Electronic Journal of Differential Equations, Vol. 2010(2010), No. 157, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS IN THE α-NORM FOR PARTIAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE WITH FINITE DELAY

KHALIL EZZINBI, HATEM MEGDICHE, AMOR REBEY

Abstract. In this work, we prove results on the local existence of mild solution and global continuation in the α-norm for some class of partial neutral differential equations. We suppose that the linear part generates a compact analytic semigroup. The nonlinear part is just assumed to be continuous. We use the compactness method, to show the main result of this work.

1. Introduction

In this work, we study the existence and global continuation of solutions in the α-norm for partial differential equations of neutral type with finite delay. The following model provides an example of such a situation

$$\begin{gathered}\frac{\partial}{\partial t}[v(t,x)-av(t-r,x)]\\ =\frac{\partial^2}{\partial x^2}[v(t,x)-av(t-r,x)]+f(\frac{\partial}{\partial x}v(t-r,x))\quad \text{for } t\geq 0,\ x\in[0,\pi]\\ v(t,0)=av(t-r,0),\quad v(t,\pi)=av(t-r,\pi)\quad \text{for } t\geq 0,\\ v(t,x)=v_0(t,x)\quad \text{for } -r\leq t\leq 0,\ x\in[0,\pi],\end{gathered}\tag{1.1}$$

where a and r are positive constants, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, and v_0 is a given initial function from $[-r,0]\times[0,\pi]$ to $\mathbb{R}$. Equation (1.1) can be written in the following abstract form for partial differential equations

$$\begin{gathered}\frac{d}{dt}Du_t=-ADu_t+F(t,\ u_t)\quad \text{for } t\geq 0,\\ u_0=\varphi,\quad \varphi\in C_\alpha,\end{gathered}\tag{1.2}$$

where $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space X, $C_\alpha := C([-r,0];D(A^\alpha))$, $0<\alpha<1$, denotes the space of continuous functions from $[-r,0]$ into $D(A^\alpha)$, and the operator A^α is the fractional α-power of A. This operator $(A^\alpha, D(A^\alpha))$ will be described later. For $x\in C([-r,b];D(A^\alpha)), b>0$, and $t\in[0,b]$, x_t denotes, as usual, the element of C_α defined by $x_t(\theta)=x(t+\theta)$ for $\theta\in[-r,0]$. F is a continuous function from $\mathbb{R}_+\times C_\alpha$ with values in X and D

2000 *Mathematics Subject Classification.* 34K30, 47D06.
Key words and phrases. Neutral equation; analytic semigroup; fractional power; phase space; mild solution; Sadovskii's fixed point theorem.

Submitted June 28, 2010. Published October 29, 2010.

is a bounded linear operator from $C_X := C([-r,0];X)$ into X defined by $D\varphi = \varphi(0) - D_0\varphi$, for $\varphi \in C_X$, where D_0 is a bounded linear operator given by:

$$D_0\varphi = \int_{-r}^{0} d\eta(\theta)\varphi(\theta) \quad \text{for } \varphi \in C_X,$$

where $\eta : [-r,0] \to \mathcal{L}(X)$ is of bounded variation and non-atomic at zero. That is, there is a continuous nondecreasing function $\delta : [0,r] \to [0,+\infty[$ such that $\delta(0) = 0$ and

$$\Big\| \int_{-s}^{0} d\eta(\theta)\varphi(\theta) \Big\| \le \delta(s)\|\varphi\| \quad \text{for } \varphi \in C_X\,; s \in [0,r]. \tag{1.3}$$

There is an extensive literature of differential equations of neutral type motivated by physical applications. Xia and Wu (1996), Hale (1994), and Wu (1996) studied the neutral partial functional differential equation

$$\frac{\partial}{\partial t} Du_t = K\frac{\partial^2}{\partial x^2} Du_t + f(u_t) \quad \text{for } x \in S^1, \tag{1.4}$$

where K is a positive constant and X be the space $C(S^1,\mathbb{R})$. Let $A = K\frac{\partial^2}{\partial x^2}$ with domain $C^2(S^1,\mathbb{R})$, then A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on X and the associated integrated form of (1.4) subject to the initial condition $u_0 = \varphi \in C([-r,0],X)$ is

$$D(u_t) = T(t)D(\varphi) + \int_0^t T(t-s)f(u_s)ds \quad \text{for } t \ge 0. \tag{1.5}$$

Wu [16] established the existence of mild solution of (1.5). Travis and webb [14] considered partial differential equations of the form

$$\begin{gathered} \frac{d}{dt}u(t) = -Au(t) + F(t,u_t) \quad t \ge 0, \\ u_0 = \varphi, \quad \varphi \in C_\alpha, \end{gathered} \tag{1.6}$$

where $-A$ the infinitesimal generator of a compact analytic semigroup and F is only continuous with respect to a fractional power of A in the second variable.

