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EXISTENCE OF NONNEGATIVE SOLUTIONS TO POSITONE-TYPE PROBLEMS IN $\mathbb{R}^N$ WITH INDEFINITE WEIGHTS

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Abstract. We study the existence of a nonnegative solution to the following problem in $\mathbb{R}^N$, $N \geq 3$, in both the radial as well as in the non-radial case with an indefinite weight function $a(x)$:

$$-\Delta u = \lambda a(x) f(u)$$
$$u(x) \to 0 \quad \text{as } |x| \to \infty.$$

The nonlinearity f above is of "positone" type; i.e., f is monotone increasing with $f(0) > 0$. We show the existence of a nonnegative solution to the above problem for $\lambda > 0$ small enough. We also prove the existence of a nonnegative solution to the above problem in exterior as well as in annular domains. Motivated by the scalar equation, we further extend these results to the case of coupled system. Our proof involves the method of monotone iteration applied to the integral equation corresponding to the problem.

1. Introduction

Many problems in areas of Mathematical Physics such as fluid dynamics [2], wave phenomena, nonlinear field theory [4], combustion theory [3, 13] etc., lead to finding a positive solution to a nonlinear eigenvalue problem of the type

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega,$$

where λ is a positive parameter. For an excellent survey on the existence and multiplicity results for positive solutions of the above problem in a bounded smooth domain Ω and when $f(0) \geq 0$, we refer the reader to the paper of Lions [23]. More recently, motivated by applications in population genetics (see [14]), there is a lot of interest in the following variant involving a weight function $a(x)$:

$$\begin{gathered} -\Delta u = \lambda a(x) f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

In addition to the nonlinearity f, the indefinite weight function also plays a crucial role in proving the existence of positive solutions to (P_Ω). For conciseness, (1.1) with $\Omega = \mathbb{R}^n$ will denote the problem (P_Ω) with $\Omega = \mathbb{R}^N$ but with the decay condition $u(x) \to 0$ as $|x| \to \infty$.

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When $f(0) = 0$, there are many interesting results dealing with the existence of classical positive solutions in any arbitrary domain and here we give a brief summary of some of the results for the case of sign changing a. Brown and Tertikas [5] established the existence of nontrivial radial solutions to (1.1) with $\Omega = \mathbb{R}^n$ for large $\lambda > 0$ by the methods of sub–supersolution and monotone iteration. Tertikas [24], Brown and Stavrakakis [6] established the existence of a positive solution to the problem (1.1) with $\Omega = \mathbb{R}^n$ again by the construction of appropriate sub and supersolutions. Due to the appearance of similar problems in population genetics, authors in [6],[24] established the existence of positive solutions u to (P_Ω) such that $0 \leq u \leq 1$. Gámez [15] also studied the same problem with sign changing weight. He established the existence of positive solutions in $D_0^{1,2}(\mathbb{R}^N)$ by means of the approximation generated by positive solutions of the problem posed on B_R. For the existence of a positive solution to (1.1) in annular domains, we refer the reader to [16], [22] and references cited therein. In [16, 22], the authors assume the positivity of the potential $a(x)$, which is easier than our situation as we don't impose any sign condition on a.

To the best of our knowledge, there seems to be no result regarding the existence of nonnegative radial solutions to the problem (1.1) with $\Omega = \mathbb{R}^N$ when $f(0) \neq 0$, although there are results for bounded domains. The earlier results for the case $f(0) = 0$ in $\mathbb{R}^N$, do not seem to extend easily to the case $f(0) \neq 0$.

In the case $f(0) \neq 0$, in order to get an idea of the conditions to be imposed on f and a, we describe some available results for bounded domains. Cac et al. [8] studied (1.1) with $\Omega = B_1$ for sign changing $a, f(0) > 0$ assuming

- f is continuous, positive and nondecreasing in $[0, \infty)$.
- $a \in L^1(0, 1)$ and there exists an $\epsilon > 0$ such that
$$\int_0^t x^{N-1} a_+(x)dx \geq (1 + \epsilon) \int_0^t x^{N-1} a_-(x)dx,$$
for all $t \in [0, 1]$.

With the above conditions on f and a, for λ small, they showed the existence of a nonnegative solution of (1.1) with $\Omega = B_1$ using a variant of monotone iteration and a fixed point argument. Cac et al. [9] generalized the result of [8] in B_1 to bounded domains with smooth boundary relaxing the non-decreasing assumption on f, but assuming that $a \in L^s(\Omega)$ for $s > \max\{1, \frac{N}{2}\}$ and that the Dirichlet problem
$$-\Delta w = a_+(x) - (1 + \epsilon)a_-(x), \quad x \in \Omega, \quad u = 0 \quad \text{on } \partial\Omega \tag{1.2}$$
has a nonnegative solution in Ω. Hai [19] established the existence of a positive solution to the problem (1.1) with an indefinite weight a in any bounded domain Ω by applying the Leray–Schauder fixed point theorem. This was done by relaxing the condition that f is nondecreasing, but by assuming the continuity of a. Afrouzi and Brown [1] also studied the same problem (P_Ω) in a bounded domain Ω for smooth f and established the existence of a positive solution for λ small by an application of the implicit function theorem. In both [1] and [19], a condition similar to 1.2 was assumed on a_+ and a_-.

There is also a good amount of research dealing with corresponding semilinear elliptic systems, in particular, reaction–diffusion systems. Reaction–diffusion systems model many phenomena in Biology, Ecology, combustion theory, chemical reactions, population dynamics etc. A typical example of these models is

$$\begin{gathered} -\Delta u = \lambda f(v) \quad \text{in } \Omega, \\ -\Delta v = \lambda g(u) \quad \text{in } \Omega, \\ u = 0 = v \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. We refer to the works, [11, 12, 17, 21, 25] among many others, in this context. In [11], Dalmasso established the existence of positive solutions for (1.3) by Schauder fixed point theorem and de Figueirdo et al. ([12]) answer the existence question in an Orlicz space setting for $N \geq 3$. For the existence and non–existence of positive solutions to (1.3) in a ball when $N \geq 4$, we refer the reader to [17]. In [21], Hulshof and Vorst established the existence of a positive solution to (1.3) for $N \geq 1$ by variational techniques. Recently, Castro et al. [10] and Hai and Shivaji [20] have also established the existence of a positive solution to the system given in (1.3). Motivated by the work of Cac et al. [8], we explore the existence question in $\mathbb{R}^N$ for single equation as well as for systems with conditions similar to theirs. To overcome the lack of compactness posed by unbounded domains, we need to assume additional integrability conditions on a. For bounded domains, one of the main tools used to prove the existence of positive solutions is the classical Schauder fixed point theorem. In this work we get the compactness of the relevant integral operator in whole $\mathbb{R}^N$ by this additional integrability condition on a (see, (H2) Prop. 2.3 below). Using (H4) for a_+ and a_- as in [8], we employ the monotone iteration method adapted to the indefinite weight $a(x)$, to construct a subset of the cone of nonnegative functions invariant under the integral operator. We remark that (H4) works in exterior as well as in annular domains.

Let $G(x,t)$ be the Green's function for the equation $(x^{N-1}y'(x))' = 0$ subject to the Dirichlet boundary conditions on I. Let $\Gamma(x-y) = c_N|x-y|^{2-N}$ be the fundamental solution of $-\Delta$, where $c_N = \frac{1}{N(N-2)w_N}$, w_N is the volume of the unit sphere in $\mathbb{R}^{\mathbb{N}}$.

