\Big(\beta_i \frac{x - \Phi_i(t, \tau, \varepsilon)}{\mu^2\varepsilon}\Big)\Big\}. \tag{2.3}$$

Here $\Phi_i = \phi_{i0}(t) + \varepsilon\phi_{i1}(\tau)$, $\phi_{i0} = V_i t + x_i^0$ are the trajectories of noninteracting kinks, $\tau = \psi_0(t)/\varepsilon$ denotes the "fast time", $\psi_0(t) = \phi_{20}(t) - \phi_{10}(t)$, the phase corrections ϕ_{i1} are smooth functions such that

$$\phi_{i1} \to 0 \quad \text{as } \tau \to -\infty, \quad \phi_{i1} \to \phi_{i1}^\infty = \text{const}_i \quad \text{as } \tau \to +\infty \tag{2.4}$$

with a rate not less than $1/|\tau|$. Furthermore, $A_i(\tau) \in \mathcal{C}^\infty$ are exponentially vanishing as $|\tau| \to \infty$ functions, μ is a sufficiently small parameter, $\varepsilon < \mu \ll 1$, and

$$U(\eta) = \frac{d^m U_0(\eta)}{d\eta^m},$$

where $m \geq 1$ is an arbitrary number and $U_0(\eta) \in \mathcal{C}^\infty$ is a sufficiently fast vanishing function as $|\eta| \to \infty$.

The main result, which is known for the problem (1.1), (2.2), is the following.

Theorem 2.3. *Assume* (A)–(C). *Set the additional assumptions*

(D) $F(1/2+z) = F(1/2-z)$,

(E) *Let the function* $F(z)$ *be such that the inequality*

$$\int_{-\infty}^{\infty} F\left(\omega(\eta) + \omega(\theta\eta)\right) d\eta \leq \int_{-\infty}^{\infty} \left\{ \sqrt{F(\omega(\eta))} + \sqrt{F(\omega(\theta\eta))} \right\}^2 d\eta \qquad (2.5)$$

holds uniformly in $\theta \in (0, \infty)$. *Then the interaction of kinks in the problem* (1.1), (2.2) *preserves the sine-Gordon scenario with accuracy* $O_{\mathcal{D}'}(\varepsilon^2)$ *in the sense of Definition 2.1. The weak asymptotic solution of* (1.1), (2.2) *has the form* (2.3) *with a special choice of the amplitudes* A_i *and of the parameter* μ.

Remark 2.4. The symmetry (D) has been assumed to simplify the asymptotic analysis and it is not very important.

Remark 2.5. The sense of the assumption (E) is the following. The phase corrections ϕ_{i1} are solutions of a 2×2-dynamical system with a singularity which support divides the phase plane into two parts with the possible exception of the point $(0,0)$. The assumptions (2.4) are satisfied (consequently, the sine-Gordon scenario takes place) if and only if there exists a specific trajectory which goes from one half-plane to the other one trough the point $(0,0)$. When A_i in (2.3) are equal to zero, the existence of the trajectory implies the appearance of an additional very complicated assumption. This condition can be made more coarse and transformed to the simplest form (2.5). Such version can be treated as an admissible one since it is satisfied for the sine-Gordon equation for any velocities $V_i, i = 1, 2$. The same is true for the nonlinearity

$$F(u) = \sin^4(\pi u). \qquad (2.6)$$

Taking into account a freedom in the choice of the amplitudes A_i, $i = 1, 2$, the assumption (2.5) can be made weaker. However, the dynamical system with $A_i \neq 0$, $i = 1, 2$, is very complicated and it's complete analysis remains undone.

Obviously, all stated above remains true for the antikink-antikink interaction.

Let us focus attention to the kink-antikink interaction, that is to the equation (1.1) with the initial data

$$u|_{t=0} = \sum_{i=1}^{2} \omega\Big(S_i\beta_i \frac{x - x_i^0}{\varepsilon}\Big), \quad \varepsilon \frac{\partial u}{\partial t}\Big|_{t=0} = -\sum_{i=1}^{2} S_i\beta_i V_i \omega'\Big(S_i\beta_i \frac{x - x_i^0}{\varepsilon}\Big), \qquad (2.7)$$

where $S_1 = 1$, $S_2 = -1$, and the notation β_i, V_i, x_i^0 is the same as in (2.2).

The asymptotic anzatz for the solution of the problem (1.1), (2.7) differs a little bit from (2.3), namely

$$u = \sum_{i=1}^{2} \Big\{ \omega\Big(S_i\beta_i \frac{x - \Phi_i(t,\tau,\varepsilon)}{\varepsilon}\Big) + A_i(\tau)\, U\Big(S_i\mu\beta_i \frac{x - \Phi_i(t,\tau,\varepsilon)}{\varepsilon}\Big) \Big\} \qquad (2.8)$$

with the same notation and the assumption (2.4).

Technically, the construction of (2.8) is similar to the kink–kink case. However, the resulting dynamical system for the phase corrections becomes much more complicated. Moreover, it is impossible to simplify the additional assumption, which appears here also, without lose of the adequacy. For this reason, to present the additional condition, we should define some entities:

$$\lambda_1^0 = \frac{1}{a_2}\int_{-\infty}^{\infty} \omega'(\eta)\omega'(\theta\eta)d\eta, \quad a_2 = \int_{-\infty}^{\infty} \big(\omega'(\eta)\big)^2 d\eta,$$

$$\lambda_2(\sigma) = \frac{1}{a_2}\int_{-\infty}^{\infty} \eta\omega'(\eta)\omega'(\theta\eta+\sigma)d\eta,\ \bar{\lambda}_2(\sigma) = \frac{1}{a_2\theta}\int_{-\infty}^{\infty} \eta\omega'(\eta)\omega'\Big(\frac{\eta-\sigma}{\theta}\Big)d\eta,$$

$$L(\sigma) = \sigma - \bar{\lambda}_2(\sigma) + \theta\lambda_2(\sigma), \quad L_1 = L'|_{\sigma=0},$$

$$B_{\triangle}^0 = \frac{2}{a_2}\int_{-\infty}^{\infty} \Big\{F\big(\omega(\eta)-\omega(\theta\eta)\big) - F\big(\omega(\eta)\big) - F\big(\omega(\theta\eta)\big)\Big\}d\eta,$$

$$\mathcal{F}_0 = (1+\theta-2\theta\lambda_1^0)^{-1}, \quad N = (\lambda_1^0 + 2\theta\lambda_2'|_{\sigma=0})(b_2+\theta b_1),$$

$$\nu = V_2 - V_1, \quad b_i = \frac{V_i}{\nu}, \quad \theta = \frac{\beta_1}{\beta_2}, \quad M = 2L_1 - 1 + \theta{\lambda_1^0}^2,$$

$$R = 1 - 2\lambda_1^0(b_2^2+\theta b_1^2) - \frac{2}{\nu^2\mathcal{F}_0}\Big(\lambda_1^0 + \frac{B_{\triangle}^0}{2\beta_1\beta_2}\Big).$$

Theorem 2.6. *Assume* (A)–(D). *Moreover, let*

(E1) $\theta \neq 1$ *and* $N^2 + MR > 0$,

(E2) $L_1 \neq 0$ *and* $L(\sigma) > 0$ *for* $\sigma > 0$.

