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PERIODIC SOLUTIONS OF NEUTRAL DELAY INTEGRAL EQUATIONS OF ADVANCED TYPE

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Abstract. We study the existence of continuous periodic solutions of a neutral delay integral equation of advanced type. In the analysis we employ three fixed point theorems: Banach, Krasnosel'skii, and Krasnosel'skii-Schaefer. Krasnosel'skii-Schaefer fixed point theorem requires an *a priori* bound on all solutions. We employ a Liapunov type method to obtain such bound.

1. Introduction

In this article we use three fixed point theorems to obtain continuous periodic solutions to the integral equation

$$x(t) = f(t, x(t), x(t-h)) - \int_t^{\infty} c(t,s) g(s, x(s), x(s-h)) ds, \tag{1.1}$$

where $f, g : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, and the delay h is a positive constant. Equations such as (1.1) are known as neutral delay integral equations of advanced type [1, p. 295].

Several applications in physical sciences lead to neutral functional differential equations such as

$$x'(t) = ax(t) + \alpha x'(t-h) - q(x(t), x(t-h)) + r(t)\,; \tag{1.2}$$

see for example [1, 3, 6, 7, 9, 11] and their references. By integrating the above equation we obtain the integral equation

$$x(t) = \alpha x(t-h) - \int_t^{\infty} [q(x(s), x(s-h)) - a\alpha x(s-h)] e^{a(t-s)} ds + p(t)\,, \tag{1.3}$$

which is a particular case of the equation to be studied here. Although integrals from t to ∞ are not common in these type of problems, they are found when studying unstable manifolds for ordinary differential equations, Coddington and Levinson [4, 1955, p. 331]. Studies of integrals of this and other types can be found in [1, 3, 5, 8] and their references.

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The three fixed point theorems: Banach, Krasnosel'skii, and Krasnosel'skii-Schaefer will be used in this article. The latter theorem is a combination of Krasnosel'skii theorem and Schaefer fixed point theorem, as explained in Burton and Kirk [2]. Advantages and disadvantages of each method will be stated at the end of their corresponding sections.

Krasnosel'skii-Schaefer's theorem requires *a priori* bounds for solutions of an auxiliary equation. To obtain such bounds, we construct a Lyapunov-like functional, assuming certain sign conditions on c and its derivatives. Since one might define various Liapunov functionals, and obtain the required bounds under various conditions, the Krasnosel'skii-Schaefer theorem has a potential for providing results better than ours. The same three theorems were used by the first author in [10] for studying (1.1) with $\int_{t-h}^{t}$ instead of $\int_{t}^{\infty}$. There, the Lyapunov functional and some assumptions are different from the ones here. We start by stating the classical theorems.

Theorem 1.1 (Krasnosel'skii Theorem [13])**.** *Let M be a closed convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose A and B map M into S such that*

(i) *$A\Phi + B\Psi \in M$ for all $\Phi, \Psi \in M$;*
(ii) *A is continuous and AM is contained in a compact set;*
(iii) *B is a contraction.*

Then there exists a $\Phi \in M$ with $\Phi = A\Phi + B\Phi$.

Theorem 1.2 (Schaefer Theorem [2])**.** *Let $(S, \|\cdot\|)$ be a normed space, and let H a continuous mapping from S into S which is compact on each bounded subset X of S. Then either*

(i) *the equation $x = \lambda Hx$ has a solution for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

Theorem 1.3 (Krasnosel'skii-Schaefer Theorem [2])**.** *Let $(S, \|\cdot\|)$ be a Banach space. Suppose $B : S \to S$ is a contraction map, and $A : S \to S$ is continuous and maps bounded sets into compact sets. Then either*

(i) *$x = \lambda B(\frac{x}{\lambda}) + \lambda Ax$ has a solution in S for $\lambda = 1$, or*
(ii) *the set of all such solutions, $0 < \lambda < 1$, is unbounded.*

As general assumptions, we have the following:

(H1) For a positive constant T, we assume
$$\begin{gathered} f(t+T, x, y) = f(t, x, y), \quad g(t+T, x, y) = g(t, x, y), \\ c(t+T, s+T) = c(t, s). \end{gathered}$$

Note that f and g are periodic in the variable t, with the same period (T-periodic for short). Using this property, we define
$$f_0 := \sup_{0 \le t \le T} |f(t, 0, 0)|, \quad g_0 := \sup_{0 \le t \le T} |g(t, 0, 0)|. \tag{1.4}$$