Let I denote any of the following intervals: $(0,\infty)$, (R_1,∞), (R_1,R_2), for some $R_1, R_2 > 0$. We will work with the following set of hypotheses for the radial case:

- (H1) $f : \mathbb{R} \to (0,\infty)$ is continuous, non-decreasing and $f(0) > 0$.
- (H2) There exists some $0 < \sigma \leq 1$ such that $\int_I t^{1+\sigma}|a(t)|\,dt < \infty$.
- (H3) $\int_I |a(t)|\,dt < \infty$.
- (H4) there exists $\mu > 0 : \int_I G(x,t)t^{N-1}a_+(t)\,dt \geq (1+\mu)\int_I G(x,t)t^{N-1}a_-(t)\,dt$, for all $x \in I$.

For the non-radial case, we assume

- (N1) $f : \mathbb{R} \to \mathbb{R}^+$ is Hölder continuous, non-decreasing and $f(0) > 0$.
- (N2) a is a locally Hölder continuous function on $\mathbb{R}^{\mathbb{N}}$ and there exist $\delta > 0$, $C > 0$ such that
$$|a(x)| \leq C|x|^{-(2+\delta)} \quad \text{for all large } x.$$
- (N3) There exists $\mu > 0$ such that
$$\int_{\mathbb{R}^{\mathbb{N}}} \Gamma(x-y)(a_+(y) - (1+\mu)a_-(y))dy \geq 0, \quad \forall\, x \in \mathbb{R}^{\mathbb{N}}.$$

More precisely, in this paper we are interested in the following set of problems:

Problem 1. To establish the existence of nonnegative solutions to the following radial version of the problem (1.1) with $\Omega = \mathbb{R}^n$ for $N \geq 3$:

$$y''(x) + \frac{N-1}{x} y'(x) + \lambda a(x) f(y(x)) = 0, \quad \text{in } (0,\infty), \tag{1.4}$$

$$y'(0) = 0, \quad y(x) \to 0 \quad \text{as } x \to \infty. \tag{1.5}$$

Also, using the same approach, to show the existence of nonnegative solutions to (1.4) in the exterior domain (R_1,∞) for some $R_1 > 0$ with the boundary conditions

$$y(R_1) = 0, \quad y(x) \to 0, \quad \text{as } x \to \infty \quad \text{for } N \geq 3 \tag{1.6}$$

and in the annular region (R_1, R_2), $\quad 0 < R_1 < R_2$ with the boundary conditions

$$y(R_1) = 0 = y(R_2) \quad \text{for } N \geq 2. \tag{1.7}$$

Problem 2. To show the existence of a nonnegative pair (y_1, y_2) of solutions to the following radial coupled system by similar arguments as are given in dealing with problem 1 for $N \geq 3$:

$$\begin{gathered}
y_1''(x) + \frac{N-1}{x} y_1'(x) + \lambda a_1(x) f_1(y_2(x)) = 0, \quad \text{in } (0,\infty), \\
y_2''(x) + \frac{N-1}{x} y_2'(x) + \lambda a_2(x) f_2(y_1(x)) = 0, \quad \text{in } (0,\infty), \\
y_i'(0) = 0, \quad y_i(x) \to 0 \quad \text{as } x \to \infty, \text{ for } i = 1,2.
\end{gathered} \tag{1.8}$$

Also, using the same approach, to show the existence of nonnegative solutions to the above system in exterior as well as in annular domains.

subsection*Problem 3 To consider, without radial assumptions, the following coupled system of differential equations in $\mathbb{R}^N$, $N \geq 3$,

$$\begin{gathered}
\Delta u_1 + \lambda a_1(x) f_1(u_2(x)) = 0, \\
\Delta u_2 + \lambda a_2(x) f_2(u_1(x)) = 0, \\
u_i(x) \to 0 \quad \text{as } |x| \to \infty \text{ for } i = 1,2,
\end{gathered} \tag{1.9}$$

and show the existence of a non-negative pair of solution (u_1, u_2). We now state the main results.

Theorem 1.1. *Let f, a satisfy hypotheses* (H1)–(H4). *Then* (1.4) *posed on I, with the corresponding boundary conditions as in anyone of* (1.5), (1.6), (1.7) *has a nonnegative solution for λ small.*

Theorem 1.2. *Let f_i, a_i, $i = 1,2$ satisfy the hypotheses* (H1)–(H4). *Then the coupled system of equations* (1.8) *has a nonnegative solution for λ small.*

Theorem 1.3. *Let f_i, a_i, $i = 1,2$ satisfy the hypotheses* (N1)–(N3). *Then the coupled system of equations* (1.9) *has a nonnegative solution for λ small.*

In Section 2, we state and prove some preliminary results which are required to prove the main results. Theorem 1.1 is proved in Section 3 in $\mathbb{R}^N$ while in Section 4 the proof is given in exterior as well as in annular domains. Theorems 1.2 and 1.3 are proved in Sections 5 and 6 respectively. Finally, in Section 7 we construct some examples for the illustration of our main results.

2. Preliminary Results

The Green's function for the boundary value problem

$$(x^{N-1}y'(x))' = 0, \quad x \in (0,\infty), y'(0) = 0, \quad y(x) \to 0, \quad \text{as } x \to \infty,$$

is the function $G : [0,\infty) \times [0,\infty) \to [0,\infty)$ given by

$$G(x,t) = \frac{1}{N-2}\begin{cases} t^{2-N}, & 0 < x \le t < \infty, \\ x^{2-N}, & 0 < t \le x < \infty. \end{cases}$$

Let $C_b([0,\infty))$ denote the space of bounded continuous functions endowed with the supremum norm. Given any open set A, let $X(A)$ denote the space of bounded measurable functions on A endowed with the ess-sup norm. Define the integral operator $L : X([0,\infty)) \to C_b([0,\infty))$ by

$$(L\xi)(x) = \lambda \int_0^\infty G(x,t)t^{N-1}a(t)f(\xi(t))\,dt.$$

Let $A = \{x \in [0,\infty) : a(x) \ge 0\}$ and $B = \{x \in [0,\infty) : a(x) < 0\}$. We denote by $C_b^+(\cdot)$ the cone of all nonnegative members in $C_b(\cdot)$. Define the following operators representing the positive and negative part of L: $L^+ : X([0,\infty) \cap A) \to C_b^+([0,\infty))$, by

$$(L^+(\varphi))(x) = \lambda \int_A G(x,t)t^{N-1}a_+(t)f(\varphi(t))\,dt$$

and $L^- : X([0,\infty) \cap B) \to C_b^+([0,\infty))$ by

$$(L^-(\varphi))(x) = \lambda \int_A G(x,t)t^{N-1}a_-(t)f(\varphi(t))\,dt.$$

We note that the operator L can now be written as

$$L\varphi = L^+\varphi - L^-\varphi.$$

Using the monotonicity of f we can conclude that L^+ and L^- are both monotone operators; i.e.,

$$\varphi \le \psi \Rightarrow L^\pm\varphi \le L^\pm\psi.$$

One of the difficulties is that, in general, L does not leave the cone $C_b^+([0,\infty))$ invariant. Thus our main task ahead is to identify a closed convex set $\mathcal{C} \subset C_b^+([0,\infty))$ which is left invariant under L. This is the content of the next result (see also [8]).