Then the kink and antikink preserve mod $O_{\mathcal{D}'}(\varepsilon^2)$ their forms after the interaction. The weak asymptotic solution of (1.1), (2.7) *has the form* (2.8) *with a special choice of the amplitudes A_i and of the parameter μ.*

Remark 2.7. Apparently, kink and antikink interact in the case $\theta = 1$ preserving the sine-Gordon scenario. However, this case should be investigated separately.

Remark 2.8. The condition (E1) is much more restrictive than (E). In particular, for the sine-Gordon equation it is satisfied for $|\theta-1| \ll 1$. Moreover, (E1) is satisfied in the cases

$$\sqrt{\pi(2-\pi/4)} - 1 < -V_1 < 1 \quad \text{if } \theta \ll 1 \text{ and } V_2 > 0,$$

$$\sqrt{\pi(2-\pi/4)} - 1 < V_2 < 1 \quad \text{if } \theta \gg 1,$$

and (E1) violates when the last inequalities are broken. Furthermore, for the nonlinearity (2.6) this condition is violated for any velocities V_i, $i = 1, 2$. We note also that the actual sufficient condition should be much less restrictive than (E1) (see Remark 2). However, it remains unknown until now.

Remark 2.9. Assumption (E2) prevents the appearance of another singularity. It is verified for all examples under our consideration.

Finally we note that there is a correspondence between weak asymptotic solutions and energy relations for the equation (1.1).

Theorem 2.10. *Let the assumptions of the theorem 2.3 (the theorem 2.6) hold. Then two kinks* (2.3) *(kink–antikink pair* (2.8)*) preserve mod $O_{\mathcal{D}'}(\varepsilon^2)$ their forms* *after the interaction if and only if they satisfy the conservation law*

$$\frac{d}{dt}\int_{-\infty}^{\infty} u_t u_x dx = 0$$

and the energy relation

$$2\frac{d}{dt}\int_{-\infty}^{\infty} x\varepsilon^2 u_t u_x dx + \int_{-\infty}^{\infty} \big\{(\varepsilon u_t)^2 + (\varepsilon u_x)^2 - 2F(u)\big\} dx = 0.$$

3. Finite differences scheme

The actual numerical simulation for the problem (1.1), (2.2) or (1.1), (2.7) is realized for a finite x-interval, $x \in [0, L]$. For this reason we simulate the Cauchy problem by the following mixed problem:

$$\begin{gathered}\varepsilon^2(u_{tt} - u_{xx}) + F'(u) = 0, \quad x \in (0, L), \quad t \in (0, T),\\ u\big|_{x=0} = \nu_\ell, \quad u\big|_{x=L} = \nu_r,\\ u\big|_{t=0} = u^0\left(\frac{x}{\varepsilon}\right), \quad \varepsilon\frac{\partial u}{\partial t}\big|_{t=0} = u^1\left(\frac{x}{\varepsilon}\right),\end{gathered} \tag{3.1}$$

where u^0 is a kink-kink or kink-antikink combination of the form (2.2) or (2.7), and u^1 is the corresponding time derivative, $\nu_\ell = u^0|_{x=0}$, $\nu_r = u^0|_{x=L}$. To simulate by (3.1) the interaction phenomena, we assume that L, T, and the initial front positions x_i^0, $i = 1, 2$, are such that the intersection point of the solitary wave fronts belongs to $Q_T = (0, L) \times (0, T)$. Furthermore, let L, T, and x_i^0 be such that uniformly in $t \leq T$,

$$\big|u_\Sigma|_{x\in[0,\delta]} - \nu_\ell\big| \leq c\varepsilon^2, \quad \big|u_\Sigma|_{x\in[L-\delta,L]} - \nu_r\big| \leq c\varepsilon^2$$

for some sufficiently small $\delta > 0$. Here u_Σ is the combination of solitary waves of the form (1.4) corresponded to the initial value u^0.

Since it is impossible to create any finite difference scheme for the problem (3.1), which remains stable uniformly in $\varepsilon \to 0$ and $t \in (0, T)$, $T = \text{const}$, we will treat ε as a small but fixed constant. However, we will fix any relation between ε and finite differences scheme parameters.

To create a finite differences scheme for the equation (3.1) we should choose appropriate approximations for the differential terms and for the nonlinear term. Let us do it separately.

3.1. **Preliminary nonlinear "scheme".**

As usual, we define a mesh $Q_{T,\tau,h} = \{(x_i, t_j) = (ih, j\tau), i = 0, \dots, I,\ j = 0, \dots, J\}$ over Q_T and denote

$$\begin{gathered} y_i^j = u(x_i, t_j), \quad y_{it}^j = \frac{y_i^{j+1} - y_i^j}{\tau}, \quad y_{i\bar t}^j = \frac{y_i^j - y_i^{j-1}}{\tau},\\ y_{ix}^j = \frac{y_{i+1}^j - y_i^j}{h}, \quad y_{i\bar x}^j = \frac{y_i^j - y_{i-1}^j}{h}, \quad y_{it\bar t}^j = (y_{it}^j)_{\bar t}, \quad y_{ix\bar x}^j = (y_{ix}^j)_{\bar x}.\end{gathered}$$

Let us consider the system of nonlinear equations

$$\begin{gathered}\varepsilon^2(y_{it\bar t}^j - y_{ix\bar x}^{j+1}) + F'(y_i^{j+1}) = 0, \quad i = 1, \dots, I-1, \quad j = 2, 3, \dots,\\ y_0^j = \nu_\ell, \quad y_I^j = \nu_r, \quad j = 0, 1, \dots,\\ y_i^0 = u^0\left(\frac{x_i}{\varepsilon}\right), \quad \varepsilon y_{it}^0 = \widetilde{u}^1\left(\frac{x_i}{\varepsilon}, \tau\right), \quad i = 0, \dots, I,\end{gathered} \tag{3.2}$$

where $\widetilde{u}^1(x_i/\varepsilon,\tau)$ is such that last equality in (3.1) is approximated with accuracy $O(\tau^2)$. Obviously, the local approximation accuracy of (3.2) is $O(\tau^2+h^2)$.

To simplify notation, we will write

$$y := y_i^j, \quad \hat{y} := y_i^{j+1}, \quad \check{y} := y_i^{j-1}.$$

So the short form of the equation (3.2) is

$$\varepsilon^2(y_{t\bar{t}} - \hat{y}_{x\bar{x}}) + F'(\hat{y}) = 0. \tag{3.3}$$

Our first result consists in obtaining of the boundedness condition for the problem (3.2) solution.

