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POSITIVE SOLUTIONS FOR SINGULAR STURM-LIOUVILLE BOUNDARY VALUE PROBLEMS ON THE HALF LINE

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Abstract. This article concerns the existence and multiplicity of positive solutions for the singular Sturm-Liouville boundary value problem

$$\begin{gathered}(p(t)u'(t))' + h(t)f(t,u(t)) = 0, \quad 0 < t < \infty,\\ au(0) - b\lim_{t\to 0^+} p(t)u'(t) = 0,\\ c\lim_{t\to\infty} u(t) + d\lim_{t\to\infty} p(t)u'(t) = 0.\end{gathered}$$

We use fixed point index theory to establish our main results based on a priori estimates derived by utilizing spectral properties of associated linear integral operators.

1. Introduction

In this article, we study the singular Sturm-Liouville boundary value problem on the half line

$$\begin{gathered}(p(t)u'(t))' + h(t)f(t,u(t)) = 0, \quad 0 < t < \infty,\\ au(0) - b\lim_{t\to 0^+} p(t)u'(t) = 0,\\ c\lim_{t\to\infty} u(t) + d\lim_{t\to\infty} p(t)u'(t) = 0,\end{gathered} \tag{1.1}$$

problem

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ ($\mathbb{R}^+ := [0,\infty)$), h is nonnegative on $\mathbb{R}^+$ and belongs to a weighted Lebesgue space on $\mathbb{R}^+$, $p \in C(\mathbb{R}^+) \cap C^1(0,\infty)$ with $p > 0$ on $(0,\infty)$ and $\int_0^\infty \frac{\mathrm{d}s}{p(s)} < \infty$, $a,b,c,d \geq 0$ with $\rho := bc + ad + ac\int_0^\infty \frac{\mathrm{d}s}{p(s)} > 0$.

Boundary value problems on the half line arise in studying radially symmetric solutions of nonlinear elliptic equations and in various applications, such as an unsteady flow of gas through a semi-infinite porous media, theory of drain flows, and plasma physics (see for example [2, 3, 5, 14]). This explains the reason that the last two decades have witnessed an overgrowing interest in the research of such problems, with many papers in this direction published. We refer the interested reader to [1, 7, 8, 9, 10, 11, 12, 13, 15, 16, 18, 19, 20] and references therein. In [11], by using fixed point theorems in a cone, Lian et al. considered problem (1.1) and obtained

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a set of sufficient conditions that guarantee existence, uniqueness, and multiplicity of positive solutions for (1.1). An interesting feature in [11] is that the nonlinearity f may be sign-changing. In [19], by using fixed point index theory, Zhang et al. studied the existence of positive solutions for (1.1) with $h(t)f(t,u(t))$ replaced by the semipositone nonlinearity $f(t,u(t))+q(t)$, generalizing and improving some results due to Liu [8] and Zhang et al. [20].

Motivated by the works cited above, we discuss the existence and multiplicity of positive solutions for (1.1). We use fixed point index theory to establish our main results based on a priori estimates derived by utilizing spectral properties of associated linear integral operators. This means that both our methodology and results in this paper are different from those in [8, 10, 11, 19, 20].

The article is organized as follows. Section 2 contains some preliminary results, including spectral properties of two linear integral operators. In Section 3, we state and prove our main results. Four examples are given in Section 4 to illustrate applications of Theorems 3.1–3.3.

2. Preliminaries

Let $E=\{u\in C(\mathbb{R}^+):\lim_{t\to\infty}u(t)\text{ exists}\}$ be equipped with the supremum norm $\|\cdot\|$ and $P=\{u\in E: u(t)\ge 0, t\in\mathbb{R}^+\}$. Then $(E,\|\cdot\|)$ is a real Banach space and P a cone on E. For simplicity, we denote $\xi(t)$ and $\eta(t)$ by

$$\xi(t):=b+a\int_0^t\frac{\mathrm{d}s}{p(s)},\quad \eta(t):=d+c\int_t^\infty\frac{\mathrm{d}s}{p(s)}.$$

Clearly, $c\xi(t)+a\eta(t)=\rho$. Let

$$G(t,s):=\frac{1}{\rho}\begin{cases}\xi(t)\eta(s), & 0\le t\le s<\infty,\\ \xi(s)\eta(t), & 0\le s\le t<\infty.\end{cases}\tag{2.1}$$

Green

We assume the following conditions hold throughout this article.