Proposition 2.1. *Assume* (H1) *holds and there exist $\xi, \eta \in C_b^+([0,\infty))$ such that $0 \le \xi \le \eta$, $\xi = L^+\xi - L^-\eta$ and $\eta = L^+\eta - L^-\xi$. Then $\mathcal{C} = \{g \in C_b([0,\infty)) : \xi \le g \le \eta\}$ is a closed convex set and is invariant under L.*

Proof. It is easy to see that $\mathcal{C}$ is a closed convex set in $C_b([0,\infty))$. Now we show that $\mathcal{C}$ is invariant under L. This is because for any $g \in \mathcal{C}$,

$$Lg = L^+g - L^-g \le L^+\eta - L^-\xi = \eta.$$

Hence $Lg \le \eta$ and similarly using the monotonicity property of L^+ and L^- we have $Lg \ge \xi$. Therefore $\mathcal{C}$ is invariant under L. □

Remark 2.2. We remark that there seems to be a gap in the arguments given by Cac et al. [8] in proving the invariance of L, though the invariance of L can be obtained there as in the above proposition.

Now we construct ξ and η as required in Proposition 2.1 by an iteration technique introduced by Cac et al. [8]. We may think of the following proposition as the indefinite version of the standard monotone iteration process that yields a solution once a pair of sub and super solutions is given. Indeed the proof uses the fact that L is a difference of two monotone operators.

Proposition 2.3. *Let the hypotheses* (H1)–(H3) *hold. Suppose we have bounded measurable functions ξ_0 and η_0 on $[0,\infty)$ such that they satisfy*

1. $0 \le \xi_0 \le \eta_0$ *on A,* $0 \le \eta_0 \le \xi_0$ *on B;*
2. $L\eta_0 \le \eta_0$ *on A,* $L\eta_0 \le \xi_0$ *on B;*
3. $L\xi_0 \ge \xi_0$ *on A,* $L\xi_0 \ge \eta_0$ *on B.*

Then there exist $\xi, \eta \in C_b^+([0,\infty))$ satisfying the requirements of Proposition 2.1.

Proof. For any integer $n \ge 0$ we define

$$\xi_{n+1}(x) = \begin{cases} L\xi_n(x) & x \in A, \\ L\eta_n(x) & x \in B \end{cases} \quad \text{and} \quad \eta_{n+1}(x) = \begin{cases} L\eta_n(x) & x \in A, \\ L\xi_n(x) & x \in B. \end{cases}$$

By induction, it is easy to check that if the pair (ξ_n, η_n) satisfies (1)–(3), then so does (ξ_{n+1}, η_{n+1}). Therefore,

$$L\xi_n(x) \le L\eta_n(x), \quad L\eta_n(x) \le L\eta_{n-1}(x), \quad L\xi_n(x) \ge L\xi_{n-1}(x),$$

for all $n \ge 1$ and all $x \in [0,\infty)$. Combining all the above inequalities, we obtain

$$0 \le L\xi_0 \le L\xi_1 \dots \le L\xi_n \le L\xi_{n+1} \le \dots \le L\eta_{n+1} \le L\eta_n \le \dots \le L\eta_0.$$

Thus we can find ξ, η such that $L\xi_n(x) \uparrow \xi(x)$ and $L\eta_n(x) \downarrow \eta(x)$ pointwise on $[0,\infty)$. Since $L\xi_0, L\eta_0$ are bounded and $L\xi_0 \le \xi \le \eta \le L\eta_0$, ξ and η are bounded. From the hypothesis (H2), $|ta(t)|$ is integrable on $[0,\infty)$ and clearly $G(x,t)t^{N-2}$ is uniformly bounded on $[0,\infty) \times [0,\infty)$. Hence by the Lebesgue dominated convergence theorem we obtain

$$\xi(x) = \lim_n L\xi_{n+1}(x) = \lim_n (L^+(L\xi_n) - L^-(L\eta_n))(x) = (L^+\xi - L^-\eta)(x).$$

We note that given any bounded measurable function ψ on $[0,\infty)$, $L^{\pm}\psi$ is a bounded continuous function on $[0,\infty)$. Therefore the last equation implies that ξ is bounded and continuous on $[0,\infty)$. In a similar fashion we can show that $\eta = L^+\eta - L^-\xi$ in $[0,\infty)$ and hence η is also bounded and continuous on $[0,\infty)$. $\square$

Proposition 2.4. *Let f, a satisfy* (H1)–(H3). *Then under the assumptions of Proposition 2.1, L has a fixed point in $\mathcal{C}$.*

Proof. We show that $\{Lg : g \in \mathcal{C}\}$ is an equicontinuous family. Since $0 \le g(x) \le \|\eta\|_\infty$, for all $x \in [0,\infty)$ and f is continuous, there exists $K > 0$ such that $|f(g(x))| \le K$, for all $x \in [0,\infty), g \in \mathcal{C}$. By the Lipschitzness of $G(x,t)t^{N-1}$ in the x variable for every t; i.e.,

$$|G(x,t)t^{N-1} - G(y,t)t^{N-1}| \le C|x-y|, \quad \text{for } x, y \in \mathbb{R} \text{ and } \forall t \in \mathbb{R},$$

for any given $\epsilon > 0$ there exists $\delta > 0$ such that for all $|x-y| < \delta$ we have

$$\begin{aligned} |Lg(x) - Lg(y)| &\le \lambda \int_0^\infty |G(x,t)t^{N-1} - G(y,t)t^{N-1}|\,|f(g(t))|\,|a(t)|\,dt \\ &\le \lambda K \epsilon \int_0^\infty |a(t)|\,dt. \end{aligned}$$

Therefore, using (H3), we obtain from the last inequality that $\{Lg : g \in \mathcal{C}\}$ is an equicontinuous family. In view of Proposition 2.1, we also obtain that $\{Lg : g \in \mathcal{C}\}$ is uniformly bounded. By Arzela–Ascoli theorem, for any given $\epsilon > 0$, and $M > 0$ there exists an $N \in \mathbb{N}$ such that

$$|Lg_n(x) - Lg_m(x)| \le \frac{\epsilon}{2}, \quad \forall\, x \in [0, M], \forall\, n, m \ge N. \tag{2.1}$$

Now we claim that $L : \mathcal{C} \to \mathcal{C}$ is a compact operator; i.e., for any bounded sequence $\{g_n\} \subset \mathcal{C}, \{Lg_n\}$ has a subsequence (which we again denote by $\{Lg_n\}$), which converges in $\mathcal{C}$. Using the explicit form of $G(x,t)$ and keeping in mind the hypothesis (H2) we rewrite

$$\begin{aligned} Lg_n(x) &= \lambda\Big[\int_0^x G(x,t)t^{N-1}a(t)f(g_n(t))\,dt + \int_x^\infty G(x,t)t^{N-1}a(t)f(g_n(t))\,dt\Big] \\ &= \frac{\lambda}{(N-2)}\Big[\int_0^x \big(\frac{t}{x}\big)^{(N-2-\sigma)} x^{-\sigma}t^{1+\sigma}a(t)f(g_n(t))\,dt + \int_x^\infty ta(t)f(g_n(t))\,dt\Big]. \end{aligned}$$

Therefore, from (H2) and the last inequality we have

$$|Lg_n(x)| \le K\frac{\lambda}{(N-2)}x^{-\sigma}\|t^{(1+\sigma)}a\|_{L^1([0,\infty))}, \quad \forall x \in (0,\infty).$$