This work is motivated by the paper of Travis and Webb [14], where the authors studied the existence and continuability in the α-norm for equation (1.2) but in the case where $D_0 = 0$, they assumed that $F : C_\alpha \to X$ is continuous. In [2] the authors obtained the local and the global existence of solution of Eq. (1.2) for $\alpha = 0$ in the case when the linear part is non densely defined Hille-Yosida. Recently, in [1] Adimy and Ezzinbi have developed a basic theory of partial neutral functional differential equations in fractional power spaces, they proved the existence and regularity of the solution of Eq. (1.2) where the nonlinear part satisfies Lipschitz conditions.

The present paper is organized as follows. In the first section, we introduce some notations and necessary preliminaries. In Section 2, we study the local existence and global continuation of mild solutions of (1.2). Finally, to illustrate our results, we give in Section 3 an application.

2. Existence of local mild solutions

In this section we study the existence of mild solutions for the abstract Cauchy problem (1.2). Before that, we state the following assumption.

(H1) $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\ge 0}$ on a Banach space X and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.

Note that if $0 \in \rho(A)$ is not satisfied, one can substitute the operator A by the operator $(A - \sigma I)$ with σ large enough such that $0 \in \rho(A - \sigma)$. This allows us to define the fractional power A^α for $0 < \alpha < 1$, as a closed linear invertible operator with domain $D(A^\alpha)$ dense in X. The closedness of A^α implies that $D(A^\alpha)$, endowed with the graph norm of A^α; i.e., the norm $|x| = \|x\| + \|A^\alpha x\|$, is a Banach space. Since A^α is invertible, its graph norm $|\cdot|$ is equivalent to the norm $\|x\|_\alpha = \|A^\alpha x\|$. Thus, $D(A^\alpha)$ equipped with the norm $\|\cdot\|_\alpha$, is a Banach space, which we denote by X_α. For $0 < \beta \le \alpha < 1$, the imbedding $X_\alpha \hookrightarrow X_\beta$ is compact if the resolvent operator of A is compact. Also, the following properties are well known.

Theorem 2.1 ([11]). *Let $0 < \alpha < 1$ and assume that (H1) holds. Then*

(i) *$T(t) : X \longrightarrow D(A^\alpha)$ for every $t > 0$,*
(ii) *$T(t)A^\alpha x = A^\alpha T(t)x$ for every $x \in D(A^\alpha)$ and $t \ge 0$,*
(iii) *for every $t > 0$ the operator $A^\alpha T(t)$ is bounded on X and there exists $M_\alpha > 0$ such that*
$$\|A^\alpha T(t)\| \le M_\alpha e^{\omega t} t^{-\alpha}, \tag{2.1}$$
(iv) *There exists $N_\alpha > 0$ such that*
$$\|(T(t) - I)A^{-\alpha}\| \le N_\alpha t^\alpha \quad \text{for } t > 0. \tag{2.2}$$

In the sequel, we denote by $C_\alpha := C([-r, 0]; X_\alpha)$ the Banach space of all continuous function from $[-r, 0]$ to X_α endowed with the norm
$$\|\varphi\|_{C_\alpha} := \sup_{\theta \in [-r,0]} \|\varphi(\theta)\|_\alpha \quad \text{for } \varphi \in C_\alpha.$$

Definition 2.2. Let $\varphi \in C_\alpha$. A continuous function $u : [-r, +\infty[\to X_\alpha$ is called a mild solution of (1.2) if

(i) $D(u_t) = T(t)D(\varphi) + \int_0^t T(t-s)F(s, u_s)ds$ for $t \ge 0$,
(ii) $u_0 = \varphi$.

Besides (H1), we consider the hypothesis:

(H2) The semigroup $(T(t))_{t \ge 0}$ is compact on X.
(H3) If $x \in X_\alpha$ and $\theta \in [-r, 0]$ then $\eta(\theta)x \in X_\alpha$ and $A^\alpha \eta(\theta)x = \eta(\theta)A^\alpha x$.

Remark 2.3. Assumption **(H3)** implies that if $\varphi \in C_\alpha$ then
$$D_0(\varphi) \in X_\alpha, \quad A^\alpha D_0(\varphi) = D_0(A^\alpha \varphi), \tag{2.3}$$
where
$$(A^\alpha \varphi)(\theta) = A^\alpha(\varphi(\theta)) \quad \text{for } \theta \in [-r, 0],\ \varphi \in C_\alpha.$$

The main result of this section is the following theorem.

Theorem 2.4. *Assume that the hypothesis* (H1)–(H3) *hold true. Let U be an open subset of the Banach space C_α. If $F : [0, a] \times U \to X$ is continuous, then for each $\varphi \in U$ there exist $t_1 := t_1(\varphi)$ with $0 < t_1 \le a$ and a mild solution $u \in C([-r, t_1]; X_\alpha)$ of* (1.2).