Lemma 3.1. *Let ε be a sufficiently small constant and let*

$$\frac{\tau}{\varepsilon^2} \le \textit{const}. \tag{3.4}$$

Suppose that the system (3.2) is solvable for any $j = 2,\dots J$. Then uniformly in j,

$$\begin{aligned} &\|\varepsilon y_t\|^2 + \|\varepsilon\hat{y}_x\|^2 + 2\|\sqrt{F(\hat{y})}\|^2 + \frac{\tau}{\varepsilon^2}\left\{|||\varepsilon^2 y_{t\bar{t}}|||^2(j) + |||\varepsilon^2 y_{x\bar{x}}|||^2(j)\right\} \\ &\le \Big\{\|\varepsilon y_t^0\|^2 + \|\varepsilon y_x^1\|^2 + 2\|\sqrt{F(y^1)}\|^2\Big\} e^{c\, t_j\tau/\varepsilon^2}(1+O(\tau)), \end{aligned} \tag{3.5}$$

where $\|\cdot\|$ and $|||\cdot|||(j)$ are the $\mathcal{L}^2$ norms, namely

$$\|f\|^2 = h\sum_{i=1}^{I-1}|f_i|^2, \quad |||f|||^2(j) = \tau\sum_{k=1}^{j}\|f^k\|^2.$$

Here and in what follows c denotes any const > 0 *which does not depend on h, τ, and ε.*

For the proof see Appendix. As a consequence of this lemma and the identity

$$y_i^j = y_i^0 + \tau\sum_{k=0}^{j-1} y_{it}^k$$

we obtain the inequality

$$\|y^j\|^2 \le 2\|y^0\|^2 + 2t_j\tau\sum_{k=0}^{j-1}\|y_t^k\|^2 \le 2\|y^0\|^2 + 2\frac{t_j^2}{\varepsilon^2}c_0, \tag{3.6}$$

where $c_0 > 0$ denotes the right-hand side in (3.5). Obviously, this estimate is very rough. However, it can be improved a little for the specific initial data (2.2) and (2.7).

Lemma 3.2. *Let the assumptions of Lemma 3.1 be satisfied. Then for the initial data $u^0(x_i/\varepsilon)$, $\widetilde{u}^1(x_i/\varepsilon,\tau)$, which approximate the Cauchy data (2.2) or (2.7), the following estimate holds uniformly in j,*

$$\sqrt{\varepsilon}\Big\{\|y_t^j\| + \|y_x^j\| + \|y^j\|\Big\} \le c. \tag{3.7}$$

For the proof it is sufficient to note that the $\mathcal{L}^2$-norms of εy_t^0, εy_x^1, and $\sqrt{F(y^1)}$ are of the value $O(\sqrt{\varepsilon})$.

Furthermore, to investigate the stability let us consider the auxiliary problem:

$$\varepsilon^2(z_{it\bar{t}}^j - z_{ix\bar{x}}^{j+1}) + F'(y_i^{j+1} + z_i^{j+1}) - F'(y_i^{j+1}) = \mathcal{F}_i^j, \tag{3.8}$$

$$z_0^j = 0, \quad z_I^j = 0, \quad z_i^0 = \psi_i^1, \quad \varepsilon z_{it}^0 = \psi_i^2,$$

where ψ_i^l, $\mathcal{F}_i^j$ are such that

$$\|\varepsilon\psi_x^1\|^2 \le c\mu^{k_1}, \quad \|\psi^2\|^2 + \|\tau\psi_x^2\|^2 \le c\mu^{k_2}, \quad \varepsilon^{-1/2}\max_j \|\mathcal{F}^j\|^2 \le c\mu^{k_3}.$$

Lemma 3.3. *Let the assumptions of Lemma 3.2 be satisfied. Then uniformly in j,*

$$\|\varepsilon z_t^j\|^2 + \|\varepsilon z_x^{j+1}\|^2 \le c \max_{l=1,2,3} \mu^{k_l}\ e^{c\,t_j/\varepsilon^{3/2}}. \tag{3.9}$$

For the proof see Appendix.

3.2. **Linearization.** Now we should verify the solvability of the equations in (3.2) for any fixed $j \ge 1$, that is, of the equation

$$y^{j+1} - \tau^2 y_{x\bar{x}}^{j+1} + \frac{\tau^2}{\varepsilon^2} F'(y^{j+1}) = G^j, \quad G^j = y^j + \tau y_{\bar{t}}^j, \tag{3.10}$$

as well as select a way to linearize the nonlinearity. To this aim let us construct the sequence of functions $\varphi(s) := \{\varphi_0(s), \dots, \varphi_I(s)\}$, $s \ge 0$, such that $\varphi(0) = y^j$ and $\varphi(s)$ for $s \ge 1$ satisfies the equation

$$\begin{gathered}\varphi(s) - \tau^2\varphi_{x\bar{x}}(s) \\ + \frac{\tau^2}{\varepsilon^2}\Big\{F'\big(\varphi(s-1)\big) + F''\big(\varphi(s-1)\big)\big(\varphi(s) - \varphi(s-1)\big)\Big\} = G^j, \\ \varphi_0(s) = \nu_\ell, \quad \varphi_I(s) = \nu_r.\end{gathered} \tag{3.11}$$

The solvability of the algebraic system (3.11) is obvious for sufficiently small τ and $\tau/\varepsilon^2 \le$ const. To simplify the notation we write $\varphi := \varphi(s)$, $\bar{\varphi} := \varphi(s-1)$, $\bar{\bar{\varphi}} := \varphi(s-2)$. Let also

$$\mathrm{w} := \varphi - \bar{\varphi}, \quad \bar{\mathrm{w}} := \bar{\varphi} - \bar{\bar{\varphi}}.$$

In view of the identity

$$F'(\varphi) = F'(\bar{\varphi}) + F''(\bar{\varphi})\mathrm{w} + \frac{1}{2}F'''(\vartheta_i)\mathrm{w}^2,$$

where ϑ_i is an intermediate point between φ_i and $\bar{\varphi}_i$, we rewrite (3.11) as

$$\varphi - \tau^2\varphi_{x\bar{x}} + \frac{\tau^2}{\varepsilon^2}\{F'(\varphi) - \frac{1}{2}F'''(\vartheta_i)\mathrm{w}^2\} = G^j. \tag{3.12}$$

Next let us assume the existence of the exact solutions y^k of (3.10) for all $k = 2, 3, \dots, j$. Moreover, for the specific initial data (2.2) and (2.7) we can assume also that y^0 and y^1 satisfy the equation (3.10) (in fact, it is not so important). Then, subtracting one equation (3.12) from the another, we obtain the following equations for the sequence of the auxiliary functions $\mathrm{w} \equiv \mathrm{w}(s)$:

$$\mathrm{w} - \tau^2\mathrm{w}_{x\bar{x}} + \frac{\tau^2}{\varepsilon^2}F''(\bar{\varphi})\mathrm{w} = \frac{\tau^2}{2\varepsilon^2}F'''(\vartheta_i)\bar{\mathrm{w}}^2 \quad \text{for} \quad s > 1, \tag{3.13}$$

$$\mathrm{w} - \tau^2\mathrm{w}_{x\bar{x}} + \frac{\tau^2}{\varepsilon^2}F''(y)\mathrm{w} = \tau f \quad \text{for} \quad s = 1, \tag{3.14}$$

where $f = 2y_{\bar{t}}^j - y_{\bar{t}}^{j-1}$. Applying the standard techniques we verify the estimates for φ and w (for the proof see Appendix).

Lemma 3.4. *Let the assumption* (3.4) *be satisfied and* τ *be sufficiently small. Then*

$$\begin{aligned}\|\varphi\|^2 + \frac{\tau^2}{\varepsilon^2}\|\varepsilon\varphi_x\|^2 \le (1+c\tau)\big\{\|y^j\|^2 + c\sqrt{\tau}(\|y^j\|^2 + \|\varepsilon y_t^j\|^2)\big\} \\ + c\big\{\|\mathrm{w}\|^2 + \tau^2\|\mathrm{w}_x\|^2\big\}^2,\end{aligned} \tag{3.15}$$

$$g(1) \le c\tau^{3/2}, \quad g(s) \le c\tau g^2(s-1) \quad \text{for } s > 1, \tag{3.16}$$

where

$$g(s) := \|\mathrm{w}(s)\|^2 + \tau^2\|\mathrm{w}_x(s)\|^2,$$

and $c > 0$ *denotes a constant which dos not depend on* h, τ, *or* ε.