From the above estimate, given $\epsilon > 0$ we can choose $M > 0$ large so that

$$|Lg_n(x)| \le \frac{\epsilon}{4}, \forall\, x > M.$$

In particular, this implies

$$|Lg_n(x) - Lg_m(x)| \le \frac{\epsilon}{2}, \forall\, x > M, \forall\, n, m \in \mathbb{N}.$$

Using (2.1) with this choice of M along with the last estimate, it follows that $\{Lg_n\}$ is a uniformly Cauchy sequence in $\mathcal{C}$. It now follows that $L : \mathcal{C} \to \mathcal{C}$ is a compact operator. Since L is clearly continuous, by Schauder's fixed point theorem L has a fixed point, i.e., $L\varphi = \varphi$ for some $\varphi \in \mathcal{C}$. $\square$

3. Proof of Theorem 1.1 in $\mathbb{R}^N$

It is easy to see that a fixed point of L in $C_b^+([0,\infty))$ is a solution of (1.4)–(1.5). Therefore, in view of Propositions 2.1, 2.3 and 2.4 to obtain such a fixed point it is sufficient to construct ξ_0 and η_0 satisfying conditions (1)–(3) of Proposition 2.3. Let

$$\xi_0(x) = \begin{cases} 0 & \text{for } x \in A, \\ \alpha & \text{for } x \in B \end{cases} \quad \text{and} \quad \eta_0(x) = \begin{cases} \alpha & \text{for } x \in A, \\ 0 & \text{for } x \in B. \end{cases}$$

Then condition (1) is satisfied if $\alpha \ge 0$. Now the condition (2) is

$$L\eta_0 = L^+(\alpha) - L^-(0) \le \alpha \quad \text{in } [0,\infty)$$

while (3) is

$$L\xi_0 = L^+(0) - L^-(\alpha) \ge 0 \quad \text{in } [0,\infty).$$

Letting $z_\pm(x) = \int_0^\infty G(x,t)t^{N-1}a_\pm(t)\,dt$ these last two conditions become respectively

$$\lambda[z_+(x)f(\alpha) - z_-(x)f(0)] \le \alpha, \tag{3.1}$$

$$\lambda[z_+(x)f(0) - z_-(x)f(\alpha)] \ge 0. \tag{3.2}$$

Define $w(x) = z_+(x) - (1+\mu)z_-(x)$. Then from (H4) we have that $z_+(x) \geq (1+\mu)z_-(x)$ in $[0,\infty)$. Also if

$$f(\alpha) \leq (1+\mu)f(0) \tag{3.3}$$

holds, then (3.2) is satisfied. We can indeed choose such an α using the continuity of f and claim that (3.1) can also be satisfied for the same α provided λ is small enough. We make the following easy estimate:

$$|\int_0^\infty G(x,t)t^{N-1}a(t)\,dt| \leq \frac{1}{(N-2)}\int_0^\infty t|a(t)|\,dt = \beta \quad \text{(say)}.$$

Noting that

$$z_+(x) - z_-(x) = \int_0^\infty G(x,t)t^{N-1}a(t)\,dt$$

we obtain

$$z_+(x) \leq z_-(x) + \beta. \tag{3.4}$$

Hence, using (3.3),

$$\begin{aligned} f(\alpha)z_+(x) - f(0)z_-(x) &\leq [f(\alpha)-f(0)]z_-(x) + f(\alpha)\beta \\ &\leq [f(\alpha)-f(0)]\beta + f(\alpha)\beta \\ &\leq f(0)\beta(1+2\mu). \end{aligned}$$

Therefore, (3.1) is satisfied if for example

$$\lambda \leq \frac{\alpha}{f(0)\beta(1+2\mu)} = \lambda_0.$$

Remark 3.1. We note that with minor changes to the proof, in the above argument for getting an inequality like (3.4), one can replace (H4) by

(H4)' There exists $\mu > 0$ such that

$$\int_0^t x^{N-1}a_+(x)dx \geq (1+\mu)\int_0^t x^{N-1}a_-(x)dx, \quad \forall\, t \in [0,\infty).$$

4. Proof of Theorem 1.1 in exterior as well as in annular domains

In this section, we consider problems (1.4), (1.6) and (1.4), (1.7) which correspond to problem on exterior and annular domain respectively and show the existence of nonnegative radial solutions for λ small. The Green's function $G : [R_1,\infty)\times[R_1,\infty) \to [0,\infty)$ for the boundary value problem

$$\begin{gathered} (x^{N-1}y'(x))' = 0, \\ y(R_1) = 0, \quad y(x) \to 0, \quad \text{as } x \to \infty,\ N \geq 3 \end{gathered}$$

is

$$G(x,t) = \frac{(tx)^{2-N}}{(N-2)}\begin{cases} x^{N-2} - R_1^{N-2}, & R_1 \leq x \leq t < \infty, \\ t^{N-2} - R_1^{N-2}, & R_1 \leq t \leq x < \infty. \end{cases}$$

Similarly, the Green's function $G : [R_1,R_2]\times[R_1,R_2] \to [0,\infty)$ for the boundary value problem

$$(x^{N-1}y'(x))' = 0, \quad y(R_1) = 0 = y(R_2),$$

for $N \geq 3$ is

$$G(x,t) = \frac{1}{(N-2)(R_1^{2-N} - R_2^{2-N})}\begin{cases} (R_2^{2-N} - t^{2-N})(x^{2-N} - R_1^{2-N}), & x \leq t, \\ (R_1^{2-N} - t^{2-N})(x^{2-N} - R_2^{2-N}), & t \leq x, \end{cases}$$

and for $N = 2$, it is

$$G(x,t) = (\log \frac{R_2}{R_1})^{-1} \begin{cases} \log \frac{R_2}{t} \log \frac{x}{R_1}, & x \le t, \\ \log \frac{R_1}{t} \log \frac{x}{R_2}, & t \le x. \end{cases}$$

If $R_2 = \infty$, by $y(R_2) = 0$ we mean that $y(x) \to 0$ as $x \to \infty$.

As before, we can define the integral operator $L : X([R_1, R_2]) \to C_b^+([R_1, R_2])$ by

$$(L\xi)(x) = \lambda \int_{R_1}^{R_2} G(x,t) t^{N-1} a(t) f(\xi(t))\, dt,$$

where G is the Green's function as given above. It is easy to check that all the details in the proof of Theorem 1.1 in $\mathbb{R}^N$ can be modified easily to give the proof in the case of an exterior or an annular domain that we are considering. Hence we omit the details.

5. Coupled radial system

In this section, we find a nonnegative solution for the coupled system (1.8) in $\mathbb{R}^N$ for $N \ge 3$. We assume that f_i, a_i satisfy the hypotheses (H1)–(H4) for $i = 1, 2$.