Proof. The proof of this result is based on the Sadovskii's fixed-point theorem. Let $\varphi \in U$ and $0 < t_1 \le a$. We choose $0 < \rho \le a$ to be small enough such that $\{\psi \in C_\alpha : \|\psi - \varphi\|_{C_\alpha} \le \rho\} \subset U$. Now we consider the set
$$\Omega := \{u \in C([-r, t_1]; X_\alpha) : u_0 = \varphi \text{ and } \|u_t - \varphi\|_{C_\alpha} \le \rho \text{ for } t \in [0, t_1]\},$$

where $C([-r,t_1];X_\alpha)$ is endowed with the uniform convergence topology. It is easy to check that Ω is nonempty and bounded. By using the triangular inequality, it is clear that $\lambda u_1+(1-\lambda)u_2\in\Omega$, for any $u_1,u_2\in\Omega$ and $\lambda\in]0,1[$. Then Ω is convex. Also, Ω is closed in $C([-r,t_1];X_\alpha)$. To prove that, consider a convergent sequence $(u^n)_{n\geq 0}$ of Ω with $\lim_{n\to+\infty}u^n=u$ in Ω. Then, for any n in $\mathbb{N}$, we have

$$\|u_0-\varphi\|_{C_\alpha}\leq\|u_0-u_0^n\|_{C_\alpha}+\|u_0^n-\varphi\|_{C_\alpha},$$

letting n to $+\infty$, yields $\|u_0-\varphi\|_{C_\alpha}=0$, then $u_0=\varphi$. In addition for any $t\in[0,t_1]$, $n\in\mathbb{N}$

$$\begin{aligned}\|u_t-\varphi\|_{C_\alpha}&\leq\|u_t-u_t^n\|_{C_\alpha}+\|u_t^n-\varphi\|_{C_\alpha}\\&\leq\|u_t-u_t^n\|_{C_\alpha}+\rho.\end{aligned}$$

Letting n to $+\infty$, we deduce that $\|u_t-\varphi\|_{C_\alpha}\leq\rho$. Consequently, $u\in\Omega$. We have Ω is a nonempty, bounded, convex and closed subset of $C([-r,t_1];X_\alpha)$ when t_1 is given by (2.6).

Let the mapping $H:\Omega\to C([-r,t_1];X_\alpha)$ be defined by

$$H(u)(t)=\begin{cases}D_0(u_t)+T(t)D(\varphi)+\int_0^t T(t-s)F(s,u_s)ds & \text{if } t\in[0,t_1],\\ \varphi(t) & \text{if } t\in[-r,0].\end{cases}$$

We will prove now the continuity of H. Let $(u^n)_{n\geq 1}$ be a convergent sequence in Ω with $\lim_{n\to\infty}u^n=u$. Using (2.3), and that D_0 is a bounded linear operator, there exists a positive constant M such that

$$\|D_0(u_t^n)-D_0(u_t)\|_\alpha\leq M\|u_t^n-u_t\|_{C_\alpha}. \tag{2.4}$$

On the other hand the set $\Lambda=\{(s,u_s^n),(s,u_s):s\in[0,t_1],n\geq 1\}$ is compact in $[0,t_1]\times C_\alpha$. By Heïne's theorem implies that F is uniformly continuous in Λ. Accordingly, since $(u^n)_{n\geq 1}$ converge to u, we have

$$\|H(u^n)-H(u)\|_\infty\leq M_\alpha\int_0^{t_1}\frac{e^{\omega s}}{s^\alpha}ds\sup_{s\in[0,t_1]}\|F(s,u_s^n)-F(s,u_s)\|\to 0 \text{ as } n\to+\infty. \tag{2.5}$$

Using (2.5) and the estimate (2.4), we obtain that $(Hu^n)_{n\geq 1}$ converge to Hu. This yields the continuity of H.

We will show that there exists $t_1:=t_1(\varphi)\in]0,a]$ such that $H(\Omega)\subseteq\Omega$. Let $u\in\Omega$. We have the following translation property

$$(Hu)_t(\theta)=\begin{cases}D_0u_{t+\theta}+T(t+\theta)D\varphi\\ +\int_0^{t+\theta}T(t+\theta-s)F(s,u_s)ds & \text{if } t+\theta\in[0,t_1],\\ \varphi(t+\theta) & \text{if } t+\theta\in[-r,0].\end{cases}$$

Choose $\gamma>0$ such that

$$\|\varphi(t+\theta)-\varphi(\theta)\|_\alpha\leq\frac{\rho}{5}\min\{1,\frac{1}{\operatorname{var}_{[-r,0]}(\eta)}\},$$

for $t\in[0,\gamma]$ and $\theta\in[-r,0]$ such that $t+\theta\in[-r,0]$. This implies in particular that $\|(H(u))_t(\theta)-\varphi(\theta)\|_\alpha\leq\rho$, for $t\in[0,\gamma]$ and $\theta\in[-r,0]$ such that $t+\theta\in[-r,0]$.

