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EXISTENCE OF SOLUTIONS TO INDEFINITE QUASILINEAR ELLIPTIC PROBLEMS OF P-Q-LAPLACIAN TYPE

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Abstract. We study the indefinite quasilinear elliptic problem

$$-\Delta u - \Delta_p u = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a sufficiently smooth boundary, q, s are subcritical exponents, $a(\cdot)$ changes sign and $b(x) \geq 0$ a.e. in Ω. Our proofs are variational in character and are based either on the fibering method or the mountain pass theorem.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with a sufficiently smooth boundary $\partial\Omega$. We consider the stationary nonlinear equation

$$-\Delta_q u - \Delta_p u = f(x, u) \quad \text{in } \Omega \tag{1.1}$$

with Dirichlet boundary condition

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $p, q \in (1, N)$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function.

Solutions to (1.1) are the steady state solutions of the reaction diffusion system

$$u_t = \operatorname{div}(A(u)\nabla u) + f(x, u), \tag{1.3}$$

where $A(u) = (|\nabla u|^{q-2} + |\nabla u|^{p-2})$. This system has a wide range of applications in physics and related sciences like chemical reaction design [2], biophysics [12] and plasma physics [19]. The function u describes the concentration of a substance, $\operatorname{div}(A(u)\nabla u)$ corresponds to the diffusion with diffusion coefficient $A(u)$ and $f(\cdot, \cdot)$ represents the reaction.

Equation (1.1) also arises in the study of soliton-like solutions of the nonlinear Schrödinger equation

$$i\psi_t = -\Delta\psi - \Delta_p\psi + f(x, \psi)$$

which was considered by Derrick [9] as a model for elementary particles.

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When $p = q = 2$, (1.1) is a normal Schrodinger equation which has been extensively studied, we refer to [3, 6, 7]. Recently, the problem when $m = 2 \neq q$ and

$$f(x, u) = V'(u)$$

was studied in [4] where it is proved that (1.1)-(1.2) admits a weak solution with a prescribed value of topological charge. The eigenvalue problem

$$-\Delta u + V(x)u + \varepsilon^r(-\Delta_p u + W'(u)) = \mu u$$

was considered in [5] and the behavior of the eigenvalues as $\varepsilon \to 0$ was examined. In [8] the case where $m \neq p$ and

$$f(x, u) = \lambda a(x)|u|^{\gamma-2}u - b(x)|u|^{m-2}u - c(x)|u|^{p-2}u$$

is studied and a bifurcation result is also presented. A solution is also provided in [13] under the assumption that

$$f(x, u) = g(x, u) - b(x)|u|^{m-2}u - c(x)|u|^{p-2}u \tag{1.4}$$

where the function $g(\cdot, \cdot)$ does not satisfy the Ambrosetti-Rabinowitz condition. The $C^{1,\delta}$-regularity of the solutions of problem (1.1) was shown in [14]. Constraint minimization is employed in [20] with constraint functional

$$\int_{\mathbb{R}^N} [b(x)|u|^q - c(x)|u|^p u] dx = \lambda$$

when $f(\cdot, \cdot)$ satisfies (1.4), in order to show that (1.1) admits a solution for $\lambda \in (0, \lambda_0)$, $\lambda_0 > 0$. Sufficient conditions for the existence of two solutions to problem ((1.1) are provided in [17].

In this article we study the problem

$$-\Delta u - \Delta_p u = a(x)|u|^{q-2}u - b(x)|u|^{s-2}u \quad \text{in } \Omega, \tag{1.5}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where the exponents q, s are subcritical and $a(\cdot), b(\cdot)$ are essentially bounded functions, $a(\cdot)$ changes sign while $b(\cdot) \geq 0$ a.e. in Ω. Our proofs are variational in character and rely either on the fibering method of Pohozaev [18] or on the mountain pass theorem of Ambrosetti-Rabinowitz [1].

By symmetry, we will only consider the cases where $p < 2$.

2. Preliminaries and main results

We make the following hypotheses concerning the data of problem (1.5)-(1.6):

(H0) $1 < s, q < 2^*$.
(H1) $a(\cdot) \in L^\infty(\Omega)$ and $a_+ := \max\{a, 0\} \neq 0$.
(H2) $b(\cdot) \in L^\infty(\Omega)$ and $b(x) \geq 0$ a.e. in Ω.

We will seek weak solutions in the space

$$E := H_0^1(\Omega),$$

supplied with the norm $\|v\|_E = \|\nabla v\|_2$. The energy functional $\Phi : E \to \mathbb{R}$ associated with (1.5)-(1.6) is

$$\Phi(v) := \frac{1}{p}\|\nabla v\|_p^p + \frac{1}{2}\|\nabla v\|_2^2 - \frac{1}{q}A(v) + \frac{1}{s}B(v), \tag{2.1}$$

where

$$A(v) := \int_\Omega a(x)|v|^q dx \text{ and } B(v) := \int_\Omega b(x)|v|^s dx.$$

to find nonnegative critical points for $\Phi(\cdot)$ we use the fibering method. So we decompose the function $u \in E$ as $u = rv$, where $r \in \mathbb{R}$, $v \in E$, and define the extended functional $F(\cdot,\cdot)$ associated with $\Phi(\cdot)$ as

$$F(r,v) := \Phi(rv) = \frac{|r|^p}{p}\|\nabla v\|_p^p + \frac{|r|^2}{2}\|\nabla v\|_2^2 - \frac{|r|^q}{q}A(v) + \frac{|r|^s}{s}B(v). \tag{2.2}$$

If $u = rv$ is a critical point of $\Phi(\cdot)$, then we must have

$$F_r(r,v) = 0. \tag{2.3}$$

Clearly, (2.3) is equivalent to

$$r^2\|\nabla v\|_2^2 + r^p\|\nabla v\|_p^p = r^q A(v) - r^s B(v). \tag{2.4}$$

Let $r := r(v)$ be a positive solution of (2.4). We define the reduced functional $\hat{\Phi}(v) := \Phi(r(v)v)$, $v \in E$, which, in view of (2.4), has the following equivalent expressions

$$\begin{aligned}
\hat{\Phi}(v) &:= r^2(\frac{1}{2} - \frac{1}{p})\|\nabla v\|_2^2 + r^q(\frac{1}{p} - \frac{1}{q})A(v) + r^s(\frac{1}{s} - \frac{1}{p})B(v) && (2.5)\\
&= r^q(\frac{1}{p} - \frac{1}{q})\|\nabla v\|_p^p + r^2(\frac{1}{2} - \frac{1}{q})\|\nabla v\|_2^2 + r^s(\frac{1}{s} - \frac{1}{q})B(v) && (2.6)\\
&= r^p(\frac{1}{p} - \frac{1}{s})\|\nabla v\|_p^p + r^2(\frac{1}{2} - \frac{1}{s})\|\nabla v\|_2^2 + r^q(\frac{1}{s} - \frac{1}{q})A(v) && (2.7)\\
&= r^p(\frac{1}{p} - \frac{1}{2})\|\nabla v\|_p^p + r^q(\frac{1}{2} - \frac{1}{q})A(v) + r^s(\frac{1}{s} - \frac{1}{2})B(v). && (2.8)
\end{aligned}$$

The fibering method is based on the following fact.

Lemma 2.1. *Let $H : E \to \mathbb{R}$ be a functional which is continuously Fréchet-differentiable in $E\backslash\{0\}$ and satisfies the conditions:*

$$\langle H'(v), v\rangle \neq 0 \quad \text{if } H(v) = 1,$$

and $H(0) = 0$. If $v \neq 0$ is a conditional critical point of $\hat{\Phi}(\cdot)$ under the constraint $H(v) = 1$, then $u := r(v)v$ is a nonzero critical point of $\Phi(\cdot)$.

For more details we refer to [11]. The constraint functional we are going to use is

$$H(v) := \|\nabla v\|_p^p + \|\nabla v\|_2^2$$

which clearly satisfies the two conditions in Lemma 2.1. Let

$$S^1 := \{v \in E : H(v) = 1\}. \tag{2.9}$$

Note that, because of assumption (H_1), the set

$$G_1 := \{v \in E : A(v) > 0\}$$

is nonempty.

We distinguish the following cases:

Case 1: $q < \min\{p, s, 2\}$. We will work as in [15, 16]. From (2.4) we see that

$$r^{p-q}\|\nabla v\|_p^p + r^{2-q}\|\nabla v\|_2^2 + r^{s-q}B(v) = A(v), \tag{2.10}$$

which admits a unique solution $r(\upsilon) > 0$ for every $\upsilon \in G_1$. It is easy to check that $r(\upsilon)\upsilon = r(k\upsilon)k\upsilon$ for every $k > 0$. The implicit function theorem, see [21] , shows that $r(\cdot) \in C^1(G_1)$. If $\upsilon \in S^1$ then the Hölder inequality implies that $\|\nabla\upsilon\|_2^2 \geq \theta$ for some $\theta > 0$ and so, by (2.10), $r(\cdot)$ is bounded on $G_1 \cap S^1$ because $A(\cdot)$ is bounded on S^1 by the Rellich theorem. Consequently, $\hat{\Phi}(\cdot)$ is bounded on $G_1 \cap S^1$. Let

$$M = \inf_{u \in G_1 \cap S^1} \hat{\Phi}(u).$$

By (2.6), $M < 0$. Suppose that $\{\upsilon_n\}$ is a minimizing sequence for $\hat{\Phi}(\cdot)$ in $G_1 \cap S^1$. Then, at least for a subsequence, we have that $\upsilon_n \to \tilde{\upsilon}$ weakly in E, and so we may assume that $A(\upsilon_n) \to A(\tilde{\upsilon})$ and $B(\upsilon_n) \to B(\tilde{\upsilon})$. Exploiting the weak lower semicontinuity of the norms we get that

$$0 \leq \|\nabla\tilde{\upsilon}\|_2^2 \leq \liminf \|\nabla\upsilon_n\|_2^2, \quad 0 \leq \|\nabla\tilde{\upsilon}\|_p^p \leq \liminf \|\nabla\upsilon_n\|_p^p.$$

Since $\{r(\upsilon_n)\}_{n\in\mathbb{N}}$ is bounded we may also assume that $r(\upsilon_n) \to \tilde{r}$. Therefore,

$$\Phi(\tilde{r}\tilde{\upsilon}) \leq \liminf \Phi(r_n\upsilon_n) = M < 0,$$

implying that $\tilde{r} > 0$ and $\tilde{\upsilon} \neq 0$. On the other hand, by (2.10)

$$r(\upsilon_n)^{p-q}\|\nabla\upsilon_n\|_p^p + r(\upsilon_n)^{2-q}\|\nabla\upsilon_n\|_2^2 + r(\upsilon_n)^{s-q}B(\upsilon_n) = A(\upsilon_n). \tag{2.11}$$