Also note that c is not necessarily periodic in t or in s; for example $c(t, s) = \phi(t-s)$ satisfies the above condition, without being periodic. However, $\int_t^\infty c(t, s)\, ds$ is T-periodic, which is used in the definition
$$\bar{c} := \sup_{0 \le t \le T} \int_t^\infty |c(t, s)|\, ds\,. \tag{1.5}$$

(H2) The kernel c satisfies $\int_t^\infty |c(t,s)|\,ds < \infty$ for $t \geq 0$, and
$$\lim_{u\to t} \int_{\max\{u,t\}}^\infty |c(u,s)-c(t,s)|\,ds = 0\,. \tag{1.6}$$
This condition is satisfied for example by $c(t,s)=\exp(t-s)$.

(H3) There are constants f_1, f_2 such that
$$|f(t,x_1,y_1)-f(t,x_2,y_2)| \leq f_1|x_1-x_2| + f_2|y_1-y_2|, \tag{1.7}$$
for $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(H4) There are constants g_1, g_2 such that
$$|g(t,x_1,y_1)-g(t,x_2,y_2)| \leq g_1|x_1-x_2| + g_2|y_1-y_2| \tag{1.8}$$
for $t, x_1, x_2, y_1, y_2 \in \mathbb{R}$.

Let $(P_T, \|\cdot\|)$ denote the space of continuous T-periodic functions with the norm $\|x\| = \sup_{0\leq t\leq T}|x(t)|$. On this space, we define the operators
$$Ax(t) = -\int_t^\infty c(t,s)g(s,x(s),x(s-h))\,ds, \quad Bx(t) = f(t,x(t),x(t-h))\,. \tag{1.9}$$
All three fixed point theorems in this article use B as a contraction, which requires (1.7) with additional conditions on f_1 and f_2.

2. Using Banach's Fixed point theorem

Theorem 2.1. *Assume* (H1)-(H4) *and*
$$f_1 + f_2 + (g_1+g_2)\bar{c} < 1\,. \tag{2.1}$$
Then (1.1) *has a unique continuous T-periodic solution.*

Proof. First note that every fixed point of $(A+B)$ in P_T is a periodic solution of (1.1). Let x be a function in P_T. Then the continuity and periodicity of Bx follows from the continuity and periodicity of f and x. Next we show that Ax is T-periodic. Using the change of variable $u = s-T$, the equality $c(t+T, u+T) = c(t,u)$, and that $g(s,x(s),x(s-h))$ is periodic in s, we have
$$\begin{aligned} Ax(t+T) &= \int_{t+T}^\infty c(t+T,s)g(s,x(s),x(s-h))\,ds \\ &= \int_t^\infty c(t+T,u+T)g(u+T,x(u+T),x(u+T-h))\,du \\ &= \int_t^\infty c(t,u)g(u,x(u),x(u-h))\,du = Ax(t)\,. \end{aligned}$$
Next we show that Ax is continuous. Using that $g(s,x(s),x(s-h))$ is continuous and periodic in s, we define $\bar{g}_x := \sup_{0\leq s\leq T}|g(s,x(s),x(s-h))| < \infty$. Then for $u \geq t$, we have
$$\begin{aligned} &|Ax(u)-Ax(t)| \\ &= \Big|\int_u^\infty c(u,s)g(s,x(s),x(s-h))\,ds - \int_t^\infty c(t,s)g(s,x(s),x(s-h))\,ds\Big| \\ &\leq \bar{g}_x \int_t^u |c(t,s)|\,ds + \bar{g}_x \int_u^\infty |c(u,s)-c(t,s)|\,ds\,. \end{aligned}$$

Similarly for $t \geq u$, we have

$$|Ax(u) - Ax(t)| \leq \bar{g}_x \int_u^t |c(u,s)|\,ds + \bar{g}_x \int_t^\infty |c(u,s) - c(t,s)|\,ds\,.$$

Taking the limit as $u \to t$, by (1.6), the right-hand side approaches zero. This shows the continuity of Ax. Furthermore, when considering x in a bounded set $M \subset P_T$, the constant $\bar{g}_x$ can be made independent of x. Thus AM becomes a family of equi-continuous functions.