Choose $s\in]0,r]$ such that $\delta(s)\leq 1/5$ and $\|T(t)D\varphi-D\varphi\|_\alpha\leq\rho/5$, for $t\in[0,s]$.

If $0 \le t+\theta \le s$, then

$$\begin{aligned}
(H(u))_t(\theta)-\varphi(\theta) &= \int_{-r}^{-s} d\eta(\tau)\Big(\varphi(t+\theta+\tau)-\varphi(\tau)\Big)\\
&\quad + \int_{-s}^{0} d\eta(\tau)\Big(u_{t+\theta}(\tau)-\varphi(\tau)\Big) + T(t+\theta)D(\varphi)-D(\varphi)\\
&\quad + \varphi(0)-\varphi(\theta)+\int_0^{t+\theta} T(t+\theta-s)F(s,u_s)ds.
\end{aligned}$$

As F is continuous, we can choose $\rho>0$ small enough such that there exists $N>0$ so that $\|F(t,\psi)\|\le N$, for $t\in[0,\rho]$ and $\|\psi-\varphi\|_{C_\alpha}\le\rho$. Then, if $t_1\le\rho$ we obtain

$$\|\int_0^{t+\theta} T(t+\theta-s)F(s,u_s)ds\|_\alpha \le M_\alpha N\int_0^t e^{\omega s}s^{-\alpha}ds.$$

We can take γ such that $\int_0^\gamma e^{\omega s}s^{-\alpha}ds\le\frac{\rho}{5M_\alpha N}$. We deduce that

$$\begin{aligned}
\|(H(u))_t(\theta)-\varphi(\theta)\|_\alpha &\le \mathrm{var}_{[-r,0]}(\eta)\sup_{\tau\in[-r,-s]}\|\varphi(t+\theta+\tau)-\varphi(\tau)\|_\alpha\\
&\quad + \delta(s)\|u_{t+\theta}-\varphi\|_{C_\alpha}+\|\varphi(0)-\varphi(\theta)\|_\alpha\\
&\quad + \|T(t+\theta)D(\varphi)-D(\varphi)\|_\alpha + M_\alpha N\int_0^t e^{\omega s}s^{-\alpha}ds.
\end{aligned}$$

Finally, we choose

$$t_1=\min\{\gamma,s,\rho\}. \tag{2.6}$$

Then, for $0\le t+\theta\le t_1$, we obtain $\|(H(u))_t(\theta)-\varphi(\theta)\|_\alpha\le\rho$. So, we have proved that there exists $t_1:=t_1(\varphi)\in]0,a]$ such that $H(\Omega)\subseteq\Omega$. Consider now the mapping $H_1:\Omega\to C([-r,t_1];X_\alpha)$ defined by

$$H_1(u)(t)=\begin{cases} D_0(u_t) & \text{if } t\in[0,t_1],\\ \varphi(t)-D\varphi & \text{if } t\in[-r,0].\end{cases}$$

Also define $H_2:\Omega\to C([-r,t_1];X_\alpha)$ by

$$H_2(u)(t)=\begin{cases} T(t)D(\varphi)+\int_0^t T(t-s)F(s,u_s)ds & \text{if } t\in[0,t_1],\\ D\varphi & \text{if } t\in[-r,0].\end{cases}$$

It is clear that $H=H_1+H_2$. If we prove that H_1 is a strict contraction and H_2 is compact. Apply the Sadovskii's fixed theorem to obtain the existence of a fixed point of H on Ω.

(1) Let $u,v\in\Omega$. Then for each $t\in[0,t_1]$, we have

$$\begin{aligned}
H_1u(t)-H_1v(t) &= D_0(u_t-v_t)\\
&= \int_{-r}^0 d\eta(\theta)(u(t+\theta)-v(t+\theta))\\
&= \int_{-s}^0 d\eta(\theta)(u(t+\theta)-v(t+\theta)).
\end{aligned}$$

According to (2.3), we have

$$A^\alpha D_0(u_t-v_t)=\int_{-s}^0 d\eta(\theta)A^\alpha(u(t+\theta)-v(t+\theta)),$$

which implies

$$\|D_0(u_t - v_t)\|_\alpha \le \delta(s) \sup_{-r\le t\le t_1} \|u(t) - v(t)\|_\alpha$$

Consequently,

$$\sup_{-r\le t\le t_1} \|H_1u(t) - H_1v(t)\|_\alpha \le \delta(s) \sup_{-r\le t\le t_1} \|u(t) - v(t)\|_\alpha.$$

Since $\delta(s) \le 1/5$, H_1 is therefore a strict contraction in Ω.