Combining the estimates (3.16), we immediately conclude that the terms of the w-sequence vanish very rapidly,

$$\|\mathrm{w}(1)\|^2 \le c\tau^{3/2}, \quad \|\mathrm{w}(2)\|^2 \le c\tau^4, \quad \|\mathrm{w}(3)\|^2 \le c\tau^9, \dots \tag{3.17}$$

By (3.15), the terms of φ-sequence are bounded uniformly in s, and

$$\|\varphi(s)\|^2 \le \|y^j\|^2\big(1 + O(\sqrt{\tau})\big). \tag{3.18}$$

Furthermore, for any $n > 0$,

$$\|\varphi(s+n) - \varphi(s)\| \le \sum_{i=1}^{n}\|\mathrm{w}_{s+i}\| \le \|\mathrm{w}_{s+1}\|\sum_{i=1}^{\infty}\frac{\|\mathrm{w}_{s+i}\|}{\|\mathrm{w}_{s+1}\|} \le c\|\mathrm{w}_{s+1}\|.$$

This implies the main statement of this subsection.

Theorem 3.5. *Let assumption* (3.4) *be satisfied and* $\varepsilon = const$. *Then for sufficiently small* τ *the sequence* φ *converges in the* $\mathcal{L}_h^2$ *sense to the solution of the equation* (3.10). *Moreover,*

$$\|y^{j+1} - \varphi(2)\| \le c\tau^{9/2}, \tag{3.19}$$

where $c > 0$ *dos not depend on* h, τ, *or* ε.

3.3. **Algorithm for the numerical simulation.** Since the accuracy $O(\tau^{9/2})$ is much less than the accuracy of the finite differences scheme (3.2), we obtain the following algorithm for the numerical simulation of the problem (3.1) solution:

For a fixed $j = 1, 2, \dots, [T/\tau]$, $T = \text{const}$:

(i) define $\varphi(0) := y^j$,
(ii) calculate $\varphi(s)$, $s = 1, 2$, accordingly with (3.11),
(iii) define $y^{j+1} := \varphi(2)$, redefine $j := j + 1$, and come back to (i).

By the estimates (3.7) and (3.19), this algorithm allows to calculate a bounded in $\mathcal{L}^2(Q_{T,h,\tau})$ numerical solution of problem (3.1).

Note that this result can be improved. Moreover, it turns out that the algorithm is absolutely stable. To prove this we state firstly the proposition (for the sketch see Appendix)

Lemma 3.6. *Let assumption* (3.4) *be satisfied and* $\varepsilon = const$. *Then, uniformly in* s,

$$\|\varepsilon\varphi_t(s)\| \le const, \quad \|\varepsilon\varphi_x(s)\| \le const. \tag{3.20}$$

Moreover, uniformly in j,

$$\|\varepsilon^2 y_{xt}^j\| + \|\varepsilon^2 y_{x\bar{x}}^{j+1}\| \le \frac{c}{\sqrt{\varepsilon}}, \tag{3.21}$$

where c does not depend on τ, h, ε, and

$$\|\varepsilon y_t^{j+1} - \varepsilon\varphi_t(s)\| \le c\varepsilon^{p(s)}, \quad \|\varepsilon y_x^{j+1} - \varepsilon\varphi_x(s)\| \le c\varepsilon^{p(s)} \tag{3.22}$$

with some $p(s)$ which tends to infinity as $s \to \infty$.

An immediate consequence of lemmas 3.1–3.6 is the following result.

Theorem 3.7. *Let assumption* (3.4) *be satisfied and $\varepsilon = const$. Then the solution by the above described finite differences scheme converges to the solution of* (3.1) *as $\tau, h \to 0$, in the $W_2^1(Q_T)$ sense.*

Finally, in view of the boundedness of the sequence $\varphi^j(s)$, $s = 1, 2$, $j = 2, 3, \dots$, it is easy to establish our last statement.

Theorem 3.8. *Under the assumptions of Theorem 3.7 the above described finite differences scheme is stable in the $W_2^1(\Omega_{T,\tau,h})$ sense.*

4. Results of numerical simulation

The numerical algorithm has been implemented as a program and tested using the sine-Gordon equation in the cases of one, two, and three solitary waves.

Example 4.1. Let us apply the above described algorithm to the equation (1.1) with the nonlinearity (2.6). The kink type solution can be found explicitly in this case,

$$\omega(\eta) = \frac{1}{\pi}\operatorname{arccot}\big(-\sqrt{2}\pi\eta\big). \tag{4.1}$$

For the kink–kink interaction we consider the mixed problem (3.1) with $\varepsilon = 0.1$ over the space interval $[0.5, 2]$. The first kink (at the left) is specified by $\beta_1 = 15$ and it moves to the right with the velocity $V_1 = \sqrt{1 - (1/15)^2} \approx 0.99778$. The second kink (at the right) is specified by $\beta_2 = 20$ and it moves to the left with the velocity $V_2 = -\sqrt{1 - (1/20)^2} \approx -0.99875$. The initial front positions are $x_1^0 = 1$ and $x_2^0 = 1.5$. All calculations have been done for the mesh with the parameters $h = 7.5 \cdot 10^{-5}$ and $\tau = 2 \cdot 10^{-5}$. To explain the selection of h and τ so small, let us note that the single kink of the sine-Gordon equation varies over $[0, h]$ as $\exp(\beta_2 h/\varepsilon) = \exp(200h)$. So the selected node density implies the variation like $\exp(1.5 \cdot 10^{-2})$. Such range is a little bit excessively small for the single kink movement, but it is adequate for the process of interaction. For the same argument we set $\tau \approx h/4$.

The result of the numerical simulation is depicted in Fig. 1. It is easy to see that the solitary waves preserve the kink shape during all the time except a small neighborhood of the time instant of interaction. Let us note finally that the sufficient condition E) is satisfied for the nonlinearity (2.6) for any parameters V_1 and V_2.

We now turn the problem of kink-antikink interaction. One can prove that the condition E1) is violated for the nonlinearity (2.6) for any velocities V_1, V_2. However, our hypothesis that E1) is excessively restrictive, is verified numerically for some pairs of the parameters V_1, V_2. The plot in Fig. 2 depicts the evolution of the kink–antikink pair with the same parameters as above. Again, the solitary waves preserve their forms during all the time except a small neighborhood of the time instant $t^* = 0.5/(V_1 - V_2)$ of the interaction. In fact, the waves lose the kink shape at $t \approx 0.25$ and give it back at $t \approx 0.26$ (see Fig. 3).