Now we show that $A+B$ is a contraction. Let x, y be functions in P_T. By (1.7),

$$|Bx(t) - By(t)| = |f(t,x(t),x(t-h)) - f(t,y(t),y(t-h))| \leq (f_1+f_2)\|x-y\|. \tag{2.2}$$

By (1.8) and (1.5),

$$\begin{aligned}
|Ax(t) - Ay(t)| &= \Big| \int_t^\infty c(t,s)\big(g(s,x(s),x(s-h)) - g(s,y(s),y(s-h))\big)\,ds\Big| \\
&\leq (g_1+g_2)\|x-y\| \int_t^\infty |c(t,s)| \\
&\leq (g_1+g_2)\|x-y\|\bar{c}\,.
\end{aligned}$$

From the two inequalities above and (2.1), the operator $(A+B)$ is a contracting mapping in the Banach space P_T. Therefore, there is a unique fixed point which is a solution of (1.1). This completes the proof. $\square$

3. Using Krasnosel'skii's Fixed point theorem

We look for periodic solutions of (1.1) in

$$M_r = \{x \in P_T : \|x\| \leq r\}\,,$$

which is a closed and convex subset of P_T.

Theorem 3.1. *Under the hypotheses of Theorem 2.1, Krasnosel'skii's Theorem also provides continuous periodic solutions to* (1.1)*, and at least one solution satisfies*

$$\|x\| \leq \frac{f_0 + g_0\bar{c}}{1 - (f_1+f_2+(g_1+g_2)\bar{c})}\,.$$

Proof. We will show that items (i)–(iii) of Theorem 1.1 are satisfied.

Part (i): Let x, y be functions in M_r. That Bx and Ax are continuous and periodic follows from the arguments in the proof of Theorem 2.1. We need to show that $\|Bx + Ay\| \leq r$, for certain value r. From (1.4), (1.8) and (1.5), we have

$$\begin{aligned}
|Ay(t)| &= \Big| \int_t^\infty c(t,s)\big(g(s,0,0) + g(s,y(s),y(s-h)) - g(s,0,0)\big)\,ds\Big| \\
&\leq \big(g_0 + (g_1+g_2)r\big)\bar{c}\,.
\end{aligned}$$

From (1.4) and (1.7),

$$|Bx(t)| = \big|f(t,0,0) + f(t,x(t),x(t-h)) - f(t,0,0)\big| \leq f_0 + (f_1+f_2)r$$

Combining the two inequalities above,

$$\frac{|Ay(t)+Bx(t)|}{r} \leq \frac{f_0 + g_0\bar{c}}{r} + f_1 + f_2 + (g_1+g_2)\bar{c}\,.$$

As $r \to \infty$, by (2.1), the right-hand side on the above inequality approaches $(f_1 + f_2 + (g_1+g_2)\bar{c})$ which is less than 1. Then, there exists r_0 such that $r \ge r_0$ implies $|Ay(t) + Bx(t)| \le r$; i, e., $Ay(t)+Bx(t) \in M_r$. Using (2.1), we can select

$$r_0 = \frac{f_0 + g_0\bar{c}}{1-(f_1+f_2+(g_1+g_2)\bar{c})}\,.$$

Part (ii): First we show that A is a continuous mapping on M_r. Let $x, y \in M_r$. By (1.8) and (1.5), we have

$$\begin{aligned}
|Ax(t) - Ay(t)| &\le \int_t^\infty |c(t,s)|\,|g(s,x(s),x(s-h)) - g(s,y(s),y(s-h))|\,ds \\
&\le \int_t^\infty |c(t,s)|(g_1+g_2)\|x-y\|\,ds \\
&\le \bar{c}(g_1+g_2)\|x-y\|\,.
\end{aligned}$$

This shows that A is a continuous mapping on M_r.

Now, we show that AM_r is a compact set, by means of the Arzela-Ascoli's theorem. Since $AM_r \subset M_r$, the functions in AM_r are unformly bounded by r. In the proof of Theorem 2.1 we showed that when M_r is bounded, the functions in AM_r are equi-continuous.

Part (iii): That B maps M_r into M_r and that B is a contracting operator follows from the proof Theorem 2.1, (2.2), and (2.1).