By taking the limit as $n \to +\infty$, we obtain

$$0 < \tilde{r}^{p-q}\|\nabla\tilde{\upsilon}\|_p^p + \tilde{r}^{2-q}\|\nabla\tilde{\upsilon}\|_2^2 + \tilde{r}^{s-q}B(\tilde{\upsilon}) \leq A(\tilde{\upsilon}), \tag{2.12}$$

which implies that $\tilde{\upsilon} \in G_1$. In view of (1.5),

$$r(\tilde{\upsilon})^{p-q}\|\nabla\tilde{\upsilon}\|_p^p + r(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2 + r(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) = A(\tilde{\upsilon}), \tag{2.13}$$

and so (2.12) shows that $\tilde{r} \leq r(\tilde{\upsilon}\ )$. If we assume that $\tilde{r} < r(\tilde{\upsilon})$, then, since the function $t \to \Phi(t\tilde{\upsilon})$, $t \in (0, r(\tilde{\upsilon}))$, is strictly decreasing, we have

$$\hat{\Phi}(\tilde{\upsilon}) = \Phi(r(\tilde{\upsilon})\tilde{\upsilon}) < \Phi(\tilde{r}\tilde{\upsilon}) \leq M. \tag{2.14}$$

Then

$$\hat{\Phi}(\frac{\tilde{\upsilon}}{\|\tilde{\upsilon}\|_E}) = \hat{\Phi}(\tilde{\upsilon}) = M,$$

a contradiction. Therefore, $\tilde{r} = r(\tilde{\upsilon})$. Then, by (2.11) and (2.13),

$$\lim_{n\to\infty}\{\|\nabla\upsilon_n\|_p^p + r(\upsilon_n)^{2-p}\|\nabla\upsilon_n\|_2^2\} = \|\nabla\tilde{\upsilon}\|_p^p + r(\tilde{\upsilon})^{2-p}\|\nabla\tilde{\upsilon}\|_2^2, \tag{2.15}$$

which implies that $\|\nabla\upsilon_n\|_p^p \to \|\nabla\tilde{\upsilon}\|_p^p$ and $\|\nabla\upsilon_n\|_2^2 \to \|\nabla\tilde{\upsilon}\|_2^2$. Consequently, $\tilde{\upsilon} \in S^1$ and $\hat{\Phi}(\tilde{\upsilon}) = M$. Since $|\tilde{\upsilon}|$ is also a minimizer of $\hat{\Phi}(\cdot)$, we may assume that $\tilde{\upsilon} \geq 0$. Lemma 2.1 implies that $u := r(\tilde{\upsilon})\tilde{\upsilon}$ is a solution to (1.5)-(1.6). By [14, Theorem 1], $u \in C^{1,\delta}(\Omega)$ for some $\delta \in (0,1)$. Therefore we have the following result.

Theorem 2.2. *Assume that* (H0)-(H2) *are satisfied and* $q < \min\{p, s, 2\}$. *Then* (1.5)-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Case 2: $p < q < 2 < s$. Let

$$Q(r,\upsilon) := r^{q-p}A(\upsilon) - r^{s-p}B(\upsilon) - r^{2-p}\|\nabla\upsilon\|_2^2. \tag{2.16}$$

Then (2.4) is equivalent to

$$Q(r,\upsilon) = \|\nabla\upsilon\|_p^p. \tag{2.17}$$

We see that for $\upsilon \in G_1$ the function $Q(\cdot,\upsilon)$ has a unique critical point $r_* := r_*(\upsilon)$ satisfying

$$(q-p)A(\upsilon) = (s-p)r_*(\upsilon)^{s-q}B(\upsilon) + (2-p)r_*(\upsilon)^{2-q}\|\nabla\upsilon\|_2^2. \tag{2.18}$$

In view of (2.16), we get the following equivalent expressions for (2.18), that will be needed in the sequel,

$$Q(r_*(\upsilon),\upsilon) = \frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon) + \frac{s-2}{2-p}r_*(\upsilon)^{s-p}B(\upsilon), \tag{2.19}$$

$$Q(r_*(\upsilon),\upsilon) = \frac{s-q}{s-p}r_*(\upsilon)^{s-p}A(\upsilon) + \frac{2-s}{s-p}r_*(\upsilon)^{2-p}\|\nabla\upsilon\|_2^2, \tag{2.20}$$

$$Q(r_*(\upsilon),\upsilon) = \frac{s-q}{q-p}r_*(\upsilon)^{s-p}B(\upsilon) + \frac{2-q}{q-p}r_*(\upsilon)^{s-p}\|\nabla\upsilon\|_2^2. \tag{2.21}$$

Let

$$G_2 := \{\upsilon \in G_1 : \|\nabla\upsilon\|_p^p < Q(r_*(\upsilon),\upsilon)\}. \tag{2.22}$$

Equation (2.17) has two positive solutions $r_1(\upsilon)$, $r_2(\upsilon)$ with $r_1(\upsilon) < r_*(\upsilon) < r_2(\upsilon)$ for every $\upsilon \in G_2$. Let $r := r_2(\upsilon)$. Then

$$r^{p-q+1}Q_r(r,\upsilon) = (q-p)A(\upsilon) - (s-p)r^{s-q}B(\upsilon) - (2-p)r^{2-q}\|\nabla\upsilon\|_2^2,$$

which, combined with (2.18), gives

$$r^{p-q+1}Q_r(r,\upsilon) = (2-p)\|\nabla\upsilon\|_2^2(r_*^{2-q} - r^{2-q}) + (s-p)B(\upsilon)(r_*^{s-q} - r^{s-q}) < 0.$$

By the implicit function theorem $r(\cdot)$ is continuously differentiable. Let

$$G_3 := \big\{\upsilon \in G_1 : \|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon)\big\} \tag{2.23}$$

and assume that $G_3 \neq \emptyset$. Since $q > p$ and $r(\upsilon) > r_*(\upsilon)$, we see that $G_3 \subseteq G_2$ and so $G_2 \neq \emptyset$. If $\upsilon \in G_3$, then

$$\|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon), \tag{2.24}$$

and so

$$\|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}r(\upsilon)^{q-p}A(\upsilon).$$

Thus

$$\frac{2-p}{p}r(\upsilon)^p\|\nabla\upsilon\|_p^p + \frac{q-2}{q}r(\upsilon)^qA(\upsilon) < 0. \tag{2.25}$$

By (2.25) and (2.8) we conclude that

$$\hat\Phi(\upsilon) < r^p(\frac{1}{p} - \frac{1}{2})\|\nabla\upsilon\|_p^p + r^q(\frac{1}{2} - \frac{1}{q})A(\upsilon) < 0.$$

On the other hand, if $\upsilon \in G_2 \cap S^1$, by (2.10)

$$r(\upsilon) \le \Big(\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)}, \tag{2.26}$$

and so $r(\cdot)$ is bounded on $G_2 \cap S^1$. Consequently, $\hat\Phi(\upsilon)$ is also bounded on $G_2 \cap S^1$. Let

$$M := \inf_{\upsilon \in G_2 \cap S^1} \hat\Phi(\upsilon) < 0.$$

Suppose that $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_2 \cap S^1$. Then there exists $\tilde{\upsilon} \in E$ such that, at least for a subsequence, $A(\upsilon_n) \to A(\tilde{\upsilon})$, $B(\upsilon_n) \to B(\tilde{\upsilon})$,

$$\begin{aligned} 0 &\le \|\nabla\tilde{\upsilon}\|_2 \le \liminf \|\nabla\upsilon_n\|_2 \le 1, \\ 0 &\le \|\nabla\tilde{\upsilon}\|_p \le \liminf \|\nabla\upsilon_n\|_p \le 1. \end{aligned}$$

We must have $\tilde{\upsilon} \ne 0$ because, otherwise, $0 = \Phi(0) \le \liminf_{n\to\infty} \Phi(r(\upsilon_n)\upsilon_n) = M$, a contradiction. Since $\{r(\upsilon_n)\}_{n\in\mathbb{N}}$ is bounded and $r_*(\upsilon_n) < r(\upsilon_n)$, $n \in \mathbb{N}$, we may assume that $r_*(\upsilon_n) \to \tilde{r}_*$ and $r(\upsilon_n) \to \tilde{r} > 0$. If $A(\tilde{\upsilon}) = 0$, then, by (2.26), we obtain that $\tilde{r} = 0$ which is a contradiction. Thus, $A(\tilde{\upsilon}) > 0$ and so $\tilde{\upsilon} \in G_1$. Also, $\tilde{r}_* > 0$ by (2.18). We claim that $\tilde{\upsilon} \in G_3$. Indeed, by (2.17),

$$\begin{aligned} \|\nabla\tilde{\upsilon}\|_p^p &\le \limsup_{n\to\infty} \|\nabla\upsilon_n\|_p^p \le \limsup_{n\to\infty} Q(r_*(\upsilon_n),\upsilon_n) \\ &\le \limsup_{n\to\infty}\{r_*(\upsilon_n)^{q-p}A(\upsilon_n) - r_*(\upsilon_n)^{s-p}B(\upsilon_n)\} - \liminf_{n\to\infty} r_*(\upsilon_n)^{2-p}\|\nabla\upsilon_n\|_2^2 \\ &\le \tilde{r}_*^{q-p}A(\tilde{\upsilon}) - \tilde{r}_*^{s-p}B(\tilde{\upsilon}) - \tilde{r}_*^{2-p}\|\nabla\tilde{\upsilon}\|_2^2 = Q(\tilde{r}_*,\tilde{\upsilon}), \end{aligned} \tag{2.27}$$

implying that

$$\|\nabla\tilde{\upsilon}\|_p^p \le Q(r_*(\tilde{\upsilon}),\tilde{\upsilon}). \tag{2.28}$$

If we assume the equality

$$\|\nabla\tilde{\upsilon}\|_p^p = Q(r_*(\tilde{\upsilon}),\tilde{\upsilon}), \tag{2.29}$$

then by using (2.4) for $\upsilon = \upsilon_n$ and passing to the limit as $n \to +\infty$, we obtain

$$\begin{aligned} &\|\nabla\tilde{\upsilon}\|_p^p \\ &\le \limsup_{n\to\infty} \|\nabla\tilde{\upsilon}_n\|_p^p \le \limsup_{n\to\infty} Q(r(\upsilon_n),\upsilon_n) \\ &\le \limsup_{n\to\infty}\{r(\upsilon_n)^{q-p}A(\upsilon_n) - r(\upsilon_n)^{s-p}B(\upsilon_n)\} - \liminf_{n\to\infty} r(\upsilon_n)^{2-p}\|\nabla\upsilon_n\|_2^2 \\ &\le \tilde{r}^{q-p}A(\tilde{\upsilon}) - \tilde{r}^{s-p}B(\tilde{\upsilon}) - \tilde{r}^{2-p}\|\nabla\tilde{\upsilon}\|_2^2 = Q(\tilde{r},\tilde{\upsilon}). \end{aligned} \tag{2.30}$$

In view of (2.27), (2.29) and (2.30), we conclude that $\tilde{r} = \tilde{r}_* = \tilde{r}_*(\tilde{\upsilon})$. On the other hand, by replacing υ by υ_n in (2.18) and passing to the limit we obtain

$$(q-p)A(\tilde{\upsilon}) \ge (s-p)r_*(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) + (2-p)r_*(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2.$$

Since $r_*(\tilde{\upsilon})$ satisfies

$$(q-p)A(\tilde{\upsilon}) = (s-p)r_*(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) + (2-p)r_*(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2,$$

we deduce that $\|\nabla\upsilon_n\|_2^2 \to \|\nabla\tilde{\upsilon}\|_2^2$ and

$$(q-p)\,A(\tilde{\upsilon}) = (s-p)\,r_*(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) + (2-p)\,r_*(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2. \tag{2.31}$$

Thus,

$$A(\tilde{\upsilon}) = \frac{s-p}{q-p}\tilde{r}^{s-q}B(\tilde{\upsilon}) + \frac{2-p}{q-p}\tilde{r}^{2-q}\|\nabla\tilde{\upsilon}\|_2^2. \tag{2.32}$$