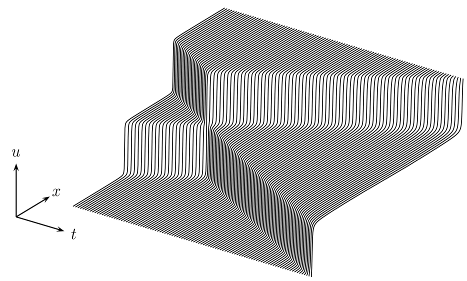

FIGURE 1. Evolution of the kink-kink pair

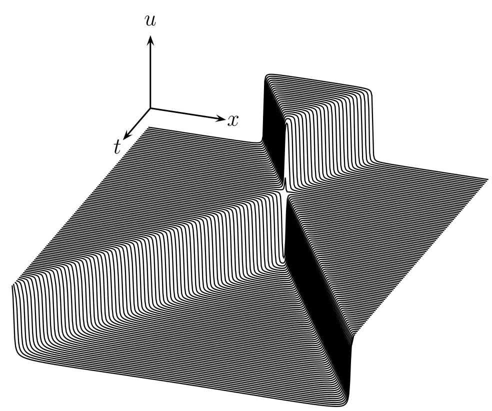

FIGURE 2. Evolution of the kink-antikink pair

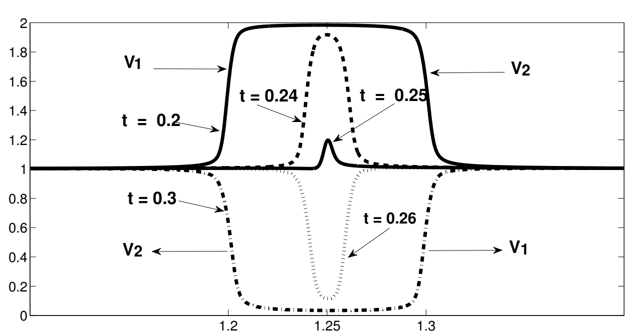

FIGURE 3. Evolution of the kink-antikink pair for some values of time

Example 4.2. For the nonlinearity

$$F(u) = \frac{1}{4\pi^2}\{2 - \cos(2\pi u) - \cos(4\pi u)\} \tag{4.2}$$

the explicit kink type solution does not have any representation in elementary functions. By this reason we solve numerically the Cauchy problem

$$\frac{d\omega}{d\eta} = \sqrt{2F(\omega)}, \quad \eta > 0, \quad \omega|_{\eta=0} = \frac{1}{2} \tag{4.3}$$

and, by the condition C), define ω with negative argument as $\omega(\eta) = 1 - \omega(-\eta)$. To calculate the solution of (4.3) we use the Runge-Kutta method of the forth order with the mesh step $h_\eta = 0.01$.

Next we set the same as above Cauchy problems for the kink–kink and kink-antikink pairs and apply the numerical algorithm with similar mesh parameters. Numerically, the results of calculations for the nonlinearities (4.2) and (2.6) are a little bit different. However, at first sight they are the same and we refer the readers again to the plots in Figures 1-3.

5. Conclusion

Summarizing all stated above, we can deduce that there exists a class of nonlinearities such that kink–kink and kink–antikink pairs preserve the sine-Gordon scenario of interaction at least in the leading term in the asymptotic sense. Apparently, this class can be specified by the assumptions A) - C).

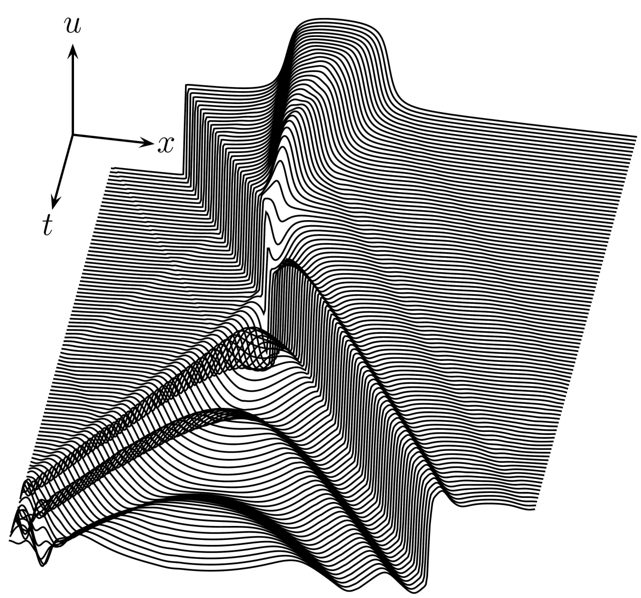

Figure 4. Evolution of the kink-kink–antikink triplet

As for multi-wave interactions, the situation is more complicated and much more interesting (we thank Vladimir Danilov for the suggestion to investigate this problem more in detail). According to the popular hypothesis (see Introduction), we expected that two kinks and one antikink will lose the structure after the triple interaction. This has been realized and the plot in Fig. 4 depicts the evolution of

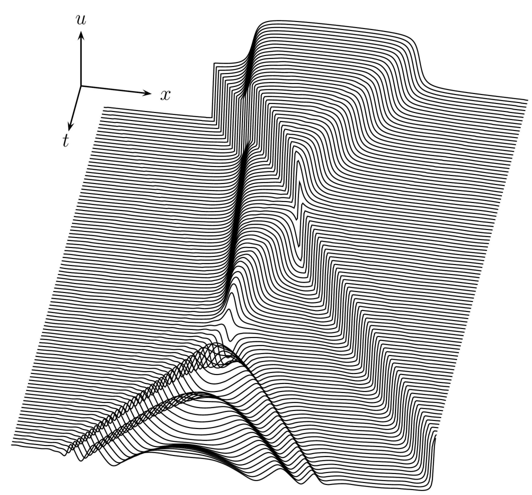

FIGURE 5. Evolution of the kink–kink and kink–antikink pairs

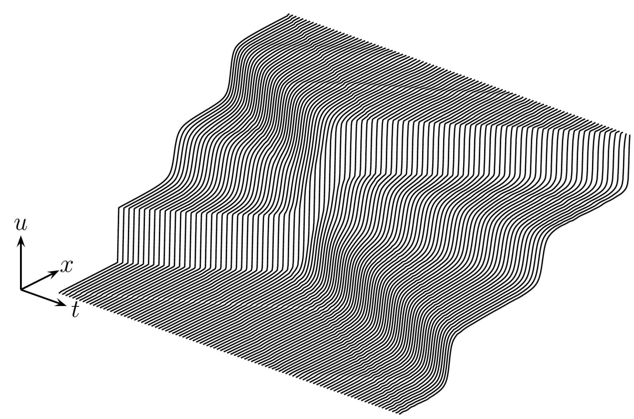

FIGURE 6. Evolution of the kink triplet

such solution for the nonlinearity (2.6). The initial positions and the velocities of the solitary waves are the following: $x_1^0 = 0.5$, $V_1 = 0.99999$, $x_2^0 = 1$, $V_2 = 0.15$, $x_3^0 = 1.5$, $V_3 = -0.69999$. The first unexpected phenomenon appeared when we checked coupled interactions of the same waves (that is for trajectories which intersect by pairs). It turns out that the solution structure goes to ruin after the second interaction (see Fig. 5). Since the pairs of the same waves interact preserving the structure, this result seems to be very strange and we can not explain it.