Define the integral operator $L : X([0,\infty)) \times X([0,\infty)) \to C_b([0,\infty)) \times C_b([0,\infty))$ by

$$L(\xi, \eta) = (L_1 \eta, L_2 \xi),$$

where

$$L_1\eta(x) = \lambda \int_0^\infty G(x,t) t^{N-1} a_1(t) f_1(\eta(t))\, dt,$$
$$L_2\xi(x) = \lambda \int_0^\infty G(x,t) t^{N-1} a_2(t) f_2(\xi(t))\, dt.$$

Let

$$A_1 = \{x \in [0,\infty) : a_1(x) \ge 0\}, \quad B_1 = \{x \in [0,\infty) : a_1(x) < 0\}$$
$$A_2 = \{x \in [0,\infty) : a_2(x) \ge 0\}, \quad B_2 = \{x \in [0,\infty) : a_2(x) < 0\}.$$

Define the following operators representing the positive and negative parts of L_i for $i = 1, 2$: $L_i^+ : X([0,\infty) \cap A_i) \to C_b^+([0,\infty))$ by

$$(L_i^+\varphi)(x) = \lambda \int_{A_i} G(x,t) t^{N-1} (a_i)_+(t) f_i(\varphi(t))\, dt,$$

and $L_i^- : X([0,\infty) \cap B_i) \to C_b^+([0,\infty))$ by

$$(L_i^-\varphi)(x) = \lambda \int_{B_i} G(x,t) t^{N-1} (a_i)_-(t) f_i(\varphi(t))\, dt.$$

We note that the operator L_i can now be written as

$$L_i\varphi = L_i^+\varphi - L_i^-\varphi, \quad i = 1, 2.$$

Using the monotonicity of f_i we can conclude that both L_i^+ and L_i^- are monotone operators; i.e.,

$$\varphi \le \psi \Longrightarrow L_i^\pm \varphi \le L_i^\pm \psi.$$

We denote any $g \in C_b([0,\infty)) \times C_b([0,\infty))$ by $g = (g^1, g^2)$ where $g^i \in C_b([0,\infty))$. For $\xi = (\xi^1, \xi^2), \eta = (\eta^1, \eta^2) \in C_b^+([0,\infty)) \times C_b^+([0,\infty))$ by $\xi \le \eta$ we mean the relations $\xi^1 \le \eta^1$ and $\xi^2 \le \eta^2$ hold.

Proposition 5.1. *Let f_1 and f_2 be nondecreasing functions. Assume that there exist $\xi, \eta \in C_b^+([0,\infty)) \times C_b^+([0,\infty))$ such that $0 \le \xi \le \eta$ and*

$$\xi^1 = L_1^+\xi^2 - L_1^-\eta^2, \quad \eta^1 = L_1^+\eta^2 - L_1^-\xi^2,$$
$$\xi^2 = L_2^+\xi^1 - L_2^-\eta^1, \quad \eta^2 = L_2^+\eta^1 - L_2^-\xi^1.$$

Then $\mathcal{C} = \{g \in C_b^+([0,\infty)) \times C_b^+([0,\infty)) : \xi \le g \le \eta\}$ is a closed convex set and is invariant under L.

Proof. It is easy to see that $\mathcal{C}$ is a closed convex set in $C_b^+([0,\infty)) \times C_b^+([0,\infty))$. Now we show that $\mathcal{C}$ is invariant under L. This is because for any $g = (g^1, g^2) \in \mathcal{C}$,

$$\begin{aligned} Lg = (L_1g^2, L_2g^1) &= (L_1^+g^2 - L_1^-g^2, L_2^+g^1 - L_2^-g^1) \\ &\le (L_1^+\eta^2 - L_1^-\xi^2, L_2^+\eta^1 - L_2^-\xi^1) \\ &= (\eta^1, \eta^2) = \eta. \end{aligned}$$

Hence $Lg \le \eta$ and similarly using the monotonicity of L_i^+ and L_i^- we have $Lg \ge \xi$. Therefore, $\mathcal{C}$ is invariant under L. □

Proposition 5.2. *Let f_i, a_i satisfy the hypotheses* (H1)–(H3)*. Suppose there exist bounded measurable functions $\xi_0 = (\xi_0^1, \xi_0^2)$ and $\eta_0 = (\eta_0^1, \eta_0^2)$ on $[0,\infty) \times [0,\infty)$ satisfying*

(1) $0 \le \xi_0^1 \le \eta_0^1$ *on* A_2, $0 \le \eta_0^1 \le \xi_0^1$ *on* B_2;
(2) $L_1\eta_0^2 \le \eta_0^1$ *on* A_2, $L_1\eta_0^2 \le \xi_0^1$ *on* B_2;
(3) $L_1\xi_0^2 \ge \xi_0^1$ *on* A_2, $L_1\xi_0^2 \ge \eta_0^1$ *on* B_2;
(4) $0 \le \xi_0^2 \le \eta_0^2$ *on* A_1, $0 \le \eta_0^2 \le \xi_0^2$ *on* B_1;
(5) $L_2\eta_0^1 \le \eta_0^2$ *on* A_1, $L_2\eta_0^1 \le \xi_0^2$ *on* B_1;
(6) $L_2\xi_0^1 \ge \xi_0^2$ *on* A_1, $L_2\xi_0^1 \ge \eta_0^2$ *on* B_1.

Then there exist $\xi, \eta \in C_b^+([0,\infty)) \times C_b^+([0,\infty))$ satisfying the requirements of Proposition 5.1.

Proof. For any integer $n \ge 0$ we define

$$\xi_{n+1}^1(x) = \begin{cases} L_1\xi_n^2 & \text{for } x \in A_2, \\ L_1\eta_n^2 & \text{for } x \in B_2, \end{cases} \quad \eta_{n+1}^1(x) = \begin{cases} L_1\eta_n^2 & \text{for } x \in A_2, \\ L_1\xi_n^2 & \text{for } x \in B_2, \end{cases}$$
$$\xi_{n+1}^2(x) = \begin{cases} L_2\xi_n^1 & \text{for } x \in A_1, \\ L_2\eta_n^1 & \text{for } x \in B_1, \end{cases} \quad \eta_{n+1}^2(x) = \begin{cases} L_2\eta_n^1 & \text{for } x \in A_1, \\ L_2\xi_n^1 & \text{for } x \in B_1. \end{cases}$$

Then following the lines of Proposition 2.3, we obtain

$$0 \le L_1\xi_0^2 \le L_1\xi_1^2 \le \dots \le L_1\xi_n^2 \le L_1\xi_{n+1}^2 \le \dots \le L_1\eta_{n+1}^2 \le L_1\eta_n^2 \le \dots L_1\eta_0^2.$$
$$0 \le L_2\xi_0^1 \le L_2\xi_1^1 \le \dots \le L_2\xi_n^1 \le L_2\xi_{n+1}^1 \le \dots \le L_2\eta_{n+1}^1 \le L_2\eta_n^1 \le \dots L_2\eta_0^1.$$

We can find $\xi^1, \xi^2, \eta^1, \eta^2$ on $[0,\infty)$, such that $L_1\xi_n^2 \uparrow \xi^1$, $L_2\xi_n^1 \uparrow \xi^2$, $L_1\eta_n^2 \downarrow \eta^1$, and $L_2\eta_n^1 \downarrow \eta^2$. Also we have that ξ^i and η^i are continuous and

$$\xi^1 = L_1^+\xi^2 - L_1^-\eta^2, \quad \eta^1 = L_1^+\eta^2 - L_1^-\xi^2,$$
$$\xi^2 = L_2^+\xi^1 - L_2^-\eta^1, \quad \eta^2 = L_2^+\eta^1 - L_2^-\xi^1.$$

□

Proposition 5.3. *Let f_i, a_i satisfy the hypotheses* (H1)–(H3)*. Then under the assumption of Proposition 5.1, L has a fixed point in $\mathcal{C}$.*