On the other hand, (2.5) and (2.32) imply that

$$M = \lim_{n\to\infty}\hat{\Phi}(\upsilon_n) = \frac{(s-q)(s-p)}{pqs}\tilde{r}^sB(\tilde{\upsilon}) + \frac{(2-p)(2-q)}{2pq}\tilde{r}^2\|\nabla\tilde{\upsilon}\|_2^2 > 0,$$

a contradiction. Therefore, $\tilde{\upsilon} \in G_3$ proving the claim. We shall show next that $\tilde{r} = r(\tilde{\upsilon})$. Let $t > 0$ be such that $t\tilde{\upsilon} \in S^1$. Since for $t > 0$

$$r_*(t\tilde{\upsilon})t\tilde{\upsilon} = r_*(\tilde{\upsilon})\tilde{\upsilon}, \tag{2.33}$$

by (2.17), (2.23) and (2.33), we have

$$\|\nabla\tilde{v}\|_p^p < Q(r_*(\tilde{v}),\tilde{v}) = Q(tr_*(t\tilde{v}),\tilde{v}) = t^{-p}Q(r_*(t\tilde{v}),t\tilde{v}).$$

Thus

$$\|t\nabla\tilde{v}\|_p^p \le Q(r_*(t\tilde{v}),t\tilde{v}),$$

which implies $t\tilde{v} \in G_2 \cap S^1$. Furthermore, by (2.17), $r(t\tilde{v})$ satisfies

$$Q(tr(t\tilde{v}),\tilde{v}) = \|\nabla\tilde{v}\|_p^p = Q(r(\tilde{v}),\tilde{v}), \tag{2.34}$$

which gives

$$tr(t\tilde{v}) = r(\tilde{v}). \tag{2.35}$$

In view of (2.30),

$$Q(r(\tilde{v}),\tilde{v}) = \|\nabla\tilde{v}\|_p^p \le Q(\tilde{r},\tilde{v}),$$

implying that $\tilde{r} \le r(\tilde{v})$. If we assume that $\tilde{r} < r(\tilde{v})$, then, since the function $z \to \Phi(z\tilde{v})$ is strictly decreasing in $(\tilde{r}, r(\tilde{v}))$, by (2.35) we obtain

$$M = \liminf_{n\to\infty} \Phi(r(v_n)v_n) \ge \Phi(\tilde{r}\tilde{v}) > \Phi(r(\tilde{v})\tilde{v}) = \Phi(r(t\tilde{v})t\tilde{v}) = \hat{\Phi}(t\tilde{v}),$$

which is a contradiction. Thus $\tilde{r} = r(\tilde{v})$. Then (2.15) holds, and so $\tilde{v} \in S^1$ and $\hat{\Phi}(\tilde{v}) = M$. As in the previous case we may assume that $\tilde{v} \ge 0$. Lemma 2.1 implies that $u := r(\tilde{v})\tilde{v}$ is a solution to (1.5)-(1.6).

Therefore, we have proved the following result.

Theorem 2.3. *Assume that conditions* (H0)-(H2) *are satisfied,* $p < q < 2 < s$ *and the set* G_3 *defined in* (2.23) *is not empty. Then the problem* (1.5)-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Remark 2.4. We will now give some conditions which guarantee that $G_3 \ne \emptyset$. Suppose that $\operatorname{supp} a^+) \subseteq \operatorname{supp} b)$. Then there exists $v \in S^1$ such that $B(v) > 0$. Since $r_*(v)^{2-q} < r(v)^{2-q}$, (2.18) yields

$$(q-p)A(v) < (s-p)r_*(v)^{s-q}B(v) + (2-p)r(v)^{2-q}\|\nabla v\|_2^2, \tag{2.36}$$

and so

$$r_*(v)^{s-q} > \frac{q-p}{s-p}\frac{A(v)}{B(v)} - \frac{2-p}{s-p}r(v)^{2-q}\frac{\|\nabla v\|_2^2}{B(v)}.$$

Consequently,

$$\begin{aligned} &\frac{p}{q}\frac{2-q}{2-p}r_*(v)^{q-p}A(v) \\ &> \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{s-p}\frac{A(v)}{B(v)} - \frac{2-p}{s-p}r(v)^{2-q}\frac{\|\nabla v\|_2^2}{B(v)}\Big)^{(q-p)/(s-q)}A(v). \end{aligned} \tag{2.37}$$

On the other hand, (2.10) implies that

$$r(v) \le \Big(\frac{A(v)}{B(v)}\Big)^{1/(s-q)}, \tag{2.38}$$

which combined with (2.37) gives

$$\begin{aligned} &\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{s-p}\frac{A(v)}{B(v)} - \frac{2-p}{s-p}r(v)^{2-q}\frac{\|\nabla v\|_2^2}{B(v)}\Big)^{(q-p)/(s-q)}A(v) \\ &> \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{s-p}\frac{A(v)}{B(v)} - \frac{2-p}{s-p}\Big(\frac{A(v)}{B(v)}\Big)^{(2-q)/(s-q)}\frac{\|\nabla v\|_2^2}{B(v)}\Big)^{\frac{q-p}{s-q}}A(v). \end{aligned}$$

If $a^+(\cdot)$ is large enough then

$$\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)}-\frac{2-p}{s-p}A(\upsilon)^{(2-q)/(s-q)}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)^{\frac{2-q}{s-q}+1}}\Big)^{(q-p)/(s-q)}A(\upsilon)>\|\nabla\upsilon\|_p^p, \tag{2.39}$$

implying that $\upsilon\in G_3$.

Suppose now that $\big(\operatorname{supp}a^+)\setminus\operatorname{supp}b)\big)^o\neq\emptyset$. Then there exists $\upsilon\in S^1$ with $B(\upsilon)=0$. From (2.18) we see that

$$r_*(\upsilon)=\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)}, \tag{2.40}$$

and so

$$\frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon)=\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon).$$

Consequently, if $a^+(\cdot)$ is large enough,

$$\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\Big)^{\frac{q-p}{2-q}}A(\upsilon)^{\frac{2-p}{2-q}}>\|\nabla\upsilon\|_2^{2(2-p)/(2-q)}, \tag{2.41}$$

implying that $G_3\neq\emptyset$.

Case 3: $p<s<q<2$. In this case we define

$$Q(r,\upsilon):=r^{q-p}A(\upsilon)-r^{s-p}B(\upsilon)-r^{2-p}\|\nabla\upsilon\|_2^2.$$

Let $\upsilon\in G_1$ and assume that $B(\upsilon)>0$. For $r\geq 0$ let

$$F(r,\upsilon):=r^{p-s}Q(r,\upsilon)=r^{q-s}A(\upsilon)-B(\upsilon)-\|\nabla\upsilon\|_2^2r^{2-s}. \tag{2.42}$$

Then, $F(0,\upsilon)=-B(\upsilon)<0$ and $\lim_{r\to+\infty}F(r,\upsilon)=-\infty$. It is easy to see that $F(\cdot,\upsilon)$ attains its maximum at

$$\bar r(\upsilon)=\Big(\frac{q-s}{2-s}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)} \tag{2.43}$$

with

$$F(\bar r(\upsilon),\upsilon)=\frac{2-q}{2-s}\bar r^{q-s}A(\upsilon)-B(\upsilon). \tag{2.44}$$

Consequently, $Q(r,\upsilon)>0$ for some $r>0$ if and only if $F(\bar r(\upsilon),\upsilon)>0$, and this holds if

$$\bar r(\upsilon)>\hat r(\upsilon):=(\frac{2-s}{2-q}\frac{B(\upsilon)}{A(\upsilon)})^{1/(q-s)}. \tag{2.45}$$

Suppose that (2.45) holds. Then it is easy to see that the function

$$r\mapsto r^{p-s+1}Q_r(r,\upsilon)=(q-p)r^{q-s}A(\upsilon)-(2-p)\|\nabla\upsilon\|_2^2r(\upsilon)^{2-s}-(s-p)B(\upsilon),$$

has two positive roots $r_{1*}(\upsilon)$ and $r_{2*}(\upsilon)$ with $r_{1*}(\upsilon)<r_{2*}(\upsilon)$. Clearly, $r_{1*}(\upsilon)$ is a point of local minimum of $Q(.,\upsilon)$ while $r_{2*}(\upsilon)$ is a point of global maximum of $Q(.,\upsilon)$. Define $r_*(\upsilon):=r_{2*}(\upsilon)$. We claim that

$$\bar r(\upsilon)<r_*(\upsilon). \tag{2.46}$$

Indeed,

$$r^{s-p}F_r(r,\upsilon)=Q_r(r,\upsilon)+(p-s)\frac{Q(r,\upsilon)}{r},$$

and since $F_r(\bar{r}(\upsilon), \upsilon) = 0$ and $Q(\bar{r}(\upsilon), \upsilon) = \bar{r}(\upsilon)^{s-p} F(\bar{r}(\upsilon), \upsilon) > 0$ we get

$$Q_r(\bar{r}(\upsilon), \upsilon) = (s - p)\frac{Q(\bar{r}(\upsilon), \upsilon)}{\bar{r}(\upsilon)} > 0,$$

proving the claim.

Next, let $\upsilon \in G_1$ and assume that $B(\upsilon) = 0$. Clearly $Q(\cdot, \upsilon)$ attains its maximum at

$$r_*(\upsilon) := \Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla \upsilon\|_2^2}\Big)^{1/(2-q)} \tag{2.47}$$

with

$$Q(r_*(\upsilon), \upsilon) = \frac{2-q}{2-p} r_*(\upsilon)^{q-p} A(\upsilon). \tag{2.48}$$

Since $r_*(\upsilon)$ satisfies the equation $Q_r(\cdot, \upsilon) = 0$, that is

$$(q-p)A(\upsilon)r_*(\upsilon)^{q-s} = (s-p)B(\upsilon) + (2-p)\|\nabla \upsilon\|_2^2 r_*(\upsilon)^{2-s}, \tag{2.49}$$

we have that

$$r_*(\upsilon) \le \Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla \upsilon\|_2^2}\Big)^{1/(2-q)}. \tag{2.50}$$

If $\upsilon \in G_2$ and the condition (2.45) is satisfied, then (2.4) has two positive solutions $r_1(\upsilon)$, $r_2(\upsilon)$ with $r_1(\upsilon) < r_*(\upsilon) < r_2(\upsilon)$. Define $r(\upsilon) := r_2(\upsilon)$. Since $Q_r(r, \upsilon) < 0$ for all $r > r_*(\upsilon)$, by the implicit function theorem, $r \in C^1(G_2)$. We will assume that the set

$$G_4 := \{\upsilon \in G_1 : \|\nabla \upsilon\|_p^p \le \frac{p}{s}\frac{2-s}{2-p}\big(\frac{s}{q}\frac{2-q}{2-s}\bar{r}(\upsilon)^{q-s}A(\upsilon) - B(\upsilon)\big)\bar{r}(\upsilon)^{s-p}\} \tag{2.51}$$

is not empty. Thus,

$$\bar{r}(\upsilon) > \Big(\frac{q}{s}\frac{2-s}{2-q}\frac{B(\upsilon)}{A(\upsilon)}\Big)^{1/(q-s)}.$$