Moreover, it turned out that three kinks interact according the sine-Gordon scenario. We refer the readers to the plots in Fig. 6 and Fig. 7 which depict

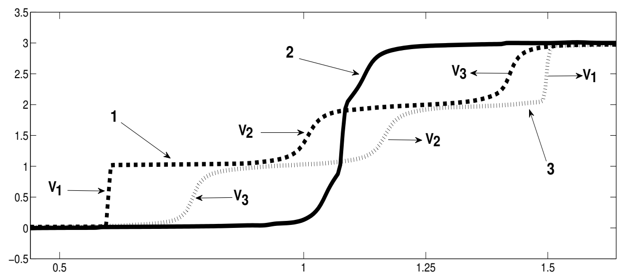

FIGURE 7. The profile evolution before (curve 1), after (curve 3) and at the time instant of interaction (curve 2)

the evolution of the kinks with the parameters $x_1^0 = 0.5$, $V_1 = 0.99999$, $x_2^0 = 1$, $V_2 = 0.15$, $x_3^0 = 1.5$, $V_3 = -0.69999$.

So it is clear now that the problem of multi–wave interaction for the sine-Gordon type equation should be investigated more in detail. It will be done later.

6. Appendix

In what follows we use the notation

$$\|f\|_p = \Big(h\sum_{i=1}^{I-1}|f_i|^p\Big)^{\frac{1}{p}}, \quad \|f\|_{(\ell)} = \big(\|f\|_2^2 + \|\partial_x^\ell f\|_2^2\big)^{\frac{1}{2}}$$

for the discrete analogs of the $L^p(0,L)$ and $W_2^l(0,L)$ norms, where W_2^l is the Sobolev space. Again, for simplicity we write $\|f\| := \|f\|_p$ if $p = 2$.

Our main tools are the discrete versions of the Hölder inequality

$$h\Big|\sum_{i=0}^{N} f_i g_i\Big| \le \|f\|_p\|g\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 < p, q < \infty$$

and the Gagliardo-Nirenberg inequality

$$\|\partial_x^r f\|_p \le c\|f\|_2^{1-\theta}\|f\|_{(\ell)}^{\theta}, \quad \theta\ell = \frac{1}{2} + r - \frac{1}{p}, \tag{6.1}$$

which is the multiplicative form of the embedding theorem for $x \in \mathbb{R}^1$ (see e.g. [11]). Here c is a constant which does not depend on h.

Proof of Lemma 3.1. Let us multiply the equation (3.3) by hy_t and use the equalities

$$2y_{t\bar{t}}y_t = (y_t^2)_{\bar{t}} + \tau(y_{t\bar{t}})^2, \quad 2\hat{y}_x y_{xt} = (y_x^2)_t + \tau(y_{xt})^2.$$

Then, summing over i the result of the multiplication and "integrating by parts" we obtain the identity

$$\varepsilon^2\left\{\partial_{\bar{t}}\|y_t\|^2 + \partial_t\|y_x\|^2 + \tau\|y_{t\bar{t}}\|^2 + \tau\|y_{xt}\|^2\right\} + 2h\sum_{i=1}^{I-1}F'(\hat{y}_i)y_{it} = 0. \tag{6.2}$$

Next, taking into account the Taylor formula, we write

$$F'(\hat{y})y_t = \partial_t(F(y)) - \frac{\tau}{2}F''(\vartheta_i^j)y_t^2,$$

where ϑ_i^j is an intermediate point between y_i^j and y_i^{j+1}. In view of the assumptions A), C), the derivatives of F are bounded by a constant. Therefore, summing over j, we transform (6.2) to the following inequality:

$$\begin{aligned} &\|\varepsilon y_t^j\|^2 + \|\varepsilon y_x^{j+1}\|^2 + 2h\sum_{i=1}^{I-1} F(y_i^{j+1}) + \frac{\tau}{\varepsilon^2}|||\varepsilon^2 y_{t\bar{t}}|||^2(j) + \frac{\tau}{\varepsilon^2}|||\varepsilon^2 y_{xt}|||^2(j) \\ &\le c_0 + \frac{\tau^2}{\varepsilon^2}c_1\sum_{k=1}^{j}\|\varepsilon y_t^k\|^2, \end{aligned} \tag{6.3}$$

where $c_1 > 0$ and

$$c_0 = \|\varepsilon y_t^0\|^2 + \|\varepsilon y_x^1\|^2 + 2h\sum_{i=1}^{I-1} F(y_i^1).$$

Applying the finite differences version of the Gronwall's lemma, we arrive at the estimate (3.5). □

Proof of Lemma 3.3. Obviously, it is sufficient to consider the nonlinear term in (3.8). We write:

$$\begin{aligned} &\tau\sum_{k=1}^{j} h\sum_{i=1}^{I-1} |F'(y_i^{j+1} + z_i^{j+1}) - F'(y_i^{j+1})|\,|z_{it}^j| \\ &\le \tau\sum_{k=1}^{j}\|z_t^j\|\,\|F''(\theta^j y^{j+1} + (1-\theta^j)z^{j+1})\|\max_i |z_i^{j+1}| \\ &\le c\sqrt{\varepsilon}\tau\sum_{k=1}^{j}\|z_t^j\|\,\|z_x^{j+1}\|, \end{aligned}$$

where the specificity of the initial data (2.2), (2.7) has been used. □

The rate $ct_j/\varepsilon^{3/2}$ of the exponent in (3.9) is bad. However, we do not know how to improve the estimate.

Proof of Lemma 3.4. Multiplying (3.12) by $h\varphi$, summing over i, and "integrating by parts", we obtain the inequality

$$\|\varphi\|^2 + \tau^2\|\varphi_x\|^2 \le \frac{1}{2}\Big\{\|G^j\|^2 + (1 + 2\frac{\tau^2}{\varepsilon^2})\|\varphi\|^2 + \frac{\tau^2}{\varepsilon^2}\big(\|F'(\varphi)\|^2 + c\|\mathrm{w}\|_4^4\big)\Big\}.$$

Let us estimate G^j in the form

$$\|G^j\|^2 \le (1+\sqrt{\tau})\|y^j\|^2 + (\tau^{3/2} + \tau^2)\|y_{\bar{t}}^j\|^2. \tag{6.4}$$

Next, applying the Hölder inequality and (6.1) for $p = 4$ and $r = 0$, we obtain

$$\tau\|\mathrm{w}\|_4^4 \le c\tau\|\mathrm{w}\|^3\|\mathrm{w}\|_{(1)} \le c(\|\mathrm{w}\|^2 + \tau^2\|\mathrm{w}_x\|^2)^2. \tag{6.5}$$

Then, in view of the assumption (3.4), we obtain the estimate

$$\begin{aligned} &(1 - c\tau)\|\varphi\|^2 + 2\tau^2\|\varphi_x\|^2 \\ &\le (1+\sqrt{\tau})\|y^j\|^2 + c(\sqrt{\tau} + \tau)\|\varepsilon y_{\bar{t}}^j\|^2 + c\tau + c(\|\mathrm{w}\|^2 + \tau^2\|\mathrm{w}_x\|^2)^2, \end{aligned}$$

which is equivalent to (3.15).