(2) We will show that the $\mathrm{Im}(H_2) := \{H_2(u), u \in \Omega\}$, is relatively compact. By the Arzela-Ascoli theorem it suffices to prove that the set $\{H_2(u)(t) : u \in \Omega\}$ is a relatively compact in X_α for each $t \in [0, t_1]$, and $H_2(\Omega)$ is an equicontinuous family of functions on $[0, t_1]$.

(i) To prove the first assertion, it is sufficient to show that the set $\{H_2u(t) : u \in \Omega\}$ is relatively compact for each $t \in]0, t_1]$. Let $t \in]0, t_1]$ fixed, and $\beta > 0$ such that $\alpha < \beta < 1$, we have

$$\begin{aligned}
\|(A^\beta H_2u)(t)\| &\le \|A^{\beta-\alpha}T(t)A^\alpha D(\varphi)\| + \|\int_0^t A^\beta T(t-s)F(s,u_s)ds\| \\
&\le M_{\beta-\alpha}e^{\omega t}t^{\alpha-\beta}\|D(\varphi)\|_\alpha + M_\beta N \int_0^t e^{\omega s}s^{-\beta}ds < +\infty.
\end{aligned}$$

Then for fixed $t \in]0, t_1]$, $\{(A^\beta H_2u)(t)\}$ is bounded in X, and appealing to the compactness of $A^{-\beta} : X \to X_\alpha$, we deduce that $\{H_2(u)(t) : u \in \Omega\}$ is relatively compact set in X_α.

(ii) On the other hand, for every $0 \le t_0 < t \le t_1$, one has

$$\begin{aligned}
H_2u(t) - H_2u(t_0) &= (T(t) - T(t_0))D\varphi + \int_{t_0}^t T(t-s)F(s,u_s)ds \\
&\quad + \int_0^{t_0} (T(t-s) - T(t_0-s))F(s,u_s)ds \\
&= (T(t) - T(t_0))D\varphi + \int_{t_0}^t T(t-s)F(s,u_s)ds \\
&\quad + (T(t-t_0) - I)\int_0^{t_0} T(t_0-s)F(s,u_s)ds.
\end{aligned}$$

We obtain that

$$\begin{aligned}
\|H_2u(t) - H_2u(t_0)\|_\alpha &\le \|(T(t) - T(t_0))D\varphi\|_\alpha + M_\alpha N \int_{t_0}^t \frac{e^{\omega s}}{s^\alpha}ds \\
&\quad + \|(T(t-t_0) - I)\int_0^{t_0} A^\alpha T(t_0-s)F(s,u_s)ds\|.
\end{aligned}$$

It is clear that the first part tend to zero as $|t - t_0| \to 0$, since for $t_0 > 0$ the set

$$\Big\{\int_0^{t_0} A^\alpha T(t_0-s)F(s,u_s)ds,\ u \in \Omega\Big\}$$

is relatively compact in X, there is a compact set $\widetilde{K}$ in X such that

$$\int_0^{t_0} A^\alpha T(t_0-s)F(s,u_s)ds \in \widetilde{K} \text{for } u \in \Omega.$$

By Banach-Steinhaus's theorem, we have

$$\Big\|(T(t-t_0)-I)\int_0^{t_0} A^\alpha T(t_0-s)F(s,u_s)ds\Big\| \to 0 \quad \text{as } t\to t_0,$$

uniformly in $u\in\Omega$. This implies

$$\lim_{t\to t_0^+}\sup_{u\in\Omega}\|H_2(u)(t)-H_2(u)(t_0)\|_\alpha = 0.$$

Using similar argument for $0\le t<t_0\le b$, we can conclude that $\{H_2u(t), u\in\Omega\}$ is equicontinuous.

Finally, the Sadovskii's fixed-point theorem implies that H has a fixed point u in Ω. The fact that u is a mild solutions of Equation (1.2). This completes the proof. □

To define the mild solution in its maximal interval of existence, we add the following condition

(H4) $F:[0,+\infty[\times C_\alpha \to X$ is continuous and takes bounded sets of $[0,+\infty[\times C_\alpha$ into bounded sets in X.

Theorem 2.5. *Assume that the hypotheses of Theorem 2.4 hold and F satisfies **(H4)**. If u is a mild solution of* (1.2) *on $[-r,t_{\max}[$, then either $t_{\max}=+\infty$ or $\limsup_{t\to t_{\max}}\|u_t\|_{C_\alpha}=+\infty$.*

To prove this result, we need the following lemma.