Proof. In view of Proposition 5.1, using similar arguments as in Proposition 2.4, one can see easily that $L : \mathcal{C} \to \mathcal{C}$ is continuous operator. For applying Schauder's fixed point theorem, it suffices to show that L is compact. Let $g_n = (g_n^1, g_n^2)$ be a bounded sequence in $C_b^+([0,\infty)) \times C_b^+([0,\infty))$. By Proposition 2.4, there exists a subsequence of g_n, which we still denote by g_n, such that $Lg_n(x) \to (g^1(x), g^2(x))$, where $g_1, g_2 \in C_b^+([0,\infty))$. Hence by Schauder's fixed point theorem there exists $\varphi = (\varphi_1, \varphi_2) \in \mathcal{C}$ such that $L\varphi = \varphi$. $\square$

Proof of Theorem 1.2. It is easy to see that a fixed point of L in $C_b^+([0,\infty)) \times C_b^+([0,\infty))$ is a solution of (1.8). In order to obtain such a fixed point it is enough to construct ξ_0 and η_0 satisfying conditions (1)–(6) of Proposition 5.2. Motivated by the scalar case, let

$$\xi_0^1(x) = \begin{cases} 0 & \text{for } x \in A_2, \\ \alpha & \text{for } x \in B_2, \end{cases} \quad \eta_0^1(x) = \begin{cases} \alpha & \text{for } x \in A_2, \\ 0 & \text{for } x \in B_2, \end{cases}$$

$$\xi_0^2(x) = \begin{cases} 0 & \text{for } x \in A_1, \\ \alpha & \text{for } x \in B_1, \end{cases} \quad \eta_0^2(x) = \begin{cases} \alpha & \text{for } x \in A_1, \\ 0 & \text{for } x \in B_1, \end{cases}$$

where $\alpha > 0$, $\xi_0 = (\xi_0^1, \xi_0^2)$ and $\eta_0 = (\eta_0^1, \eta_0^2)$. Let

$$\beta_i = \int_0^\infty \frac{t|a_i(t)|}{N-2}\, dt$$

for $i = 1, 2$ and $\lambda \le \bar\lambda$, where

$$\bar\lambda = \min\Big\{\frac{\alpha}{\beta_1 f_1(0)(1+2\mu)}, \frac{\alpha}{\beta_2 f_2(0)(1+2\mu)}\Big\}.$$

By similar arguments as in the proof of Theorem 1.1, with the above choice of α and $\bar\lambda$, it can be easily checked that the hypotheses of Proposition 5.2 are satisfied. So for the sake of brevity, we omit the detailed verification. $\square$

6. Coupled nonradial system

In this section, we show the existence of a nonnegative solution to the coupled system (1.9) in $\mathbb{R}^N$ for $N \ge 3$. Let B_R denote the open ball of radius R centered at 0. In this section we first recall the following result by Li and Ni [26]:

Lemma 6.1. *Let h be a locally Hölder continuous function on $\mathbb{R}^N$ with the following decay at infinity, for some $\delta > 0$ and $C > 0$:*

$$|h(x)| \le C|x|^{-(2+\delta)} \quad \text{for all large } x.$$

Let w be the Newtonian potential of h. Then $w(x)$ is well defined and has the decay property

$$|w(x)| \le C|x|^{-\delta} \quad \text{for all large } x.$$

Lemma 6.2. *Let h, w be as in Lemma 6.1, then $-\Delta w = h$ in $\mathbb{R}^N$.*

Proof. Let $|x| < R$. Then we can write $w(x) = c_N \int_{\mathbb{R}^N} \frac{h(y)}{|x-y|^{N-2}} dy = w_1(x) + w_2(x)$ where

$$w_1(x) = c_N \int_{B_R(0)} \frac{h(y)}{|x-y|^{N-2}}\, dy, \quad w_2(x) = c_N \int_{\mathbb{R}^N \setminus B_R(0)} \frac{h(y)}{|x-y|^{N-2}}\, dy.$$

We have $-\Delta w_1(x) = h(x)$ by a standard argument since h is Hölder continuous in B_R (see [18]). Further, w_2 is well defined due to the decay hypothesis on h and hence $\Delta w_2 = 0$. $\square$

Let us define the integral operator $L : X(\mathbb{R}^{\mathbb{N}}) \times X(\mathbb{R}^{\mathbb{N}}) \to C_b(\mathbb{R}^{\mathbb{N}}) \times C_b(\mathbb{R}^{\mathbb{N}})$ by

$$L(\xi, \eta)(x) = (L_1\eta(x), L_2\xi(x)),$$

where

$$\begin{aligned} L_1\eta(x) &= \lambda \int_{\mathbb{R}^{\mathbb{N}}} \Gamma(x-y)a_1(y)f_1(\eta(y))\,dy, \\ L_2\xi(x) &= \lambda \int_{\mathbb{R}^{\mathbb{N}}} \Gamma(x-y)a_2(y)f_2(\xi(y))dy. \end{aligned}$$

By the hypotheses (N1)–(N2) and Lemma 6.1 the operator L is well defined. Let

$$\begin{aligned} A_1 &= \{x \in \mathbb{R}^N : a_1(x) \geq 0\}, \quad B_1 = \{x \in \mathbb{R}^N : a_1(x) < 0\}, \\ A_2 &= \{x \in \mathbb{R}^N : a_2(x) \geq 0\}, \quad B_2 = \{x \in \mathbb{R}^N : a_2(x) < 0\}. \end{aligned}$$

Define the following operators representing the positive and negative parts of L_i for $i = 1, 2$: $L_i^+ : X(\mathbb{R}^{\mathbb{N}} \cap A_i) \to C_b^+(\mathbb{R}^{\mathbb{N}})$ by

$$(L_i^+\varphi)(x) = \lambda \int_{A_i} \Gamma(x-y)(a_i)_+(y)f_i(\varphi(y))\,dy,$$

and $L_i^- : X(\mathbb{R}^{\mathbb{N}} \cap B_i) \to C_b^+(\mathbb{R}^{\mathbb{N}})$ by

$$(L_i^-\varphi)(x) = \lambda \int_{B_i} \Gamma(x-y)(a_i)_-(y)f_i(\varphi(y))\,dy.$$

We note that the operator L_i can now be written as

$$L_i\varphi = L_i^+\varphi - L_i^-\varphi, \quad i = 1, 2.$$

Using the monotonicity of f_i, we can conclude that for $i = 1, 2$, both L_i^+ and L_i^- are monotone operators. The following proposition can be proved as above.

Proposition 6.3. *Let f_i, a_i satisfy the hypotheses* (N1)–(N2) and $\xi_0, \eta_0 \in X(\mathbb{R}^N) \times X(\mathbb{R}^N)$ satisfy assumptions (1)–(6) of Proposition 5.2. Then there exist $\xi, \eta \in C_b^+(\mathbb{R}^N) \times C_b^+(\mathbb{R}^N)$ with $\xi \leq \eta$ such that

$$\begin{aligned} \xi^1 &= L_1^+\xi^2 - L_1^-\eta^2, \quad \eta^1 = L_1^+\eta^2 - L_1^-\xi^2, \\ \xi^2 &= L_2^+\xi^1 - L_2^-\eta^1, \quad \eta^2 = L_2^+\eta^1 - L_2^-\xi^1. \end{aligned}$$

Proposition 6.4. *Let*

$$\mathcal{C} = \{g \in C_b(\mathbb{R}^{\mathbb{N}}) \times C_b(\mathbb{R}^{\mathbb{N}}) : \xi \leq g \leq \eta\}.$$

Further assume that f_i, a_i satisfy the hypotheses (N1)–(N2) *and there exist ξ, η as in Proposition 6.3. Then L has a fixed point in $\mathcal{C}$.*

Proof. With the above hypotheses, it can be shown as in Proposition 5.1, that $\mathcal{C}$ is a closed, convex set invariant under L. Since Γ is locally integrable and a_i's have the decay given in (N2), $\Gamma * a_1$ and $\Gamma * a_2$ are also integrable in $\mathbb{R}^N$. By standard arguments, it can be shown that $\{Lg_n\}$ is an equicontinuous family in $C(B_R)$ and also $L : \mathcal{C} \to \mathcal{C}$ is a continuous operator. To apply the Schauder's fixed point theorem, it suffices to show that $L : \mathcal{C} \to \mathcal{C}$ is a compact operator.