To prove the inequality (3.16) for $s > 1$ we use the estimate of the form (6.5) again; that is,

$$\frac{\tau^2}{\varepsilon^2}h\Big|\sum_{i=1}^{N-1}F'''(\vartheta_i)\bar{\mathrm{w}}_i^2\mathrm{w}_i\Big| \leq c\tau\|\mathrm{w}\|\|\bar{\mathrm{w}}\|_4^2 \leq \frac{1}{2}\|\mathrm{w}\|^2 + c\tau(\|\bar{\mathrm{w}}\|^2+\tau^2\|\bar{\mathrm{w}}_x\|^2)^2.$$

By (3.4), (3.7), to prove the estimate (3.16) for $s = 1$ it is sufficient to note that

$$\tau h\sum_{i=1}^{I-1}\Big|\frac{\tau}{\varepsilon^2}F(y_i)\mathrm{w}_i^2 - f_i\mathrm{w}_i\Big| \leq c\tau\|\mathrm{w}\|^2 + c\tau^{3/4}\|\sqrt{\varepsilon}y_{\bar{t}}\|\|\mathrm{w}\|.$$

□

Sketch of the proof for Lemma 3.6. Let us prove firstly the additional a-priori estimate (3.21). We differentiate the equation (3.2) with respect to x, multiply the result by $h\varepsilon^2 y_{xt}$, and sum over i. By the identity

$$\partial_x F'(\hat{y}_i) = F''(\vartheta_i)\hat{y}_{ix}, \tag{6.6}$$

we arrive at the estimate

$$\partial_{\bar{t}}\|\varepsilon^2 y_{xt}\|^2 + \partial_t\|\varepsilon^2 y_{x\bar{x}}\|^2 + \frac{\tau}{\varepsilon^2}\|\varepsilon^3 y_{xt\bar{t}}\|^2 + \frac{\tau}{\varepsilon^2}\|\varepsilon^3 y_{x\bar{x}t}\|^2 \leq c\|\hat{y}_x\|^2 + \|\varepsilon^2 y_{xt}\|^2.$$

Summing over j, using (3.7), and applying the Gronwall lemma, we obtain the desired a-priori estimate.

Furthermore, repeating similar manipulations with the equation (3.11) we obtain the equality

$$\begin{aligned}&\|\varepsilon\varphi_x\|^2 + \frac{\tau^2}{\varepsilon^2}\|\varepsilon^2\varphi_{x\bar{x}}\|^2 \\ &= -\tau^2 h\sum_{i=1}^{N-1}\left\{F'(\bar{\varphi}_i) + F''(\bar{\varphi}_i)\mathrm{w}_i\right\}_x\varphi_{ix} + \varepsilon^2 h\sum_{i=1}^{N-1}G_{ix}\varphi_{ix}.\end{aligned}\tag{6.7}$$

By (6.6) and the boundedness of $F^{(k)}$,

$$\begin{aligned}\tau^2 h\Big|\sum_{i=1}^{N-1}F'(\bar{\varphi}_i)_x\varphi_{ix}\Big| &= \tau^2 h\Big|\sum_{i=1}^{N-1}F''(\bar{\vartheta}_i)(\varphi_{ix}-\mathrm{w}_{ix})\varphi_{ix}\Big| \\ &\leq c\tau^2(\|\varphi_x\|^2 + \|\mathrm{w}_x\|^2) \\ &\leq c\tau(\|\varepsilon\varphi_x\|^2 + \|\varepsilon\mathrm{w}_x\|^2).\end{aligned}\tag{6.8}$$

Next we write

$$(F''(\bar{\varphi}_i)\mathrm{w}_i)_x = F''(\bar{\varphi}_i)\mathrm{w}_{ix} + F'''(\bar{\vartheta}_i)(\varphi_{ix}-\mathrm{w}_{ix})\mathrm{w}_{i+1}.$$

The Gagliardo-Nirenberg inequality for $p = 4$, $r = 1$, (3.4), and (3.16) imply

$$\begin{aligned}\tau^2 h\Big|\sum_{i=1}^{N-1}F'''(\bar{\vartheta}_i)\varphi_{ix}^2\mathrm{w}_{i+1}\Big| &\leq c\tau^2\|\varphi_x\|_4^2\|\mathrm{w}\| \\ &\leq c\tau^{11/4}\|\varphi\|^{3/4}\,\|\varphi\|_{(2)}^{5/4} \\ &\leq c\tau^{11/16}\big\{\|\sqrt{\varepsilon}\varphi\|^2 + \|\varepsilon\tau\varphi\|_{(2)}^2\big\}\end{aligned}\tag{6.9}$$

and

$$\begin{aligned}\tau^2 h\Big|\sum_{i=1}^{N-1} F'''(\bar\vartheta_i)\mathrm{w}_{ix}\mathrm{w}_{i+1}\varphi_{ix}\Big| &\le c\tau^2\|\mathrm{w}\|\|\varphi_x\|_4\|\mathrm{w}_x\|_4 \\ &\le c\tau^{17/16}\|\sqrt{\varepsilon}\varphi\|^{3/8}\|\varepsilon\tau\varphi\|_{(2)}^{5/8}\|\varepsilon\tau\mathrm{w}\|_{(2)}^{5/8} \\ &\le c\tau^{17/16}\big\{\|\sqrt{\varepsilon}\varphi\|+\|\varepsilon\tau\varphi\|_{(2)}^2+\|\varepsilon\tau\mathrm{w}\|_{(2)}^2\big\}.\end{aligned} \tag{6.10}$$

Furthermore, in view of (3.7) and (3.10),

$$\|\varepsilon G_x\| = \|\varepsilon(2y_x^j - y_x^{j-1})\| \le c\sqrt{\varepsilon}. \tag{6.11}$$

Combining (6.7)–(6.11) we arrive at the inequality

$$\|\varepsilon\varphi_x\|^2 + \frac{\tau^2}{\varepsilon^2}\|\varepsilon^2\varphi_{x\bar x}\|^2 \le c(\sqrt{\varepsilon}+\tau^{11/16}) + c\tau\Big(\|\varepsilon\mathrm{w}_x\|^2 + \frac{\tau^2}{\varepsilon^2}\|\varepsilon^2\mathrm{w}_{x\bar x}\|^2\Big). \tag{6.12}$$

To close the estimates, we should come back to the equations (3.13), (3.14) again. For $s=1$ we use the inequality similar to (6.8); that is,

$$\begin{aligned}\tau^2 h\Big|\sum_{i=1}^{N-1}(F''(y_i)\mathrm{w}_i)_x\mathrm{w}_{ix}\Big| &\le c\tau^2\big(\|\mathrm{w}_x\|^2 + \|y_x\|\|\mathrm{w}\|_4\|\mathrm{w}_x\|_4\big) \\ &\le c\tau\big(\|\varepsilon\mathrm{w}_x\|^2 + \|\varepsilon\tau\mathrm{w}_{x\bar x}\|^2\big) + c\tau^{13/10}.\end{aligned}$$

This and (3.21) yield

$$\|\varepsilon\mathrm{w}_x\|^2 + \frac{\tau^2}{\varepsilon^2}\|\varepsilon^2\mathrm{w}_{x\bar x}\|^2 \le c\sqrt{\tau}, \quad s=1. \tag{6.13}$$