We will show that $G_4 \subseteq G_2$. Indeed, let $\upsilon \in G_5$ and assume first that $B(\upsilon) > 0$. Then, since $\frac{p}{s}$, $\frac{2-s}{2-p}$ and $\frac{s}{q}$ are less than 1, (2.42), (2.44), (2.46) and (2.51) imply that

$$\begin{aligned}
\|\nabla \upsilon\|_p^p &< \Big(\frac{s}{q}\frac{2-q}{2-s}\bar{r}(\upsilon)^{q-s}A(\upsilon) - B(\upsilon)\Big)\bar{r}(\upsilon)^{s-p} \\
&< \Big(\frac{2-q}{2-s}\bar{r}(\upsilon)^{q-s}A(\upsilon) - B(\upsilon)\Big)\bar{r}(\upsilon)^{s-p} \\
&= F(\bar{r}(\upsilon), \upsilon)\bar{r}(\upsilon)^{s-p} = Q(\bar{r}(\upsilon), \upsilon) \\
&< Q(r_*(\upsilon), \upsilon),
\end{aligned}$$

and so $\upsilon \in G_2$. Next, let $\upsilon \in G_4$ and assume $B(\upsilon) = 0$. Then, from (2.46),

$$\|\nabla \upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}\bar{r}(\upsilon)^{q-p}A(\upsilon) < \frac{2-q}{2-p} r_*(\upsilon)^{q-p}A(\upsilon) = Q(r_*(\upsilon), \upsilon),$$

which shows that $\upsilon \in G_2$. Notice also that $G_4 \cap S^1 \neq \emptyset$. Since $\bar{r}(\upsilon) < r_*(\upsilon) < r(\upsilon)$ for any $\upsilon \in G_4$, we get

$$\|\nabla \upsilon\|_p^p \le \frac{p}{s}\frac{2-s}{2-p}\Big(\frac{s}{q}\frac{2-q}{2-s} r(\upsilon)^{q-s}A(\upsilon) - B(\upsilon)\Big) r(\upsilon)^{s-p},$$

which, in view of (2.8), implies that $\hat{\Phi}(\upsilon) < 0$ for $\upsilon \in G_4$. On the other hand, if $\upsilon \in G_2 \cap S^1$, then (2.10) implies that

$$r(\upsilon) \le \Big(\frac{A(\upsilon)}{\|\nabla \upsilon\|_2^2}\Big)^{1/(2-q)}, \tag{2.52}$$

and so $r(\cdot)$ is bounded on $G_2 \cap S^1$. Therefore $\hat{\Phi}(\upsilon)$ is bounded on $G_2 \cap S^1$. Let

$$M := \inf_{\upsilon \in G_2 \cap S^1} \hat{\Phi}(\upsilon) < 0.$$

Suppose that $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a minimizing sequence for $\hat{\Phi}(\cdot)$ in $\widetilde{G}_2 \cap S^1$. Then, there exist $\tilde{\upsilon} \in E$ such that, at least for a subsequence, $A(\upsilon_n) \to A(\tilde{\upsilon})$, $B(\upsilon_n) \to B(\tilde{\upsilon})$. We must have $\tilde{\upsilon} \neq 0$ because, otherwise, $0 = \Phi(0) \le \liminf_{n\to\infty} \Phi(r(\upsilon_n)\upsilon_n) = M$, a contradiction. Since $\{r(\upsilon_n)\}_{n\in\mathbb{N}}$ is bounded we get $r(\upsilon_n) \to \tilde{r}$, and $r_*(\upsilon_n) \to \tilde{r}_*$. On the other hand, $\tilde{r} > 0$ because $M = \liminf_{n\to\infty} \hat{\Phi}(\upsilon_n) < 0$. If we assume that $A(\tilde{\upsilon}) = 0$, then, by (2.52), we should have $\tilde{r} = 0$, a contradiction. Thus, $\tilde{\upsilon} \in G_1$. Also, by (2.45) and (2.46), we have

$$\tilde{r} \ge \tilde{r}_* \ge \hat{r}(\tilde{\upsilon}) := (\frac{2-s}{2-q}\frac{B(\tilde{\upsilon})}{A(\tilde{\upsilon})})^{1/(q-s)}. \tag{2.53}$$

We will show that $\tilde{\upsilon} \in G_2$. Indeed, if not, then, as in proof of the previous Theorem, $\tilde{r} = \tilde{r}_* = r_*(\tilde{\upsilon})$ where $r_*(\tilde{\upsilon})$ is the point of global maximum of $Q(\cdot, \tilde{\upsilon})$ which satisfies

$$(q-p)A(\tilde{\upsilon})r_*(\tilde{\upsilon})^{q-s} = (s-p)B(\tilde{\upsilon}) + (2-p)\|\nabla\tilde{\upsilon}\|_2^2 r_*(\tilde{\upsilon})^{2-s}.$$

Consequently, by passing to the limit in (2.49), where we have replaced υ by υ_n, $n \in \mathbb{N}$, we get $\|\nabla \upsilon_n\|_2^2 \to \|\nabla\tilde{\upsilon}\|_2^2$, where

$$(q-p)A(\tilde{\upsilon})\tilde{r}^{q-s} - (s-p)B(\tilde{\upsilon}) = (2-p)\|\nabla\tilde{\upsilon}\|_2^2 \tilde{r}^{2-s}. \tag{2.54}$$

This, however, leads to a contradiction since, (2.5), (2.54) and (2.53),

$$M = \lim_{n\to\infty} \hat{\Phi}(\upsilon_n) = \frac{(q-p)(2-q)}{2pq}\Big(\tilde{r}^{q-s}A(\tilde{\upsilon}) - \frac{q}{s}\frac{s-p}{q-p}\frac{2-s}{2-q}\frac{B(\tilde{\upsilon})}{A(\tilde{\upsilon})}\Big)\tilde{r}^s A(\tilde{\upsilon}) > 0.$$

Therefore, $\tilde{\upsilon} \in G_2$ as claimed. A similar reasoning as in Case 2 shows that $\tilde{r} = r(\tilde{\upsilon})$. Finally, by passing to the limit in (2.17) we conclude that $\tilde{\upsilon} \in S^1$ and $\hat{\Phi}(\tilde{\upsilon}) = M$. Lemma 2.1 implies that $u := r(\tilde{\upsilon})\tilde{\upsilon} \ge 0$ is a solution to (1.5)-(1.6). Therefore, we have the following result.

Theorem 2.5. *Assume that* (H0)-(H2) *are satisfied,* $p < s < q < 2$ *and the set* G_4 *defined in* (2.51) *is not empty. Then* (1.5) -(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Remark 2.6. We will now give some conditions which guarantee that $G_4 \neq \emptyset$. Suppose that $\operatorname{supp} a^+ \subseteq \operatorname{supp} b$. Then there exists $\upsilon \in S^1$ such that $B(\upsilon) > 0$.

From (2.43) we obtain

$$
\begin{aligned}
&\frac{p}{q}\frac{2-q}{2-p}\bar{r}(\upsilon)^{q-p}A(\upsilon)-\frac{p}{s}\frac{2-s}{2-p}B(\upsilon)\bar{r}(\upsilon)^{s-p}\\
&=\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)-\frac{p}{s}\frac{2-s}{2-p}B(\upsilon)\Big(\frac{q-s}{2-s}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-q)}\\
&=\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{q-p}{2-q}+1}\\
&\quad-\frac{p}{s}\frac{2-s}{2-p}B(\upsilon)\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-q)}A(\upsilon)^{\frac{s-p}{2-q}}.
\end{aligned}
$$

If we assume that

$$
\begin{aligned}
&\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{q-p}{2-q}+1}\\
&-\frac{p}{s}\frac{2-s}{2-p}B(\upsilon)\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-q)}A(\upsilon)^{(s-p)/(2-q)}>\|\nabla\upsilon\|_p^p,
\end{aligned}
\tag{2.55}
$$

then $\upsilon\in G_4$. It is easy to see that if $a^+(\cdot)$ is large enough then (2.55) is true.

On the other hand, suppose that $(\operatorname{supp}a^+)\setminus\operatorname{supp}b))^o\neq\emptyset$. Then there exists $\upsilon\in G_1$ with $B(\upsilon)=0$. From (2.43) we obtain

$$
\begin{aligned}
\frac{p}{q}\frac{2-q}{2-p}\bar{r}(\upsilon)^{q-p}A(\upsilon)&=\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)\\
&=\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{q-p}{2-q}+1}.
\end{aligned}
$$

If we assume that

$$
\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-s}{2-s}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{q-p}{2-q}+1}>\|\nabla\upsilon\|_p^p,\tag{2.56}
$$

then $\upsilon\in G_4$. Note that if $a^+(\cdot)$ is large enough then (2.56) holds.

Case 4: $p<2<q<s$. In this case we make the additional assumption:

(H3) $b(x)\geq bo>0$ a.e. in Ω.

Let

$$
Q(r,\upsilon):=r^{q-2}A(\upsilon)-r^{s-2}B(\upsilon)-r^{p-2}\|\nabla\upsilon\|_p^p.\tag{2.57}
$$

Then (2.4) is equivalent to

$$
Q(r,\upsilon)=\|\nabla\upsilon\|_2^2.\tag{2.58}
$$

For every $\upsilon\in G_1$ the function $Q(\cdot,\upsilon)$ has a unique critical point $r_*:=r_*(\upsilon)$ which corresponds to a global maximum with

$$
(q-2)r_*^{q-p}A(\upsilon)+(2-p)\|\nabla\upsilon\|_p^p=(s-2)r_*^{s-p}B(\upsilon).\tag{2.59}
$$

Thus,

$$
r_*(\upsilon)\geq\big(\frac{q-2}{s-2}\frac{A(\upsilon)}{B(\upsilon)}\big)^{\frac{1}{s-q}}.\tag{2.60}
$$

On combining (2.57) with (2.59) we get

$$
\begin{aligned}
Q(r_*(\upsilon),\upsilon)&=\frac{q-p}{2-p}r_*(\upsilon)^{q-2}A(\upsilon)-\frac{s-p}{2-p}r_*(\upsilon)^{s-2}B(\upsilon) &(2.61)\\
&=\frac{s-q}{s-2}r_*(\upsilon)^{q-2}A(\upsilon)-\frac{s-p}{s-2}r_*(\upsilon)^{p-2}\|\nabla\upsilon\|_p^p. &(2.62)
\end{aligned}
$$

Let

$$\tilde{G}_2 := \{v \in G_1 : \|\nabla v\|_2^2 < Q(r_*(v), v)\}.$$

Clearly, if $v \in \tilde{G}_2$, then (2.4) has exactly two positive solutions $r_1(v)$ and $r_2(v)$ with $r_1(v) < r_*(v) < r_2(v)$. As before, let $r := r_2(v)$. Since

$$r^{2-q+1}Q_r(r, v) = (q-2)A(v) - (s-2)r^{s-q}B(v) - (p-2)r^{p-q}\|\nabla v\|_p^p,$$

in view of (2.59), we obtain

$$r^{2-q+1}Q_r(r, v) = (s-2)B(v)(r_*^{s-q} - r^{s-q}) + (2-p)\|\nabla v\|_p^p(r^{p-q} - r_*^{p-q}) < 0,$$

which implies that $r(\cdot)$ is continuously differentiable. We now define

$$G_5 := \{v \in G_1 : \|\nabla v\|_2^2 < \frac{2}{q}\frac{s-q}{s-2}A(v)r_*(v)^{q-2} - \frac{2}{p}\frac{s-p}{s-2}\|\nabla v\|_p^p r_*(v)^{p-2}\}, \tag{2.63}$$

and assume that $G_5 \neq \emptyset$. Since $\frac{2}{q} < 1$ and $\frac{2}{p} > 1$ we see that $G_5 \subseteq \tilde{G}_2$, and so $\tilde{G}_2 \neq \emptyset$ as well. Furthermore, $G_5 \cap S^1 \neq \emptyset$ because r satisfies (2.35). If $v \in G_5$, then by (2.63),

$$\|\nabla v\|_2^2 < \frac{2}{q}\frac{s-q}{s-2}A(v)r(v)^{q-2} - \frac{2}{p}\frac{s-p}{s-2}\|\nabla v\|_p^p r(v)^{p-2}. \tag{2.64}$$