From Theorem 1.1, the operator $(A+B)$ has a fixed point in $M_r \subset P_T$. This is a solution of (1.1), which is periodic and bounded by r. $\square$

Remark 3.2. In our setting, the Banach and Kasnosels'kii Theorems require the same assumptions. Banach fixed point theorem provides the uniqueness of the solution, and Kasnosels'kii's theorem provides an estimate on the size of at least one solution.

4. Using Krasnosel'skii-Schaefer Fixed point theorem

In this section, we omit (H4), but use the following conditions:

(H5) The kernel $c(t,s)$ is twice differentiable, $c_{st}(t,s) \le 0$ for $t \le s$, and

$$\lim_{s\to\infty}(s-t)c(t,s) = 0\,. \tag{4.1}$$

(H6)

$$L := \sup_{0\le t\le T}\int_t^\infty |c_s(t,s)|(1+\frac{s-t}{T})ds < \infty. \tag{4.2}$$

(H7) $xg(t,x,y) \ge 0$, and for f_0, f_1, f_2 as defined above, there exist positive constants K, β such that

$$f_2|yg(t,x,y)| + (f_0+\beta)|g(t,x,y)| \le K + (1-f_1)xg(t,x,y)\,,$$

for $t, x, y \in \mathbb{R}$.

Note that (H7) implies that for each fixed x, $\lim_{|y|\to\infty}|yg(t,x,y)| < \infty$.

For the parameter λ in Krasnosel'ksii-Schaefer Theorem, we define the auxiliary equation

$$x(t) = \lambda f\Big(t, \frac{x(t)}{\lambda}, \frac{x(t-h)}{\lambda}\Big) - \lambda\int_t^\infty c(t,s)g(s,x(s),x(s-h))ds. \tag{4.3}$$

With A and B as defined in (1.9), we write the above equation as

$$x(t)=\lambda B(\frac{x}{\lambda})(t)+\lambda(Ax)(t). \tag{4.4}$$

Notice that this equation becomes (1.1) when $\lambda=1$.

Theorem 4.1. *Assume* (H1)–(H3), (H5)-(H7) *and that*

$$f_1+f_2<1\,. \tag{4.5}$$

Then (1.1) *has a continuous T-periodic solution.*

Proof. Let $x\in P_T$. Then by the same argument as in the proof of Theorem 2.1, Ax and Bx are also in P_T. That B is a contraction, also follows from the proof of Theorem 2.1, assuming (4.5).

To show that (4.4) has a solution in P_T when $\lambda=1$, we show that all solutions of (4.4) are bounded by a constant independent of the solution when $0<\lambda\le 1$.

Note that when x is T-periodic, so is $|g(u,x(u),x(u-h))|$, and it is bounded by a constant $\bar g_x$. Therefore, by (4.1),

$$\lim_{s\to\infty}|c(t,s)\int_t^s g(u,.,.)\,du|\le\lim_{s\to\infty}\bar g_x|s-t|\,|c(t,s)|=0\,,$$

where $g(u,.,.):=g(u,x(u),x(u-h))$. Using this limit, integrating by parts in (4.3), we have

$$x(t)=\lambda f\big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\big)+\lambda\int_t^\infty c_s(t,s)\int_t^s g(u,.,.)\,du\,ds\,. \tag{4.6}$$

The first term in the right-hand side is estimated using (1.4), (1.7), and

$$\begin{aligned}|\lambda f\big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\big)|&\le|\lambda f(t,0,0)+\lambda f\big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\big)-\lambda f(t,0,0)|\\&\le\lambda f_0+f_1|x(t)|+f_2|x(t-h)|\,.\end{aligned} \tag{4.7}$$

To estimate the integral in (4.6), we define the Lyapunov-like functional

$$V(t)=V(t,x(t))=\frac{\lambda}{2}\int_t^\infty c_s(t,s)\Big(\int_t^s g(u,x(u),x(u-h))\,du\Big)^2ds. \tag{4.8}$$

Differentiating with respect to t,

$$\begin{aligned}V'(t)&=\frac{\lambda}{2}\int_t^\infty\frac{d}{dt}\big[c_s(t,s)\Big(\int_t^s g(u,.,.)\,du\Big)^2\big]\,ds\\&=\frac{\lambda}{2}\int_t^\infty c_{st}(t,s)\Big(\int_t^s g(u,.,.)\,du\Big)^2-c_s(t,s)2\Big(\int_t^s g(u,.,.)\,du\Big)g(t,.,.)\,ds\,.\end{aligned}$$