To estimate $\varepsilon\mathrm{w}_x$ for $s>1$ we write firstly:

$$\tau^2 h\Big|\sum_{i=1}^{N-1}(F'''(\bar\vartheta_i)\bar{\mathrm{w}}^2)_x\mathrm{w}_x\Big| \le \frac{1}{2}\|\varepsilon\mathrm{w}_x\|^2 + c\tau^3\left\{\|\bar{\mathrm{w}}\bar{\mathrm{w}}_x\| + \|\bar\varphi_x\bar{\mathrm{w}}^2\|\right\}^2, \tag{6.14}$$

where $\bar\vartheta_i = \alpha_i\bar\varphi_i + (1-\alpha_i)\bar{\bar\varphi}_i, \alpha_i\in[0,1]$. Furthermore, by (6.1),

$$\tau^{3/2}\|\bar{\mathrm{w}}\bar{\mathrm{w}}_x\| \le c\tau^{3/8}\{\|\bar{\mathrm{w}}\|^2 + \|\varepsilon\tau\bar{\mathrm{w}}_{x\bar x}\|^2\}$$

and by (6.1) and (6.12),

$$\tau^{3/2}\|\bar\varphi_x\bar{\mathrm{w}}^2\| \le c\tau^{3/32}\|\varepsilon\tau\bar\varphi\|_{(2)}^{5/8}\|\bar{\mathrm{w}}\|^{7/4}\|\varepsilon\tau\bar{\mathrm{w}}\|_{(2)}^{1/4} \le c\tau^{3/32}\left\{\|\bar{\mathrm{w}}\|^2 + \|\varepsilon\tau\bar{\mathrm{w}}_{x\bar x}\|^2\right\}.$$

Next

$$\begin{aligned}\tau^2 h\Big|\sum_{i=1}^{N-1}(F''(\bar\varphi_i)\mathrm{w}_i)_x\mathrm{w}_x\Big| &\le c\tau^2\big\{\|\mathrm{w}_x\|^2 + \|\bar\varphi_x\|\|\mathrm{w}\|_4\|\mathrm{w}_x\|_4\big\} \\ &\le c\tau\|\varepsilon\mathrm{w}_x\|^2 + c\tau^{3/8}\|\varepsilon\bar\varphi_x\|\big\{\|\mathrm{w}\|^2 + \|\varepsilon\tau\mathrm{w}_{x\bar x}\|^2\big\}.\end{aligned}$$

Taking into account (3.16), (6.12), and denoting

$$f(s) = \big\{\|\mathrm{w}\|^2 + \|\varepsilon\mathrm{w}_x\|^2 + \frac{\tau^2}{\varepsilon^2}\big(\|\varepsilon\mathrm{w}_x\|^2 + \|\varepsilon^2\mathrm{w}_{x\bar x}\|^2\big)\big\}(s),$$

we arrive at the inequality

$$f(s) \le c\tau^{3/8}\big\{\varepsilon^{1/4} + \tau^{11/34} + \tau\sqrt{f(s-1)}\big\}f(s) + c\tau^{3/16}f^2(s-1). \tag{6.15}$$

In view of (6.13),

$$f(1) \le c\sqrt{\tau}. \tag{6.16}$$

From this and (6.15) it follows that (6.16) holds uniformly in $s \geq 1$. Therefore,

$$f(s) \leq c\tau^{3/16} f^2(s-1), \quad s > 1, \tag{6.17}$$

which implies the convergence of the sequence $\|\varepsilon\varphi_x\|(s)$ as $s \to \infty$. Moreover,

$$\|\varepsilon \mathrm{w}_x\|(s) \leq c_s \tau^{p(s)}, \quad s \geq 1,$$

where c_s does not depend on τ or ε, and $p(s) \to \infty$ as $s \to \infty$. In particular,

$$p(1) = \frac{1}{4}, \quad p(2) = \frac{19}{32}, \quad p(3) = \frac{41}{32}, \quad p(4) = \frac{85}{32}.$$

To prove the second part of Lemma 3.6 statement we do the same as above but for the derivative with respect to t. $\square$

Sketch of the proof for Theorem 3.8. By Lemma 3.3 it is sufficient to prove the stability of the $\varphi(s)$ calculations for $s = 1, 2$. To this aim let us consider the recurrence equation

$$\begin{aligned} &\Phi(s) - \tau^2\Phi(s)_{x\bar{x}} + \frac{\tau^2}{\varepsilon^2}\Big\{F'\big(\varphi(s-1) + \Phi(s-1)\big) - F'\big(\varphi(s-1)\big) \\ &+ F''\big(\varphi(s-1) + \Phi(s-1)\big)\big(\varphi(s) + \Phi(s) - \varphi(s-1) - \Phi(s-1)\big) \\ &- F''\big(\varphi(s-1)\big)\big(\varphi(s) - \varphi(s-1)\big)\Big\} = \widetilde{G}, \quad s = 1, 2 \end{aligned} \tag{6.18}$$

for the difference $\Phi(s) = \varphi_1(s) - \varphi_2(s)$ of two pairs $\varphi_i(s)$, $s = 0, 1, 2$. We assume that

$$\|\Phi(0)\|^2 + \|\varepsilon\Phi_x(0)\|^2 + \|\varepsilon\Phi_t(0)\|^2 \leq \mu^{k_3}, \tag{6.19}$$

$$\|\widetilde{G}\|^2 + \|\varepsilon\widetilde{G}_x\|^2 + \|\varepsilon\widetilde{G}_t\|^2 \leq \mu^{k_3}. \tag{6.20}$$

Multiplying (6.18) by $\Phi(s)$ and using the boundedness of $F^{(k)}$, we obtain the inequality

$$\begin{aligned} \|\Phi(s)\|^2 + \tau^2\|\Phi_x(s)\|^2 \leq \Big(\frac{1}{4} + c\frac{\tau^2}{\varepsilon^2}\Big)\|\Phi(s)\|^2 + \mu^{k_3} + c\frac{\tau^2}{\varepsilon^2}\Big(\|\Phi(s)\|_3^3 \\ + \|\Phi(s)\|_4^2\|\Phi(s-1)\| + \|\Phi(s-1)\|^2\Big). \end{aligned} \tag{6.21}$$

Next the Gagliardo-Nirenberg inequality and the assumption (3.4) imply

$$\frac{\tau^2}{\varepsilon^2}\|\Phi(s)\|_3^3 \leq c\tau^{1/4}\varepsilon^{1/2}\|\Phi(s)\|\Big(\|\Phi(s)\|^2 + \tau^2\|\Phi_x(s)\|^2\Big), \tag{6.22}$$

$$\frac{\tau^2}{\varepsilon^2}\|\Phi(s)\|_4^2 \leq c\tau^{1/4}\varepsilon^{1/2}\|\Phi(s)\|\|\Phi(s)\|^{1/2}(\tau\|\Phi_x(s)\|)^{1/2}. \tag{6.23}$$

By (3.18), $\|\Phi(s)\|$ is bounded uniformly in s: $\|\Phi(s)\| \leq 1/\sqrt{\varepsilon}$. Therefore, combining (6.21)-(6.23) we arrive at the inequality

$$\|\Phi(s)\|^2 + \tau^2\|\Phi_x(s)\|^2 \leq c\sqrt{\tau}\Big(\|\Phi(s-1)\|^2 + \tau^2\|\Phi_x(s-1)\|^2\Big) + \mu^{k_3}.$$

In view of (6.19),

$$\|\Phi(s)\|^2 + \tau^2\|\Phi_x(s)\|^2 \leq (1 + c\sqrt{\tau})^2\mu^{k_3}. \tag{6.24}$$

Repeating the same for the derivatives $\varepsilon\Phi_x(s)$, $\varepsilon\Phi_t(s)$, and taking into account Lemma 3.3 we complete the proof. $\square$

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