On the other hand, (2.7) and (2.64) show that

$$r^p(\frac{1}{p} - \frac{1}{s})\|\nabla v\|_p^p + r^2(\frac{1}{2} - \frac{1}{s})\|\nabla v\|_2^2 + r^q(\frac{1}{s} - \frac{1}{q})A(v) < 0,$$

and so $\hat{\Phi}(v) < 0$. We claim that $r(\cdot)$ is bounded above on $\tilde{G}_2 \cap S^1$. Indeed, from (2.10) we have

$$r(v) \leq (\frac{A(v)}{B(v)})^{1/(s-q)}, \tag{2.65}$$

while hypothesis (H3) implies

$$A(v) \leq cB(v)^{q/s} \tag{2.66}$$

for every $v \in E$ and some $c > 0$. At the same time if $v \in \tilde{G}_2$, then for some $\theta > 0$,

$$\theta < \|\nabla v\|_2^2 < \frac{q-p}{2-p}r_*(v)^{q-2}A(v) < \frac{q-p}{2-p}r(v)^{q-2}A(v). \tag{2.67}$$

From (2.65) and (2.67) we deduce

$$\theta < \frac{q-p}{2-p}\Big(\frac{A(v)}{B(v)}\Big)^{(q-2)/(s-q)}A(v). \tag{2.68}$$

Next, by using (2.66) and (2.68), we have

$$\theta < \frac{q-p}{2-p}c^{\frac{s-2}{s-q}}B(v)^{\frac{2}{s}}, \tag{2.69}$$

and so $B(\cdot)$ is bounded away from 0. The claim is proved by reverting to (2.65). Accordingly, $\hat{\Phi}(v)$ is also bounded on $\tilde{G}_2 \cap S^1$. Consider the variational problem

$$M = \inf_{\tilde{G}_2 \cap S^1} \hat{\Phi}(v) < 0$$

and assume that $\{v_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $\tilde{G}_2 \cap S^1$. Since $\{v_n\}_{n\in\mathbb{N}}$ is bounded, there exists $\tilde{v} \in E$ such that, at least for a subsequence, $A(v_n) \to A(\tilde{v}) \geq 0$ and $B(v_n) \to B(\tilde{v})$. By (2.69), $\tilde{v} \neq 0$. We may also assume that $r_*(v_n) \to \tilde{r}_*$ and $r(v_n) \to \tilde{r}$. Clearly, $\tilde{r} > 0$ since $M = \liminf_{n\to\infty} \hat{\Phi}(v_n) < 0$. On the other hand, $A(\tilde{\upsilon}) > 0$ because, otherwise, this would imply $\tilde{r} = 0$. Furthermore $\tilde{r}_* > 0$ by (2.60). We claim that $\tilde{\upsilon} \in G_5$. Since

$$\begin{aligned} \|\nabla \tilde{\upsilon}\|_2^2 &\le \limsup_{n\to\infty} \|\nabla \tilde{\upsilon}_n\|_2^2 \le \limsup_{n\to\infty} Q(r_*(\upsilon_n), \upsilon_n) \\ &\le \limsup_{n\to\infty} \{r_*(\upsilon_n)^{q-2} A(\upsilon_n) - r_*(\upsilon_n)^{s-2} B(\upsilon_n)\} \\ &\quad - \liminf_{n\to\infty} r_*(\upsilon_n)^{p-2} \|\nabla \upsilon_n\|_p^p \\ &\le \tilde{r}_*^{q-2} A(\tilde{\upsilon}) - \tilde{r}_*^{s-2} B(\tilde{\upsilon}) - \tilde{r}_*^{p-2} \|\nabla \tilde{\upsilon}\|_2^2 = Q(\tilde{r}_*, \tilde{\upsilon}), \end{aligned} \tag{2.70}$$

we see that

$$\|\nabla \tilde{\upsilon}\|_2^2 \le Q(r_*(\tilde{\upsilon}), \tilde{\upsilon}). \tag{2.71}$$

We shall show that strict inequality holds. Indeed, let us suppose

$$\|\nabla \tilde{\upsilon}\|_2^2 = Q(r_*(\tilde{\upsilon}), \tilde{\upsilon}). \tag{2.72}$$

Since $\tilde{r} > 0$, by applying (2.58) for $\upsilon = \upsilon_n$ and passing to the limit, we also obtain

$$\begin{aligned} \|\nabla \tilde{\upsilon}\|_2^2 &\le \limsup_{n\to\infty} \|\nabla \tilde{\upsilon}_n\|_2^2 \le \limsup_{n\to\infty} Q(r(\upsilon_n), \upsilon_n) \\ &\le \limsup_{n\to\infty} \{r(\upsilon_n)^{q-2} A(\upsilon_n) - r(\upsilon_n)^{s-2} B(\upsilon_n)\} \\ &\quad - \liminf_{n\to\infty} r(\upsilon_n)^{p-2} \|\nabla \upsilon_n\|_p^p \\ &\le \tilde{r}^{q-2} A(\tilde{\upsilon}) - \tilde{r}^{s-2} B(\tilde{\upsilon}) - \tilde{r}^{p-2} \|\nabla \tilde{\upsilon}\|_p^p = Q(\tilde{r}, \tilde{\upsilon}). \end{aligned} \tag{2.73}$$

Consequently, by (2.70), (2.72) and (2.73), we should have $\tilde{r} = \tilde{r}_* = \tilde{r}_*(\tilde{\upsilon})$. On the other hand, by replacing υ by υ_n in (2.59) and passing to the limit we obtain

$$(q-2) r_*(\tilde{\upsilon})^{q-p} A(\tilde{\upsilon}) + (2-p) \|\nabla \tilde{\upsilon}\|_p^p \le (s-2) r_*(\tilde{\upsilon})^{s-p} B(\tilde{\upsilon}).$$

Since $r_*(\tilde{\upsilon})$ satisfies

$$(q-2) r_*(\tilde{\upsilon})^{q-p} A(\tilde{\upsilon}) + (2-p) \|\nabla \tilde{\upsilon}\|_p^p = (s-2) r_*(\tilde{\upsilon})^{s-p} B(\tilde{\upsilon}),$$

we deduce that $\|\nabla \upsilon_n\|_p^p \to \|\nabla \tilde{\upsilon}\|_p^p$ where, by (2.59),

$$\frac{q-2}{2s} \tilde{r}^q A(\tilde{\upsilon}) + \frac{2-p}{2s} \tilde{r}^p \|\nabla \tilde{\upsilon}\|_p^p = \frac{s-2}{2s} \tilde{r}^s B(\tilde{\upsilon}). \tag{2.74}$$

Then, (2.8) and (2.74) yield

$$M = \lim_{n\to\infty} \hat{\Phi}(\upsilon_n) = \frac{(2-p)(s-p)}{2ps} \tilde{r}^p \|\nabla \tilde{\upsilon}\|_p^p + \frac{(q-2)(s-q)}{2ps} \tilde{r}^q A(\tilde{\upsilon}) > 0,$$

which is a contradiction. Therefore, $\tilde{\upsilon} \in \tilde{G}_2$ as claimed. A similar reasoning as in Case 2 shows that $\tilde{r} \le r(\tilde{\upsilon})$. If we assume that $\tilde{r} < r(\tilde{\upsilon})$, then, since the function

$$\psi(z) := \frac{\partial}{\partial z} \Phi(z\tilde{\upsilon}) = z\{\|\nabla \tilde{\upsilon}\|_2^2 - Q(z, \tilde{\upsilon})\}, \tag{2.75}$$

is strictly negative for $z \in (\tilde{r}, r(\tilde{\upsilon}))$, by (2.35) we obtain

$$M = \liminf_{n\to\infty} \Phi(r(\upsilon_n)\upsilon_n) \ge \Phi(\tilde{r}\tilde{\upsilon}) > \Phi(r(\tilde{\upsilon})\tilde{\upsilon}) = \Phi(r(t\tilde{\upsilon})t\tilde{\upsilon}) = \hat{\Phi}(t\tilde{\upsilon}),$$

contradicting the definition of M. Consequently, $\tilde{\upsilon} \in S^1$ and $\hat{\Phi}(\tilde{\upsilon}) = M$. Therefore $u := r(\tilde{\upsilon})\tilde{\upsilon}$ is a solution of (1.5)-(1.6).

Thus, we have proved the following result.

Theorem 2.7. *Assume that conditions* (H0)–(H3) *are satisfied,* $p < 2 < q < s$ *and the set* G_5 *defined in* (2.63) *is not empty. Then* (1.5)*)*-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Remark 2.8. We will present a condition which guarantees that $G_5 \neq \emptyset$. From (2.59),

$$\Big(\frac{q-2}{s-2}\frac{A(\upsilon)}{B(\upsilon)}\Big)^{1/(s-q)} \le r_*(\upsilon),$$

and so

$$\begin{aligned}
&\frac{2}{q}\frac{s-q}{s-2}A(\upsilon)r_*(\upsilon)^{q-2} - \frac{2}{p}\frac{s-p}{s-2}\|\nabla\upsilon\|_p^p r_*(\upsilon)^{p-2}\\
&\ge \frac{2}{q}\frac{s-q}{s-2}A(\upsilon)\Big(\frac{q-2}{s-2}\frac{A(\upsilon)}{B(\upsilon)}\Big)^{(q-2)/(s-q)} - \frac{2}{p}\frac{s-p}{s-2}\|\nabla\upsilon\|_p^p\Big(\frac{q-2}{s-2}\frac{A(\upsilon)}{B(\upsilon)}\Big)^{\frac{p-2}{s-q}}\\
&= \frac{2}{q}\frac{s-q}{s-2}\Big(\frac{q-2}{s-2}\Big)^{(q-2)/(s-q)}A(\upsilon)^{(s-2)/(s-q)}B(\upsilon)^{(2-q)/(s-q)}\\
&\quad - \frac{2}{p}\frac{s-p}{s-2}\Big(\frac{q-2}{s-2}\Big)^{\frac{p-2}{s-q}}\|\nabla\upsilon\|_p^p B(\upsilon)^{\frac{2-p}{s-q}}A(\upsilon)^{\frac{p-2}{s-q}}.
\end{aligned}$$

Since $\frac{s-2}{s-q} > \frac{p-2}{s-q}$, $G_5 \neq \emptyset$ for $a^+(\cdot)$ large enough.