Lemma 2.6 ([1])**.** *Assume that* (H1), (H3) *hold, and that there exist positive constants a,b,c such that, if $w\in C([-r,+\infty[;X_\alpha)$ is a solution of*

$$\begin{gathered} Dw_t = f(t) \quad \text{for } t\ge 0,\\ w_0=\varphi, \quad \varphi\in C_\alpha, \end{gathered} \tag{2.7}$$

where f is a continuous function from $[0,+\infty[$ to X_α. Then

$$\|w_t\|_{C_\alpha} \le (a\|\varphi\|_{C_\alpha} + \sup_{0\le s\le t}\|f(s)\|_\alpha)e^{ct} \quad \text{for } t\ge 0. \tag{2.8}$$

Proof of Theorem 2.5. Assume that $t_{\max}<+\infty$ and $\limsup_{t\to t_{\max}}\|u_t\|_{C_\alpha}<+\infty$. Let $R=\sup_{s\in[0,t_{\max}[}\|F(s,u_s)\|$ and $u:[t_0,t_{\max}[\to X_\alpha$, $t_0\in]0,t_{\max}[$, be the restriction of u to $[t_0,t_{\max}[$. Consider $t\in[t_0,t_{\max}[$ and β such that $\alpha<\beta<1$. Then

$$\begin{aligned} \|D(u_t)\|_\beta &\le \|A^{\beta-\alpha}T(t)A^\alpha D(\varphi)\| + \|\int_0^t A^\beta T(t-s)F(s,u_s)ds\| \\ &\le M_{\beta-\alpha}e^{\omega t}t^{\alpha-\beta}\|D(\varphi)\|_\alpha + M_\beta R\int_0^t e^{\omega s}s^{-\beta}ds. \end{aligned}$$

Thus, $\|D(u_t)\|_\beta$ is bounded on $[t_0, t_{\max}[$. Now, for $t_0 \le t < t+h < t_{\max}$, we have

$$\begin{aligned} D(u_{t+h}) - D(u_t) &= T(t+h)D\varphi - T(t)D\varphi + \int_0^{t+h} T(t+h-s)F(s,u_s)ds \\ &\quad - \int_0^t T(t-s)F(s,u_s)ds \\ &= T(t)[(T(h)-I)D\varphi] + (T(h)-I)\int_0^t T(t-s)F(s,u_s)ds \\ &\quad + \int_t^{t+h} T(t+h-s)F(s,u_s)ds \\ &= (T(h)-I)D(u_t) + \int_t^{t+h} T(t+h-s)F(s,u_s)ds. \end{aligned}$$

We put, for $t \in [t_0, t_{\max}[$,

$$f(t) = (T(h)-I)D(u_t) + \int_t^{t+h} T(t+h-s)F(s,u_s)ds.$$

Using the estimate (2.8), we obtain that

$$\|u_{t+h} - u_t\|_{C_\alpha} \le (a\|u_h - u_0\|_{C_\alpha} + b \sup_{0\le s\le t} \|f(s)\|_\alpha)e^{ct}.$$

On the other hand and using (2.2), for $t \in [t_0, t_{\max}[$, we have

$$\begin{aligned} \|f(t)\|_\alpha &\le \|(T(h)-I)A^{-(\beta-\alpha)}A^\beta D(u_t)\| + \|\int_t^{t+h} A^\alpha T(t+h-s)F(s,u_s)ds\| \\ &\le N_{\beta-\alpha}h^{\beta-\alpha}\|D(u_t)\|_\beta + RM_\alpha \int_t^{t+h} e^{\omega(t+h-s)}(t+h-s)^{-\alpha}ds \\ &\le N_{\beta-\alpha}h^{\beta-\alpha}\|D(u_t)\|_\beta + RM_\alpha \int_0^h e^{\omega s}s^{-\alpha}ds \\ &\le N_{\beta-\alpha}h^{\beta-\alpha}\|D(u_t)\|_\beta + RM_\alpha \max\{1, e^{\omega t_{\max}}\}\frac{h^{1-\alpha}}{1-\alpha} \to 0 \quad \text{as } h \to 0. \end{aligned}$$

Since $\|u_h - u_0\|_{C_\alpha} \to 0$ as $h \to 0$,

$$\lim_{h\to 0} \|u_{t+h} - u_t\|_{C_\alpha} = 0$$

uniformly with respect to $t \in [t_0, t_{\max}[$. Which implies that u is uniformly continuous and $\lim_{t\to t_{\max}} u(t)$ exists in X_α; the solution can be continued to the right to $t_{\max}$, which contradicts the maximality of $[-r, t_{\max}[$. This completes the proof of the theorem. □

The following result provides sufficient conditions for the existence of global solutions to (1.2).