Using that $c_{st}(t,s)\le 0$ by (H5), we have

$$V'(t)\le-\lambda g(t,.,.)\int_t^\infty c_s(t,s)\int_t^s g(u,.,.)\,du\,ds\,.$$

Integrating by parts and using (4.1) and that x is a solution of (4.3), we have

$$\begin{aligned}V'(t)&\le g(t,.,.)\Big(\lambda\int_t^\infty c(t,s)g(s,.,.)\,ds\Big)\\&=g(t,.,.)\Big(\lambda f\big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\big)-x(t)\Big)\\&=g(t,.,.)\lambda f\big(t,\frac{x(t)}{\lambda},\frac{x(t-h)}{\lambda}\big)-x(t)g(t,.,.)\,.\end{aligned}$$

By (4.7), and that $0 \le x(t)g(t,.,.)$, we have

$$\begin{aligned} V'(t) &\le |g(t,.,.)|\big(\lambda f_0 + f_1|x(t)| + f_2|x(t-h)|\big) - x(t)g(t,.,.) \\ &= (f_1 - 1)x(t)g(t,.,.) + f_2|x(t-h)|\,|g(t,.,.)| + \lambda f_0|g(t,.,.)|\,. \end{aligned}$$

Using that $\lambda \le 1$, by (H7) there exists positive constants K and β, such that

$$V'(t) \le K - \beta|g(t,x(t),x(t-h))|\,. \tag{4.9}$$

Note that $V(t)$ is T-periodic which can be shown as follows: First show that $c_s(t,s) = c_s(t+T,s+T)$ and that $\int_t^s g(u,x(u),x(u-h))\,du$ is T-periodic. Then show that $V(t)$ is T-periodic as is done for Ax in Theorem 2.1. By the periodicity of V,

$$0 = V(T) - V(0) \le KT - \beta\int_0^T |g(t,x(t),x(t-h))|\,dt\,,$$

which implies

$$\int_0^T |g(t,x(t),x(t-h))|\,dt \le \frac{KT}{\beta}\,.$$

With this inequality, and using that $|g(t,x(t),x(t-h))|$ is T-periodic, we have

$$\int_t^s |g(u,.,.)|\,du \le \big(1 + \frac{s-t}{T}\big)\int_0^T |g(u,.,.)|\,du \le \big(1 + \frac{s-t}{T}\big)\frac{KT}{\beta}\,.$$

From (4.6), (4.7), and (H6), we have

$$\begin{aligned} |x(t)| &\le \lambda f_0 + f_1|x(t)| + f_2|x(t-h)| + \int_t^\infty |c_s(t,s)|\int_t^s |g(u,.,.)|\,du\,ds \\ &\le \lambda f_0 + f_1|x(t)| + f_2|x(t-h)| + \frac{LKT}{\beta}\,. \end{aligned}$$

Computing the supremum over $[0,T]$ and using that $\lambda \le 1$, we have

$$\|x(t)\| \le \frac{f_0 + \frac{LKT}{\beta}}{1 - (f_1 + f_2)}\,.$$

Note that the right-hand side is independent of the solution; Thus the set of solutions to (4.4) is bounded for all $\lambda \in (0,1]$. The proof is complete. □

Remark 4.2. When applying Krasnosel'skii, (H4) requires $g(t,x,y)$ to satisfy a Lipschitz condition at every value of x and y. When applying Krasnosel'skii-Schaefer, (H7) replaces these conditions by $yg(t,x,y)$ being bounded. There are functions that satisfy (H4) but not (H7), for example $g(t,x,y) = \sin(t)(x/5 + y/5)$. There are also functions that satisfy (H7) but not (H4), for example $g(t,x,y) = \sin^2(t)x^{1/3}/(1+y^2)$.

Note that Krasnosel'skii-Schaefer's theorem imposes additional restrictions on the kernel $c(t,s)$, (H5)-(H6).

Remark 4.3. In this article, the operator B is a contraction. However, our results remain valid if the function f of (1.1) defines a "large contraction" or a "separate contraction". Burton [1] introduced a concept of large contraction. Then Liu and Li [12] defined a separate contraction, and showed that a large contraction is also a separate contraction. They also proved that Krasnosel'skii's theorem holds for separate contractions.

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