Case 5: $s < p < q < 2$. In this case we define

$$Q(r,\upsilon) := r^{q-p}A(\upsilon) - r^{s-p}B(\upsilon) - r^{2-p}\|\nabla\upsilon\|_2^2.$$

For $\upsilon \in G_1$, $Q(\cdot,\upsilon)$ has a unique critical point $r_* := r_*(\upsilon)$ which corresponds to a global maximum and satisfies

$$(q-p)r_*^{q-s}A(\upsilon) + (p-s)B(\upsilon) = (2-p)r_*^{2-s}\|\nabla\upsilon\|_2^2 \tag{2.76}$$

and

$$Q(r_*(\upsilon),\upsilon) = \frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon) - \frac{2-s}{2-p}r_*(\upsilon)^{s-p}B(\upsilon).$$

From (2.76) we get

$$r_*(\upsilon) \ge \Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)}. \tag{2.77}$$

Clearly, if $\upsilon \in G_2$ then (2.16) has exactly two positive solutions $r_1(\upsilon)$, $r_2(\upsilon)$ with $r_1(\upsilon) < r_*(\upsilon) < r_2(\upsilon)$. We set $r := r(\upsilon)$ to be the greater solution. We have

$$r^{p-1}Q_r(r,\upsilon) = (q-p)A(\upsilon)r^{q-2} - (s-p)r^{s-2}B(\upsilon) - (2-p)\|\nabla\upsilon\|_2^2,$$

which, on account of (2.76), yields

$$r^{p-1}Q_r(r,\upsilon) = (q-p)A(\upsilon)(r^{q-2} - r_*^{q-2}) + (p-s)B(\upsilon)(r^{s-2} - r_*^{s-2}) < 0.$$

Therefore, $r(\cdot)$ is continuously differentiable. Let

$$G_6 := \{\upsilon \in G_1 : \|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon) - \frac{p}{s}\frac{2-s}{2-p}r_*(\upsilon)^{s-p}B(\upsilon)\} \tag{2.78}$$

and assume that $G_6 \neq \emptyset$. We immediately see that $G_6 \subseteq G_2$, since $\frac{p}{q} < 1$ and so $G_2 \neq \emptyset$ as well. Moreover, $G_6 \cap S^1 \neq \emptyset$ and $\hat{\Phi}(\upsilon) < 0$ for any $\upsilon \in G_6$. Indeed, since $r(\upsilon) > r_*(\upsilon)$, by (2.78) we get

$$\|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{2-q}{2-p}r(\upsilon)^{q-p}A(\upsilon) - \frac{p}{s}\frac{2-s}{2-p}r(\upsilon)^{s-p}B(\upsilon). \tag{2.79}$$

At the same time, (2.8) and (2.79) yield

$$r^q \frac{2-p}{2p}\|\nabla \upsilon\|_p^p + r^q \frac{q-2}{2q} A(\upsilon) + r^s \frac{2-s}{2s} B(\upsilon) < 0,$$

which proves the assertion. Next, because $2 > q$, (2.26) shows that $r(\cdot)$ is bounded above on $G_2 \cap S^1$. Consequently, $\hat{\Phi}(\upsilon)$ is also bounded on $G_2 \cap S^1$. Consider the variational problem

$$M = \inf_{\upsilon \in G_2 \cap S^1} \hat{\Phi}(\upsilon) < 0.$$

If $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_2 \cap S^1$ then, there exist $\tilde{\upsilon} \in E$ such that, at least for a subsequence, $A(\upsilon_n) \to A(\tilde{\upsilon}) \geq 0$ and $B(\upsilon_n) \to B(\tilde{\upsilon}) \geq 0$, while by (2.9) we get

$$0 < \|\nabla \tilde{\upsilon}\|_2^2 \leq \liminf \|\nabla \upsilon_n\|_2^2 \leq 1.$$

Since $r(\cdot)$ is bounded on $G_2 \cap S^1$ we may assume that $r_*(\upsilon_n) \to \tilde{r}_*$ and $r(\upsilon_n) \to \tilde{r}$. Again $\tilde{r} > 0$ because, otherwise, $M = \liminf_{n\to\infty} \hat{\Phi}(\upsilon_n) = 0$, a contradiction. We also have that $A(\tilde{\upsilon}) > 0$, because, if we assume the contrary, (2.17) yields

$$r(\upsilon_n)^{2-q}\|\nabla \upsilon_n\|_2^2 \leq A(\upsilon_n),$$

and by passing to the limit,

$$\tilde{r}^{2-q}\|\nabla \tilde{\upsilon}\|_2^2 \leq \liminf_{n\to\infty}(r(\upsilon_n)^{2-q}\|\nabla \upsilon_n\|_2^2) \leq \lim_{n\to\infty} A(\upsilon_n) = A(\tilde{\upsilon}).$$

Thus, $\tilde{r} = 0$, a contradiction. Furthermore $\tilde{r}_* > 0$ due to (2.77). We claim that $\tilde{\upsilon} \in G_6$. Indeed, if not, then, by applying the same arguments as in the proof of Case 2, we would have $\tilde{r} = \tilde{r}_* = r_*(\tilde{\upsilon})$, while, along a subsequence, $\|\nabla \upsilon_n\|_2^2 \to \|\nabla \tilde{\upsilon}\|_2^2$ where, by (2.76)

$$\frac{q-p}{2p}\tilde{r}^s A(\tilde{\upsilon}) + \frac{p-s}{2p}\tilde{r}^s B(\tilde{\upsilon}) = \frac{2-p}{2p}\tilde{r}^2\|\nabla \tilde{\upsilon}\|_2^2. \tag{2.80}$$

Then (2.5) and (2.80) yield

$$M = \lim_{n\to\infty} \hat{\Phi}(\upsilon_n) = \frac{(q-p)(2-q)}{2pq}\tilde{r}^q A(\tilde{\upsilon}) + \frac{(p-s)(2-s)}{2ps}\tilde{r}^s B(\tilde{\upsilon}) > 0.$$

Therefore, $\tilde{\upsilon} \in G_2$ as claimed. A similar reasoning as in Case 2 shows that $\tilde{r} = r(\tilde{\upsilon})$. Finally, by passing to the limit in (2.17) we rederive (2.15) which implies that $\tilde{\upsilon} \in S^1$ and $\hat{\Phi}(\tilde{\upsilon}) = M$. Thus $u := r(\tilde{\upsilon})\tilde{\upsilon}$ is a solution to (1.5)-(1.6).

Therefore we have proved the following result.

Theorem 2.9. *Assume that conditions* (H0)–(H2) *are satisfied,* $s < p < q < 2$ *and the set* G_6 *defined in* (2.78) *is not empty. Then* (1.5)-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Remark 2.10. We will give some conditions which guarantee that $G_6 \neq \emptyset$. Suppose that $\operatorname{supp} a^+) \subseteq \operatorname{supp} b)$. Then there exists $\upsilon \in S^1$ such that $B(\upsilon) > 0$. From ((2.76)

$$\Big(\frac{p-s}{2-p}\frac{B(\upsilon)}{\|\nabla \upsilon\|_2^2}\Big)^{1/(2-s)} \leq r_*(\upsilon), \tag{2.81}$$

and so, in view of (2.81),

$$\begin{aligned}
&\frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon) - \frac{p}{s}\frac{2-s}{2-p}r_*(\upsilon)^{s-p}B(\upsilon)\\
&\geq \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-s)}A(\upsilon) - \frac{p}{s}\frac{2-s}{2-p}\Big(\frac{p-s}{2-p}\frac{B(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-s)}B(\upsilon)\\
&\geq \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-s)}A(\upsilon)^{\frac{2-p}{2-s}+1}\\
&\quad - \frac{p}{s}\frac{2-s}{2-p}\Big(\frac{p-s}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-s)}B(\upsilon)^{(2-p)/(2-s)}.
\end{aligned}$$

Note that if

$$\begin{aligned}
&\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-s)}A(\upsilon)^{\frac{2-p}{2-s}+1}\\
&- \frac{p}{s}\frac{2-s}{2-p}\Big(\frac{p-s}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(s-p)/(2-s)}B(\upsilon)^{\frac{2-p}{2-s}} > \|\nabla\upsilon\|_2^2,
\end{aligned} \tag{2.82}$$

then $G_6 \neq \emptyset$. It is clear that if $a^+(\cdot)$ is large compared to $b(\cdot)$ then (2.82) is satisfied.

Suppose now that $(\operatorname{supp} a^+)\setminus \operatorname{supp} b))^o \neq \emptyset$. Then there exists $\upsilon \in S^1$ with $B(\upsilon) = 0$. From (2.76) we have

$$\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)} = r_*(\upsilon), \tag{2.83}$$

and so, in view of (2.83),

$$\begin{aligned}
\frac{p}{q}\frac{2-q}{2-p}r_*(\upsilon)^{q-p}A(\upsilon) &= \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)\\
&= \frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{2-p}{2-q}}.
\end{aligned}$$

If we assume that

$$\frac{p}{q}\frac{2-q}{2-p}\Big(\frac{q-p}{2-p}\frac{1}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon)^{\frac{2-p}{2-q}} > \|\nabla\upsilon\|_2^2,$$

we have

$$A(\upsilon)^{\frac{2-p}{2-q}} > \frac{q}{p}\frac{2-p}{2-q}\Big(\frac{q-p}{2-p}\Big)^{(p-q)/(2-q)}\|\nabla\upsilon\|_2^2, \tag{2.84}$$

and so if $a^+(\cdot)$ large enough the condition (2.84) is valid implying that $G_6 \neq \emptyset$.

Case 6: $s < q < p < 2$. In this case we assume that the following condition holds:

(H4) $V := (\operatorname{supp} a^+\setminus \operatorname{supp} b)^o \neq \emptyset$.

We define

$$Q(r,\upsilon) := r^{q-p}A(\upsilon) - r^{s-p}B(\upsilon) - r^{2-p}\|\nabla\upsilon\|_2^2. \tag{2.85}$$

Let $\upsilon \in G_1$. If $B(\upsilon) = 0$, the equation (2.10) has a unique solution $r(\upsilon) > 0$, while if $B(\upsilon) > 0$, the function $Q(\cdot,\upsilon)$ has a unique critical point $r_* := r_*(\upsilon)$ which corresponds to a global maximum and satisfies

$$(p-s)B(\upsilon) = (p-q)r_*^{q-s}A(\upsilon) + (2-p)r_*^{2-s}\|\nabla\upsilon\|_2^2. \tag{2.86}$$

Clearly, if $\upsilon \in G_2$, then (2.4) has exactly two positive solutions $r_1(\upsilon)$ and $r_2(\upsilon)$ with $r_1(\upsilon) < r_*(\upsilon) < r_2(\upsilon)$. Let $r := r(\upsilon)$ be the unique solution of (2.4) in case $B(\upsilon) = 0$ or the greater solution r_2 in case $B(\upsilon) > 0$. Note that, if $B(\upsilon) > 0$ then

$$r^{p-s+1}Q_r(r,\upsilon) = (q-p)A(\upsilon)r^{q-s} - (s-p)B(\upsilon) - (2-p)r^{2-s}\|\nabla\upsilon\|_2^2$$

and so, in view of (2.86), we obtain

$$r^{p-s+1}Q_r(r,\upsilon) = (p-q)A(\upsilon)(r_*^{q-s} - r^{q-s}) - (p-2)\|\nabla\upsilon\|_2^2(r_*^{2-s} - r^{2-s}) < 0,$$

while if $B(\upsilon) = 0$, then

$$r^{p+1}Q_r(r,\upsilon) = (q-p)A(\upsilon)r^q - (2-p)\|\nabla\upsilon\|_2^2 r^2 < 0.$$