Corlllary 2.7. *Under the same assumptions as in Theorem 2.4, if there exists locally integrable functions k_1 and k_2 such that $\|F(t,\varphi)\| \le k_1(t)\|\varphi\|_{C_\alpha} + k_2(t)$ for $\varphi \in C_\alpha$ and $t \ge 0$, then* (1.2) *admits global solutions.*

3. Example

Consider the partial functional differential equation of neutral type,

$$
\begin{gathered}
\frac{\partial}{\partial t}[v(t,x)-av(t-r,x)] \\
=\frac{\partial^2}{\partial x^2}[v(t,x)-av(t-r,x)]+f\big(\frac{\partial}{\partial x}v(t-r,x)\big) \quad \text{for } t\geq 0,\ x\in[0,\pi], \\
v(t,0)=av(t-r,0),\quad v(t,\pi)=av(t-r,\pi) \quad \text{for } t\geq 0, \\
v(t,x)=v_0(t,x) \quad \text{for } -r\leq t\leq 0,\ x\in[0,\pi].
\end{gathered} \tag{3.1}
$$

Where a, r are positive constants, $f:\mathbb{R}\to\mathbb{R}$ and $v_0:[-r,0]\times[0,\pi]\to\mathbb{R}$ are continuous. Let $X=\mathrm{L}^2([0,\pi];\mathbb{R})$ and $A:D(A)\subset X\to X$ be defined by $Ay=-y''$ with domain $D(A)=H^2[0,\pi]\cap H_0^1[0,\pi]$. Then $Ay=\sum_{n=1}^{\infty}n^2(y,e_n)e_n$ for $y\in D(A)$, where $\{e_n(s)=\sqrt{2/\pi}\sin ns, n\geq 1\}$, is the orthonormal set of eigenvectors of A. For each $y\in D(A^{1/2}):=\{y\in X:\sum_{n=1}^{\infty}n(y,e_n)e_n\in X\}$ the operator $A^{1/2}$ is given by $A^{1/2}y=\sum_{n=1}^{\infty}n(y,e_n)e_n$.

Lemma 3.1. [15] *If $y\in D(A^{1/2})$, then y is absolutely continuous, $y'\in X$ and $\|y'\|_X=\|A^{1/2}y\|_X$.*

It is well known that $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X given by $T(t)y=\sum_{n=1}^{\infty}e^{-n^2t}(y,e_n)e_n$, $y\in X$. It follows from this last expression that $(T(t))_{t\geq 0}$ is a compact semigroup on X (for any $t>0$, $T(t)$ is a Hilbert Schmidt operator).

Let $u(t)=v(t,.)$ for $t\geq 0$, $\varphi(\theta)=v_0(\theta,.)$ for $\theta\in[-r,0]$, $D:C_{1/2}\to X_{1/2}$ be defined by

$$
D\varphi=\varphi(0)-a\varphi(-r)=\varphi(0)-\int_{-r}^{0}d\eta(\theta)\varphi(\theta) \quad \text{for } \varphi\in C_{1/2},
$$

with $\eta(\theta)=0$ for $-r<\theta\leq 0$ and $\eta(-r)=aI$. Let $F:C_{1/2}\longrightarrow X$ be given by

$$
(F(\varphi))(x)=f(\varphi(-r)'(x)) \quad \text{for } \varphi\in C_{1/2},\ x\in[0,\pi].
$$

Then (3.1) takes the abstract form

$$
\begin{gathered}
\frac{d}{dt}Du_t=-ADu_t+F(t,u_t) \quad \text{for } t\geq 0, \\
u_0=\varphi, \quad \varphi\in C_\alpha.
\end{gathered} \tag{3.2}
$$

Lemma 3.2. *Operator F is continuous from $C_{1/2}$ to X.*

Proof. Let $\varphi\in C_{1/2}$. We consider a sequence $(\varphi_n)_n$ convergent to φ in $C_{1/2}$. Then

$$
\begin{aligned}
\|A^{1/2}\varphi_n(-r)-A^{1/2}\varphi(-r)\|_X &\leq \sup_{\theta\in[-r,0]}\|A^{1/2}\varphi_n(\theta)-A^{1/2}\varphi(\theta)\|_X \\
&=\|\varphi_n-\varphi\|_{C_{\frac12}}\to 0 \quad \text{as } n\to+\infty.
\end{aligned}
$$

Then

$$
\int_0^{\pi}\big|\frac{\partial}{\partial x}\varphi_n(-r)(x)-\frac{\partial}{\partial x}\varphi(-r)(x)\big|^2dx\to 0 \quad \text{as } n\to+\infty.
$$

This implies

$$
\frac{\partial}{\partial x}\varphi_n(-r)\to\frac{\partial}{\partial x}\varphi(-r) \quad \text{as } n\to\infty
$$

in $L^2[0,\pi]$. Consequently, there exists $(\varphi_{n_k})_k$, $g \in L^2[0,\pi]$ such that

$$\frac{\partial}{\partial x}\varphi_{n_k}(-r)(x) \to \frac{\partial}{\partial x}\varphi(-r)(x) \quad \text{a. e., as } k \to \infty$$

and

$$|\frac{\partial}{\partial x}\varphi_n(-r)(x)| \le |g(x)| \quad \text{a.e.}$$

By the continuity of f,

$$f\Big(\frac{\partial}{\partial x}\varphi_{n_k}(-r)(x)\Big) \to f\Big(\frac{\partial}{\partial x}\varphi(-r)(x)\Big) \quad \text{as } k \to \infty.$$