Let $\{g_n\}$ be a sequence of functions in $\mathcal{C}$. It is easy to see that $Lg_n(x)$ is uniformly bounded in $\mathbb{R}^N$. Thus by Arzela–Ascoli theorem, $Lg_n(x)$ has a uniformly convergent subsequence in B_R (which we still denote by $\{Lg_n\}$) for any fixed $R > 0$. We note that, by Lemma 6.1,

$$\begin{aligned} |L_1 g_n^2(x)| &= \lambda \Big| \int_{\mathbb{R}^N} \Gamma(x-y) a_1(y) f_1(g_n^2(y))\, dy \Big| \\ &\le M\lambda \int_{\mathbb{R}^N} \Gamma(x-y)|a_1(y)|\, dy \\ &\le C|x|^{-\delta} \quad \text{for all large } x. \end{aligned}$$

Therefore, for a given $\varepsilon > 0$, we fix $R > 0$ large enough such that

$$|L_1 g_n^2(x)| \le \frac{\epsilon}{4} \quad \forall |x| > R.$$

Similarly, we get $|L_2 g_n^1(x)| \le \frac{\epsilon}{4}$ for all $|x| > R$. Thus for the sequence $\{g_n\} \subset \mathcal{C}$ for which $\{Lg_n\}$ converges in $C(\bar{B}_R)$, we have that $L_1 g_n^2(x)$ and $L_2 g_n^1(x)$ are uniformly Cauchy in $C(\mathbb{R}^N)$. Thus $Lg_n(x)$ converges to some $g = (g^1, g^2)$ in $\mathbb{R}^N$ which shows that $L : \mathcal{C} \to \mathcal{C}$ is compact. Now by Schauder's fixed point theorem there exists $\varphi = (\varphi_1, \varphi_2) \in \mathcal{C}$ such that $L\varphi = \varphi$. $\square$

Proof of Theorem 1.3. Define ξ_0 and η_0 as in the proof of Theorem 1.2. Then in a similar way, for λ small we can obtain ξ and η as required in the Proposition 6.4. Thus we have a $\varphi \in \mathcal{C}$ such that $L\varphi = (L_1\varphi_2, L_2\varphi_1) = (\varphi_1, \varphi_2)$. Since $a_1(x) f_1(\varphi_2(x))$ is bounded and integrable in $\mathbb{R}^N$, we have $\varphi_1 \in C^1(\mathbb{R}^N)$ (see, [18]). Similarly φ_2 is also in $C^1(\mathbb{R}^N)$ and by an application of Lemmas 6.1 and 6.2 (φ_1, φ_2) solves the non-radial coupled system. Indeed by classical Schauder's theory $(\varphi_1, \varphi_2) \in C^{2,\alpha}(\mathbb{R}^N) \times C^{2,\alpha}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. $\square$

Remark 6.5. In fact, we can consider the following $n \times n$ coupled system and show the existence of a nonnegative solution using the methods in this section

$$\begin{gathered} \Delta u_1 + \lambda a_1(x) f_1(u_{\sigma(1)}(x)) = 0, \\ \Delta u_2 + \lambda a_2(x) f_2(u_{\sigma(2)}(x)) = 0, \\ \dots \\ \Delta u_n + \lambda a_n(x) f_n(u_{\sigma(n)}(x)) = 0. \end{gathered}$$

Here σ is a bijection from $\{1, 2, \dots, n\}$ to itself, a_i's may change sign and f_i, a_i satisfy the hypotheses (N1)–(N3) for $i = 1, \dots, n$. For the sake of brevity, we omit the details.

Remark 6.6. $f(0) > 0$ is used in showing the existence of the solution of $-\Delta u = \lambda a(x) f(u)$, for λ small enough. If we assume $f(u) = u^p$, and $a(x)$ satisfies $(N2)$, the above equation does not have any bounded positive solution decaying at infinity for any λ. The proof follows by Pohozaev's identity (see [26]).

7. Examples

In this section, we construct some examples for the illustration of our results.

Example 7.1. Let $N \ge 3, T \ge 1$ and $A, B > 0$. Define

$$a(t) = \begin{cases} A\theta_1(t), & t \in [0, T], \\ -B\theta_2(t), & t \in (T, \infty), \end{cases}$$

where $\theta_1(t) \geq \alpha_1 > 0, \theta_2(t) > 0$ such that $\int_0^T \theta_1(t)\,dt < \infty$ and $\int_T^\infty t^{N-1}\theta_2(t)\,dt \leq \alpha_2 < \infty$. It is easy to see that $a(t)$ satisfies (H2), (H3).

Now our aim is to find A and B such that the hypothesis (H4) is satisfied; i.e., there exists $\mu > 0$ such that for all $x \in (0,\infty)$,

$$\int_0^\infty G(x,t)t^{N-1}a_+(t)\,dt \geq (1+\mu)\int_0^\infty G(x,t)t^{N-1}a_-(t)\,dt.$$

The above inequality holds if and only if

$$\int_0^T G(x,t)t^{N-1}a_+(t)\,dt \geq (1+\mu)\int_T^\infty G(x,t)t^{N-1}a_-(t)\,dt. \tag{7.1}$$

First consider the case $x \in (0,T]$. In this case, $G(x,t) = \frac{1}{N-2}t^{2-N}$ for $T < t < \infty$. Therefore, (7.1) becomes

$$A\int_0^x x^{2-N}t^{N-1}\theta_1(t)\,dt + A\int_x^T t\theta_1(t)\,dt \geq (1+\mu)B\int_T^\infty t\theta_2(t)\,dt.$$

In the above expression, the left-hand side is greater than or equal to $A\alpha_1[\frac{x^2}{N} + \frac{T^2}{2} - \frac{x^2}{2}] \geq \frac{A\alpha_1 T^2}{N}$ while the right-hand side is less than or equal to $(1+\mu)B\alpha_2$ (because $\int_T^\infty t\theta_2(t)\,dt \leq \alpha_2$). So, with the choice: $A \geq \frac{(1+\mu)B\alpha_2 N}{T^2\alpha_1}, a(t)$ satisfies (7.1) for all $x \in (0,T]$. Now let $x \in (T,\infty)$. Then $G(x,t) = \frac{1}{N-2}x^{2-N}$ for $0 < t < T$. Thus (7.1) becomes

$$\begin{aligned} &A\int_0^T x^{2-N}t^{N-1}\theta_1(t)\,dt \\ &\geq B(1+\mu)\Big[\int_T^x x^{2-N}t^{N-1}B\theta_2(t)\,dt + \int_x^\infty t\theta_2(t)\,dt\Big]. \end{aligned} \tag{7.2}$$

Similar to the previous case, we obtain the left-hand side is greater than or equal to $\frac{A\alpha_1 T^N}{N}x^{2-N}$ while the rihgt-hand side is less than or equal to $(1+\mu)B\alpha_2 x^{2-N}$ (in view of $\int_T^\infty t^{N-1}\theta_2(t)\,dt \leq \alpha_2$). Hence, with the choice: $A \geq \frac{(1+\mu)B\alpha_2 N}{T^N\alpha_1}, a(t)$ satisfies (7.1) for all $x \in (T,\infty)$. Thus, since $T \geq 1$, with the choice: $A \geq \frac{(1+\mu)B\alpha_2 N}{T^2\alpha_1}, a(t)$ satisfies (H4).