Thus $r(\cdot)$ is continuously differentiable by the implicit function theorem. We now define

$$G_7 = \{\upsilon \in G_1 : B(\upsilon) = 0\} \cup \{\upsilon \in G_1 : B(\upsilon) > 0 \text{ and } \|\nabla\upsilon\|_p^p < Q(r_*(\upsilon),\upsilon)\}.$$

In view of (H1) and (H4), we see that $G_7 \neq \emptyset$ since for any $\upsilon \in E$ with $\operatorname{supp}\upsilon \subseteq V$ there holds $A(\upsilon) > 0$ and $B(\upsilon) = 0$. We claim that G_7 is open. Indeed, let $\hat\upsilon \in G_7$ and assume that there exists a sequence $\{\upsilon_n\}_{n\in\mathbb{N}} \subseteq E\backslash G_7$ with $\upsilon_n \to \hat\upsilon$ strongly in E. Suppose, without loss of generality, that $B(\hat\upsilon) = 0$ while $B(\hat\upsilon) > 0$ for every $n \in \mathbb{N}$. Therefore,

$$\|\nabla\upsilon_n\|_p^p \geq Q(r_*(\upsilon_n),\upsilon_n) \text{ for every } n \in \mathbb{N}. \tag{2.87}$$

Since $A(\hat\upsilon) > 0$, on account of (2.86), $r_*(\upsilon_n) \to 0$. Combining (2.86) and (2.85) we obtain

$$Q(r_*(\upsilon),\upsilon) = \frac{q-s}{p-s}r_*(\upsilon)^{q-p}A(\upsilon) - \frac{2-s}{p-s}r_*(\upsilon)^{2-p}\|\nabla\upsilon\|_2^2,$$

and so $\lim_{n\to\infty} Q(r_*(\upsilon_n),\upsilon_n) = +\infty$, contradicting (2.87). It follows from (2.4) that $r(\cdot)$ is bounded and so $\hat\Phi(\cdot)$ is also bounded on $G_7 \cap S^1$. On account of (2.5) and (H4), $M < 0$.

Consider the variational problem

$$M = \inf_{\upsilon\in G_2\cap S^1} \hat\Phi(\upsilon) < 0$$

and assume that $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_7 \cap S^1$. Then there exists $\tilde\upsilon \in E$ so that $A(\upsilon_n) \to A(\tilde\upsilon) \geq 0$, $B(\upsilon_n) \to B(\tilde\upsilon) \geq 0$ and

$$0 \leq \|\nabla\tilde\upsilon\|_p^p \leq \liminf \|\nabla\upsilon_n\|_p^p \leq 1.$$

Furthermore, $r(\upsilon_n) \to \tilde r$ for a new subsequence. In particular, $\tilde r > 0$ because if $\tilde r = 0$ then, by (2.5)), $M = \lim_{n\to\infty}\hat\Phi(\upsilon_n) = 0$; a contradiction. We claim that $A(\tilde\upsilon) > 0$. Indeed, from (2.10) we have

$$\|\nabla\upsilon_n\|_p^p r(\upsilon_n)^{p-q} \leq A(\upsilon_n),$$

and by passing to the limit,

$$\|\nabla\tilde\upsilon\|_p^p r(\tilde\upsilon)^{p-q} \leq \liminf_{n\to\infty}\|\nabla\upsilon_n\|_p^p r(\upsilon_n)^{p-q} \leq \lim_{n\to\infty} A(\upsilon_n) = A(\tilde\upsilon).$$

Thus, if $A(\tilde\upsilon) = 0$ then $\tilde\upsilon = 0$. However, this leads to a contradiction because by (2.2), we should have $0 = \Phi(0) \leq \liminf_{n\to\infty}\Phi(r(\upsilon_n)\upsilon_n) = M$.

We shall show next that $\tilde\upsilon \in G_7$. Let us assume that $B(\tilde\upsilon) > 0$. Since

$$(p-s)B(\upsilon_n) = (p-q)r_*^{q-s}A(\upsilon_n) + (2-p)r_*^{2-s}\|\nabla\upsilon_n\|_2^2,$$

we see that the sequence $\{r_*(\upsilon_n)\}_{n\in\mathbb{N}}$ is bounded. Thus, up to a further subsequence, $r_*(\upsilon_n) \to \tilde{r}_* > 0$. As before, $\tilde{r} = \tilde{r}_* = r_*(\tilde{\upsilon})$. On the other hand, by passing to the limit in (2.86) we see that $\|\nabla\upsilon_n\|_2^2 \to \|\nabla\tilde{\upsilon}\|_2^2$ and

$$B(\tilde{\upsilon}) = \frac{p-q}{p-s} r_*^{q-s}(\tilde{\upsilon})A(\tilde{\upsilon}) + \frac{2-p}{p-s} r_*^{2-s}(\tilde{\upsilon})\|\nabla\tilde{\upsilon}\|_2^2.$$

Thus,

$$M = \lim_{n\to\infty} \hat{\Phi}(\upsilon_n) = \frac{(2-s)(2-p)}{2ps}\tilde{r}^2\|\nabla\tilde{\upsilon}\|_2^2 + \tilde{r}^q A(\tilde{\upsilon})\frac{(q-s)(p-q)}{psq} > 0,$$

which is a contradiction. Therefore, $\tilde{\upsilon} \in G_7$ as claimed. On the other hand, if $B(\tilde{\upsilon}) = 0$ then it is obvious that $\tilde{\upsilon} \in G_7$. Working as in Case 2 we are lead to the following result.

Theorem 2.11. *Assume that conditions* (H0)-(H2), (H4) *are satisfied and* $s < q < p < 2$. *Then* (1.5)-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Case 7: $p < q < s < 2$. In this case we define

$$Q(r,\upsilon) := r^{q-p}A(\upsilon) - r^{s-p}B(\upsilon) - r^{2-p}\|\nabla\upsilon\|_2^2.$$

We see that for $\upsilon \in G_1$ the function $Q(\cdot,\upsilon)$ has a unique critical point $r_* := r_*(\upsilon)$ satisfying

$$(q-p)A(\upsilon) = (s-p)r_*(\upsilon)^{s-q}B(\upsilon) + (2-p)r_*(\upsilon)^{2-q}\|\nabla\upsilon\|_2^2. \tag{2.88}$$

It is clear that (2.4) has two positive solutions $r_1(\upsilon)$, $r_2(\upsilon)$ with $r_1(\upsilon) < r_*(\upsilon) < r_2(\upsilon)$ for every $\upsilon \in G_2$. Let $r := r_2(\upsilon)$. Then

$$r^{p-q+1}Q_r(r,\upsilon) = (q-p)A(\upsilon) - (s-p)r^{s-q}B(\upsilon) - (2-p)r^{2-q}\|\nabla\upsilon\|_2^2,$$

which combined with (2.88), gives

$$r^{p-q+1}Q_r(r,\upsilon) = (2-p)\|\nabla\upsilon\|_2^2(r_*^{2-q} - r^{2-q}) + (s-p)B(\upsilon)(r_*^{s-q} - r^{s-q}) < 0.$$

Therefore, the implicit function theorem implies that $r(\cdot)$ is continuously differentiable. Assume that the set

$$G_8 := \{\upsilon \in G_1 : \|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{s-q}{s-p} r_*(\upsilon)^{q-p}A(\upsilon)\}$$

is not empty. Since $q > p$, and $r(\upsilon)^{q-p} > r_*(\upsilon)^{q-p}$, we see that $G_8 \subseteq G_2$ and so $G_2 \neq \emptyset$. If $\upsilon \in G_8$, then

$$\|\nabla\upsilon\|_p^p < \frac{p}{q}\frac{s-q}{s-p} r_*(\upsilon)^{q-p}A(\upsilon) < \frac{p}{q}\frac{s-q}{s-p} r(\upsilon)^{q-p}A(\upsilon)$$

and so

$$\frac{2-p}{p} r(\upsilon)^p\|\nabla\upsilon\|_p^p + \frac{q-2}{q} r(\upsilon)^q A(\upsilon) < 0. \tag{2.89}$$

Combining (2.89) with (2.7), we conclude that

$$\hat{\Phi}(\upsilon) < r^p(\frac{1}{p} - \frac{1}{s})\|\nabla\upsilon\|_p^p + r^q(\frac{1}{s} - \frac{1}{q})A(\upsilon) < 0.$$

On the other hand, if $\upsilon \in G_2 \cap S^1$, then (2.10) implies

$$r(\upsilon) \le \Big(\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)}$$

and so $r(\cdot)$ is bounded on $G_2 \cap S^1$. Consequently, $\hat{\Phi}(\upsilon)$ is also bounded on $G_2 \cap S^1$. Let

$$M := \inf_{\upsilon \in G_2 \cap S^1} \hat{\Phi}(\upsilon) < 0$$

and assume that $\{\upsilon_n\}_{n\in\mathbb{N}}$ is a minimizing sequence in $G_2 \cap S^1$. Then, there exist $\tilde{\upsilon} \in E$ such that, at least for a subsequence, $A(\upsilon_n) \to A(\tilde{\upsilon}) \geq 0$, $B(\upsilon_n) \to B(\tilde{\upsilon}) \geq 0$,

$$0 \leq \|\nabla\tilde{\upsilon}\|_2 \leq \liminf \|\nabla\upsilon_n\|_2 \leq 1,$$
$$0 \leq \|\nabla\tilde{\upsilon}\|_p \leq \liminf \|\nabla\upsilon_n\|_p \leq 1.$$

We must have $\tilde{\upsilon} \neq 0$ because, otherwise, $0 = \Phi(0) \leq \liminf_{n\to\infty} \Phi(r(\upsilon_n)\upsilon_n) = M$, a contradiction. Since $\{r(\upsilon_n)\}_{n\in\mathbb{N}}$ is bounded and $r_*(\upsilon_n) < r(\upsilon_n)$, $n \in \mathbb{N}$, we may assume that $r_*(\upsilon_n) \to \tilde{r}_*$ and $r(\upsilon_n) \to \tilde{r}$. Since $M = \liminf_{n\to\infty} \hat{\Phi}(\upsilon_n) < 0$ we obtain $\tilde{r} > 0$. We also have that $A(\tilde{\upsilon}) > 0$, because, if we assume the opposite, then by

$$\tilde{r}^{2-q}\|\nabla\tilde{\upsilon}\|_2^2 \leq \liminf_{n\to\infty}(r(\upsilon_n)^{2-q}\|\nabla\upsilon_n\|_2^2) \leq \lim_{n\to\infty} A(\upsilon_n) = A(\tilde{\upsilon})$$

we would get $\tilde{r} = 0$, a contradiction. Therefore, $\tilde{\upsilon} \in G_1$. Also, $\tilde{r}_* > 0$ by (2.88). We will show that $\tilde{\upsilon} \in G_2$.Working as in Case 2 we conclude that $\tilde{r} = \tilde{r}_* = \tilde{r}_*(\tilde{\upsilon})$. On the other hand, replacing υ by υ_n in (2.88) and passing to the limit leads to

$$(q-p)A(\tilde{\upsilon}) \geq (s-p)r_*(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) + (2-p)r_*(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2.$$

However, $r_*(\tilde{\upsilon})$ satisfies

$$(q-p)A(\tilde{\upsilon}) = (s-p)r_*(\tilde{\upsilon})^{s-q}B(\tilde{\upsilon}) + (2-p)r_*(\tilde{\upsilon})^{2-q}\|\nabla\tilde{\upsilon}\|_2^2,$$

so we deduce that $\|\nabla\upsilon_n\|_2^2 \to \|\nabla\tilde{\upsilon}\|_2^2$. From (2.31) we get

$$A(\tilde{\upsilon}) = \frac{s-p}{q-p}\tilde{r}^{s-q}B(\tilde{\upsilon}) + \frac{2-p}{q-p}\tilde{r}^{2-q}\|\nabla\tilde{\upsilon}\|_2^2. \tag{2.90}$$

Thus, (2.7) and (2.90) yield

$$M = \lim_{n\to\infty} \hat{\Phi}(\upsilon_n) = \frac{(s-q)(s-p)}{pqs}\tilde{r}^s B(\tilde{\upsilon}) + \frac{(2-p)(2-q)}{2pq}\tilde{r}^2\|\nabla\tilde{\upsilon}\|_2^2 > 0,$$

a contradiction, proving the claim. Working as in Case 2 we have $\tilde{r} = r(\tilde{\upsilon})$. Finally, by passing to the limit in (2.17) we have (2.15), which implies $\tilde{\upsilon} \in S^1$ and $\hat{\Phi}(\tilde{\upsilon}) = M$. Therefore, we have the following theorem.