Assuming that $|f(t)| \le b|t| + c$, by the Lebesgue's dominated convergence theorem, we have

$$f\Big(\frac{\partial}{\partial x}\varphi_{n_k}(-r)\Big) \to f\Big(\frac{\partial}{\partial x}\varphi(-r)\Big) \quad \text{as } k \to \infty$$

in $L^2[0,\pi]$. Since the limit does not depend on the subsequence $(\varphi_{n_k})_k$, then we obtain

$$F(\varphi_n) \to F(\varphi)$$

in $L^2[0,\pi]$ as $n \to \infty$. We deduce that F is continuous. □

Consequently, Theorem 2.4 ensures the existence of a maximal interval of existence $[-r, t_{\max}[$ and a mild solution of (3.1).

References

[1] M. Adimy and K. Ezzinbi, Existence and stability in the α-norm for partial functional differential equations of neutral type, Annali di matematica pura ed applicata, 185(3), (2006), 437-460.

[2] M. Adimy, K. Ezzinbi, M. Laklach; *Existence of solutions for a class of partial neutral differential equations*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique, 330, (2000), 957-962.

[3] M. Adimy, H. Bouzahir, K. Ezzinbi; *Existence for a class of partial functional differential equations with infinite delay*, Nonlinear Analysis, 46, (2001), 91-112.

[4] M. Adimy, H. Bouzahir, K. Ezzinbi; *Local existence and stability for some partial functional differential equations with infinite delay*, Nonlinear Analysis, 48, (2002), 323-348.

[5] M. Adimy, H. Bouzahir, K. Ezzinbi; *Existence and stability for some partial neutral functional differential equations with infinite delay*, Journal of Mathematical Analysis and Applications, 294, (2004), 438-461.

[6] J-C. Chang, H. Liu; *Existence of solutions for a class of neutral partial differential equations with nonlocal conditions in the α-norm*, Nonliner Analysis, 70, (2009), 3076-3083.

[7] K. Engel and D. Nagel; *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer-Verlag, 194, (2000).

[8] K. Ezzinbi, X. Fu, K. Hilal; *Existence and regularity in the α-norm for some neutral partial differential equations with nonlocal conditions*, Nonlinear Analysis, 67, (2006), 1613-1622.

[9] Y. Hino, S. Murakami, T. Naito; *Functional Differential Equations With Infinite Delay*, Lectures Notes, Springer-Verlag, Vol. 1473, (1991).

[10] R. H . Martin, Jr.; *Nonlinear Operators and Differental Equations in Banach space*, John Wiley and Sons, New York, (1976).

[11] A. Pazy; *Semigroups of Linear Operators and Application to Partial Differental Equation*, Applied Mathematical Sciences, Springer-Verlag, New York, Vol. 44, (1983).

[12] D. O'Regan; *A fixed point theorem for condensing operators and applications to Hammerstein integral equations in Banach spaces*, Computers and Mathematics with Applications, Vol. 30, No 9, (1995), 39-49.

[13] C. C. Travis, G. F. Webb; *Existence and Stability for Partial Functional Differential Equations*, Journal of Mathematical Analysis and Applications, 200, (1974), 395-419.

[14] C. C. Travis, G. F. Webb; *Existence, stability, and compactness in the α-norm for partial functional differential equations*, Transaction of the American Mathematical Society, 240, (1978), 129-143.
[15] C. C. Travis, G. F. Webb; *Partial differential equations with deviating arguments in the time variable*, Journal of Mathematical Analysis and Applications, 56, (1976), 397-409.
[16] J. Wu; *Theory and Applications for Partial Functional Differential Function*, Applied Mathematical Sciences, Springer-Verlag, Vol. 119, (1996).

Khalil Ezzinbi
Université Cadi Ayyad, Faculté des Sciences Semlalia, Département de Mathématiques, BP 2390, Marrakech, Maroc
E-mail address: ezzinbi@gmail.com

Hatem Megdiche
Département de Mathématiques, Faculté des Sciences de Gabès, Cité Erriadh 6072, Zrig, Gabès, Tunisie
E-mail address: megdichehatem@yahoo.fr

Amor Rebey
Institut Supérieur des Mathématiques Appliquées et de l'Informatique de Kairouan, Avenue Assad Iben Fourat - 3100 Kairouan, Tunisie
E-mail address: rebey_amor@yahoo.fr