Note that there are many examples of θ_1 and θ_2, one can choose in the above example. For instance, we can take $\theta_1(t) = 1, (1+t^2), e^{-t}$ and $\theta_2(t) = \frac{e^{-t}}{t^{N-1}}, \frac{1}{t^{\alpha+N-1}}$, with $\alpha > 1$.

Remark 7.2. For $T < 1$, one can construct examples of $\theta_1(t)$ and $\theta_2(t)$ with some different integrability conditions on θ_1, θ_2. We omit the details for the sake of brevity.

Example 7.3. Let $n \in \mathbb{Z}^+$, and define

$$a(t) = \begin{cases} \frac{e^{-t}}{1+t^{N-1}}, & t \in [2n, 2n+1), \\ \frac{-e^{-t}}{2N(1+\mu)t^{N-1}}, & t \in [2n+1, 2n+2). \end{cases}$$

By elementary calculation we observe that $a(t)$ satisfies (H2), (H3) and (H4').

Example 7.4. Two specific examples of the nonlinearity f are:

(i) $f(y) = (1+y^2)^3$ and $\alpha = \sqrt{(1+\mu)^{\frac{1}{3}} - 1}$;
(ii) $f(y) = e^y - \frac{1}{2(y+1)}$ and $\alpha > 0$ sufficiently small.

In both cases, it is easy to see that $f(\alpha) \le (1+\mu)f(0)$.

Remark 7.5. Let

$$a(t) = \begin{cases} A, & t \in [0,T], T \ge 1, \\ -\frac{e^{-t}}{t^{N-1}}, & t \in (T,\infty), \end{cases}$$

and $f(y) = (1+y^2)^3$. We note that for these choices, one can find out the value of λ_0 in Theorem 1.1. We first note that

$$\frac{1}{(N-2)}\int_0^\infty t|a(t)|\,dt \le \frac{1}{(N-2)}(A\frac{T^2}{2} + e^{-T}) \equiv \beta.$$

Let $A = \frac{(1+\mu)Ne^{-T}B}{T^2}$ in view of Example 7.1. Then we obtain

$$\lambda_0 = \frac{2(N-2)\sqrt{(1+\mu)^{\frac{1}{3}} - 1}}{[(1+\mu)BN + 2]e^{-T}(1+2\mu)}.$$

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References

[1] G. A. Afrouzi, K. J. Brown; *Positive solutions for a semilinear elliptic problem with sign–changing nonlinearity*, Nonlinear Anal., **36** (1999), pp. 507–510.

[2] D. Aronson, M. G. Crandall, and L. A. Peletier; *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Anal., **6** (1982), pp. 1001–1022.

[3] J. W. Bebernes,and D. R. Kassoy; *A mathematical analysis of blowup for thermal reactions–the spatially inhomogenous case*, SIAM J. Appl. Math., **40** (1) (1981), pp. 476–484.

[4] H. Berestycki,and P. L. Lions; *Existence of a ground state in nonlinear equations of the type Klein-Gordon, in Variational Inequalities*, (Cottle, Gianessi and J. L. Lions, editors), J. Wiley, New York, 1980.

[5] K. J. Brown and A. Tertikas; *On the bifurcation of radially symmetric steady–state solutions arising in population genetics*, SIAM J. Math. Anal., **22** (2) (1991), pp. 400–413.

[6] K. J. Brown and N. M. Stavrakakis; *On the construction of super and subsolutions for elliptic equations on all of* $\mathbb{R}^N$, Nonlinear Anal., **32** (1998), pp. 87–95.

[7] N. P. Cac, A. M. Fink, J. A. Gatica; *Nonnegative solutions of quasilinear elliptic problms with nonnegative coefficients*, Journal Math. Anal. Appl., **206** (1997), pp. 1–9.

[8] N. P. Cac, A. M. Fink, J. A. Gatica; *Nonnegative solutions of the radial Laplacian with nonlinearty that changes sign*, Proc. Amer. Math. Soc., **123** (5) (1995), pp. 1393–1398.

[9] N. P. Cac, A. M. Fink, Y. Li; *Positive solutions to semilinear problems with coefficient that changes sign, Nonlinear Anal.*, **37** (1999), pp. 501–510.

[10] A. Castro, C. Maya and R. Shivaji; *Positivity of non-negative solutions for semipositone cooperative systems*, Proc. Dyn. Sys. **3** (2001), pp. 113–120.

[11] R. Dalmasso; *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal. **39** (2000), pp. 559–568.

[12] D. G. de Figueirdo, J. Marcos do Ó, B. Ruf; *An Orlicz–space approach to superlinear elliptic systems*, J. Func. Anal., **224** (2005), pp. 471–496.

[13] A. M. Fink; *The radial Laplacian Gelfand problem*, " Delay and Differential Equations", pp. 93–98, World Scientific, New Jersey, 1992.

[14] W. H. Fleming; *A selection–migration model in population genetics*, J. Math. Biol., **2** (1975), pp. 219–233.

[15] J. L. Gámez; *Existence and bifurcation of positive solutions of a semilinear elliptic problem in* $\mathbb{R}^N$, Nonlinear Diff. Eqn. Appl., **4** (1997), pp. 341–357.

[16] X. Garaizar; *Existence of positive radial solutions for semilinear elliptic equations in the annulus*, J. Diff. Equations, **70** (1987), pp. 69–92.

[17] J. García–Melián, Julio D. Rossi; *Boundary blow–up solutions to elliptic systems of competitive type*, J. Diff. Eqns., **206** (2004), pp. 156–181.
[18] D. Gilbarg, Neil S Trudinger; *Elliptic Partial Differential Equations of Second Order*, Springer–Classics in Mathematics.
[19] D. D. Hai; *Positive solutions to a class of elliptic boundary value problems*, J. Math. Anal. Appl., **227** (1998), pp. 195–199.
[20] D. D. Hai and R. Shivaji; *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh, **134 A** (2004), pp. 137–141.
[21] D. Hulshof and R. Van der Vorst; *Differential systems with strongly indefinite variational structure*, J. Func. Anal., **114** (1993), pp. 32–58.
[22] S. S. Lin; *On the existence of positive radial solutions for nonlinear elliptic equations in annular domains*, J. Diff. Equations, **81** (1989), pp. 221–233.
[23] P. L. Lions; *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. **24** (1982), no. 4, pp. 441–467.
[24] A. Tertikas; *Existence and uniqueness of solutions for a nonlinear diffusion problem arising in population genetics*, Arch. Rat. Mech. Anal., **103** (1988), pp. 289–317.
[25] J. Tyagi; *Existence of nonnegative solutions for a class of semilinear elliptic systems with indefinite weight*, Nonlinear Analysis T.M.A., **73** (2010), pp. 2882–2889.
[26] Yi Li and W. M. Ni.; *On conformal scalar curvature equations in* $\mathbb{R}^N$, Duke Math. J., **57** (1988), pp. 895–924.

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