Theorem 2.12. *Assume that conditions* (H0)-(H2) *are satisfied,* $p < q < s < 2$ *and the set* G_3 *defined in* (2.23) *is not empty. Then* (1.5)-(1.6) *admits a non-negative solution* $u \in C^{1,\delta}(\Omega)$ *for some* $\delta \in (0,1)$.

Remark 2.13. We will now give some conditions which guarantee that $G_3 \neq \emptyset$. Suppose that $\operatorname{supp} a^+) \subseteq \operatorname{supp} b)$. Then there exists $\upsilon \in G_1$ such that $B(\upsilon) > 0$. Since $r_*(\upsilon)^{2-q} < r(\upsilon)^{2-q}$, (2.88) yields

$$(q-p)A(\upsilon) < (s-p)r_*(\upsilon)^{s-q}B(\upsilon) + (2-p)r(\upsilon)^{2-q}\|\nabla\upsilon\|_2^2, \tag{2.91}$$

and so

$$r_*(\upsilon)^{s-q} > \frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)} - \frac{2-p}{s-p}r(\upsilon)^{2-q}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)}.$$

Consequently,

$$\begin{aligned} &\frac{p}{q}\frac{s-q}{s-p}r_*(\upsilon)^{q-p}A(\upsilon)\\ &> \frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)} - \frac{2-p}{s-p}r(\upsilon)^{2-q}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)}\Big)^{(q-p)/(s-q)}A(\upsilon). \end{aligned} \tag{2.92}$$

On the other hand, (2.10) implies

$$r(\upsilon) \le \Big(\frac{A(\upsilon)}{B(\upsilon)}\Big)^{1/(s-q)},$$

which combined with (2.92) gives

$$\begin{aligned} &\frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)} - \frac{2-p}{s-p}r(\upsilon)^{2-q}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)}\Big)^{(q-p)/(s-q)}A(\upsilon)\\ &> \frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)} - \frac{2-p}{s-p}\Big(\frac{A(\upsilon)}{B(\upsilon)}\Big)^{(2-q)/(s-q)}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)}\Big)^{\frac{q-p}{s-q}}A(\upsilon). \end{aligned}$$

If $a^+(\cdot)$ is large enough, then

$$\frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{s-p}\frac{A(\upsilon)}{B(\upsilon)} - \frac{2-p}{s-p}A(\upsilon)^{(2-q)/(s-q)}\frac{\|\nabla\upsilon\|_2^2}{B(\upsilon)^{\frac{2-q}{s-q}+1}}\Big)^{(q-p)/(s-q)}A(\upsilon) > \|\nabla\upsilon\|_p^p,$$

implying that $\upsilon \in G_8$. Thus $G_8 \neq \emptyset$.

Suppose next that $(\operatorname{supp} a^+)\setminus \operatorname{supp} b))^o \neq \emptyset$. Then there exists $\upsilon \in S^1$ with $B(\upsilon) = 0$. By (2.88)

$$r_*(\upsilon) = \Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{1/(2-q)},$$

and so

$$\frac{p}{q}\frac{s-q}{s-p}r_*(\upsilon)^{q-p}A(\upsilon) = \frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon).$$

Therefore, if $a^+(\cdot)$ is large enough, then

$$\frac{p}{q}\frac{s-q}{s-p}\Big(\frac{q-p}{2-p}\frac{A(\upsilon)}{\|\nabla\upsilon\|_2^2}\Big)^{(q-p)/(2-q)}A(\upsilon) > \|\nabla\upsilon\|_p^p,$$

implying that $G_8 \neq \emptyset$.

Case 8: $q > \max\{p, s, 2\}$. In this case we shall use the mountain pass theorem.

Lemma 2.14. $\Phi(\cdot)$ *satisfies the Palais-Smale condition.*

Proof. Let $\{u_n\}_{n=1}^\infty$ be a sequence in E such that $|\Phi(u_n)| \le C$ for some $C > 0$ and every $n \in \mathbb{N}$ and $\Phi'(u_n) \to 0$ in $H^{-1}(\Omega)$. For $\varepsilon > 0$ and $\upsilon \in E$ we have

$$\begin{aligned} |\langle\Phi'(u_n), \upsilon\rangle| &= \Big|\int |\nabla u_n|^{p-2}\nabla u_n \nabla\upsilon dx + \int \nabla u_n \nabla \upsilon dx\\ &\quad - \int a(x){u_n}^{q-1}\upsilon dx + \int b(x){u_n}^{s-1}\upsilon dx\Big|\\ &\le \varepsilon\|\upsilon\|_E. \end{aligned} \tag{2.93}$$

If $\upsilon = u_n$ in (2.93), then

$$\int a(x){u_n}^q dx \le \varepsilon\|u_n\|_{1,k} + \int |\nabla u_n|^p dx + \int |\nabla u_n|^2 dx + \int b(x){u_n}^s dx. \tag{2.94}$$

By hypothesis

$$\frac{1}{p}\|\nabla u_n\|_p^p+\frac{1}{2}\|\nabla u_n\|_2^2-\frac{1}{q}\int a(x)|u_n|^q dx+\frac{1}{s}\int b(x)|u_n|^s dx\le C. \tag{2.95}$$

On combining (2.94) and (2.95) we obtain

$$\begin{aligned}&\frac{1}{p}\|\nabla u_n\|_p^p+\frac{1}{2}\|\nabla u_n\|_2^2+\frac{1}{s}\int b(x)|u_n|^s dx-\frac{1}{q}\varepsilon\|u_n\|_E\\ &-\frac{1}{q}\int|\nabla u_n|^p dx-\frac{1}{q}\int|\nabla u_n|^2 dx-\frac{1}{q}\int b(x){u_n}^s dx\le C,\end{aligned}$$

and so

$$(\frac{1}{p}-\frac{1}{q})\|\nabla u_n\|_p^p+(\frac{1}{2}-\frac{1}{q})\|\nabla u_n\|_2^2+(\frac{1}{s}-\frac{1}{q})\int b(x)|u_n|^s dx\le C+\frac{1}{q}\varepsilon\|u_n\|_E.$$

Since $q>\max\{p,2,s\}$, we deduce that

$$(\frac{1}{p}-\frac{1}{q})\|\nabla u_n\|_p^p+(\frac{1}{2}-\frac{1}{q})\|\nabla u_n\|_2^2\le C+\frac{1}{q}\varepsilon\|u_n\|_E \tag{2.96}$$

which implies that the sequence $\{u_n\}_{n=1}^\infty$ is bounded in E. By passing to a subsequence if necessary, we may assume that $u_n\to u$ weakly in E. Consequently,

$$\lim_{n\to\infty}\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle=0. \tag{2.97}$$

By taking $\upsilon=u_n-u$ in (2.93) we have

$$\begin{aligned}&\int\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)(\nabla u_n-\nabla u)dx+\int(\nabla u_n-\nabla u)(\nabla u_n-\nabla u)dx\\ &=\langle\Phi'(u_n)-\Phi'(u),u_n-u\rangle-\int|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-u)dx\\ &\quad-\int\nabla u_n\nabla(u_n-u)dx+\int|\nabla u|^{p-2}\nabla u\nabla(u_n-u)dx+\int\nabla u\nabla(u_n-u)dx\\ &\quad-\int a(x)|u|^{q-2}u(u_n-u)dx+\int b(x)|u_n|^{s-2}u_n(u_n-u)dx\\ &\quad+\int a(x)|u_n|^{q-2}u_n(u_n-u)dx+\int b(x)|u|^{s-2}u(u_n-u)dx.\end{aligned} \tag{2.98}$$

Since, at least for a subsequence, $u_n\to u$ in $L^p(\Omega)$ and $L^2(\Omega)$, (2.98) yields

$$\begin{aligned}\lim_{n\to\infty}\Big\{&\int\left(|\nabla u_n|^{p-2}\nabla u_n-|\nabla u|^{p-2}\nabla u\right)(\nabla u_n-\nabla u)dx\\ &+\int(\nabla u_n-\nabla u)(\nabla u_n-\nabla u)dx\Big\}=0.\end{aligned}$$

We now use the inequality

$$\begin{aligned}0&\le\Big\{\Big(\int|\varphi|^k dx\Big)^{1/k'}-\Big(\int|\psi|^k dx\Big)^{1/k'}\Big\}\Big\{\Big(\int|\varphi|^k dx\Big)^{1/k}-\Big(\int|\psi|^k dx\Big)^{1/k}\Big\}\\ &\le\int\left(|\varphi|^{k-2}\varphi-|\psi|^{k-2}\psi\right)(\varphi-\psi)dx,\end{aligned}$$

which holds for $\varphi,\psi\in L^k(\Omega)$ and $k'=k/(k-1)$, see [10], to conclude that $u_n\to u$ in E. □

Lemma 2.15. *(i) There exist $\rho, \alpha > 0$ such that $\Phi(u) \geq \alpha$ if $\|u\|_E = \rho$.*
(ii) There exists $u \in E$ with $\|u\| > \rho$ and $\Phi(u) < 0$.

Proof. (i) Fix $u \in E \backslash \{0\}$. Then

$$\Phi(u) \geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{q}\int a(x)|u|^q dx.$$

By the Sobolev embedding and the fact that $q > 2$ we have

$$\Phi(u) \geq \frac{1}{p}\|u\|_E^2 - \frac{c}{q}\|u\|_E^q \geq \alpha > 0,$$

whenever $\|u\|_E = \rho$ and $\rho > 0$ is small enough. Now fix $v \in G_1$. Then for $t > 0$

$$\Phi(tv) = \frac{t^p}{p}\|\nabla v\|_p^p + \frac{t^2}{2}\|\nabla v\|_2^2 - \frac{t^q}{q}\int a(x)|v|^q dx + \frac{t^s}{s}\int b(x)|v|^s dx,$$

and so $\lim_{t\to\infty} \Phi(tv) = -\infty$. Thus $\Phi(tv) < 0$ for large enough t. □

By an application of the mountain pass theorem we obtain the following result.

Theorem 2.16. *Assume that conditions* (H0)–(H4) *hold with* $q > \max\{p, s, 2\}$. *Then* (1.5)-(1.6) *admits a solution.*

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