- (H1) h is Lebesgue measurable and nonnegative on $\mathbb{R}^+$, with $\int_0^\infty G(s,s)h(s)\,\mathrm{d}s\in(0,\infty)$.
- (H2) $p\in C(\mathbb{R}^+)\cap C^1(0,\infty)$, with $p>0$ in $(0,\infty)$ and $\int_0^\infty\frac{\mathrm{d}s}{p(s)}<\infty$.
- (H3) $f\in C(\mathbb{R}^+\times\mathbb{R}^+,\mathbb{R}^+)$ is bounded on $\mathbb{R}^+\times[0,R]$ for every $R>0$.

lemma1 **Lemma 2.1.** *Let* (H1)-(H3) *hold and* $G(t,s)$ *be given in* (2.1). *Then* (1.1) *is equivalent to the fixed point equation* $u=Au$, *where* $A:P\to P$ *is defined by*

$$(Au)(t):=\int_0^\infty G(t,s)h(s)f(s,u(s))\,\mathrm{d}s,t\in\mathbb{R}^+.\tag{2.2}$$

A

continous **Lemma 2.2** ([9, 11, 19])**.** *Let* (H1)-(H3) *hold. Then* $A:P\to P$ *is a completely continuous operator.*

rmk2.1 **Remark 2.3.** Note that G satisfies the following properties, which are obtained from the monotonicity of ξ and η:

1. $G(t,s)$ is continuous and $0\le G(t,s)\le G(s,s)$ for all $t,s\in\mathbb{R}^+$,
2. $G(t,s)\ge\gamma(t)G(s,s)$ for all $t,s\in\mathbb{R}^+$, where

$$\gamma(t):=\min\Big\{\frac{\xi(t)}{\xi(\infty)},\frac{\eta(t)}{\eta(0)}\Big\},\quad t\in\mathbb{R}^+.\tag{2.3}$$

gamma

lemma2 **Lemma 2.4** ([4]). *Let $\Omega \subset E$ be a bounded open set and $A : \overline{\Omega} \cap P \to P$ a completely continuous operator. If there exists $u_0 \in P \setminus \{0\}$ such that $u - Au \neq \mu u_0$ for all $\mu \geq 0$ and $u \in \partial\Omega \cap P$, then $i(A, \Omega \cap P, P) = 0$, where i indicates the fixed point index on P.*

lemma3 **Lemma 2.5** ([4]). *Let $\Omega \subset E$ be a bounded open set with $0 \in \Omega$. Suppose $A : \overline{\Omega} \cap P \to P$ is a completely continuous operator. If $u \neq \mu Au$ for all $u \in \partial\Omega \cap P$ and $0 \leq \mu \leq 1$, then $i(A, \Omega \cap P, P) = 1$.*

To establish our main results, we need two extra completely continuous linear operators T and S, defined by

$$
\begin{gathered}
(Tu)(t) := \int_0^\infty G(t,s)h(s)u(s)\,\mathrm{d}s, \quad u \in E, \qquad (2.4) \\
(Sv)(s) := \int_0^\infty G(t,s)h(s)v(t)\,\mathrm{d}t, \quad v \in L(\mathbb{R}^+).
\end{gathered}
$$

T

Condition (H1) implies $S : L(\mathbb{R}^+) \to L(\mathbb{R}^+)$ maps every nonnegative function in $L(\mathbb{R}^+)$ to a nonnegative function. Note that T can be viewed as the dual operator of S for the reason that T can be extended to a bounded linear operator $\overline{T} : L^\infty(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$, satisfying $\overline{T}(L(\mathbb{R}^+)) \subset E$. It is easy to prove that the spectral radius of T, denoted by $r(T)$, is positive. Now the well-known Krein-Rutman [6] theorem asserts that there exist a $\varphi \in P\setminus\{0\}$ and a nonnegative $\psi \in L(\mathbb{R}^+) \setminus \{0\}$ such that $T\varphi = r(T)\varphi$ and $S\psi = r(T)\psi$, which can be written as

$$
\int_0^\infty G(t,s)h(s)\varphi(s)\,\mathrm{d}s = r(T)\varphi(t), \qquad (2.5)
$$

Krein-Rutman-1

and

$$
\int_0^\infty G(t,s)h(s)\psi(t)\,\mathrm{d}t = r(T)\psi(s). \qquad (2.6)
$$

Krein-Rutman-2

Note that ψ may be required to satisfy

$$
\int_0^\infty \psi(t)\,\mathrm{d}t = 1. \qquad (2.7)
$$

integral

rmk2.2 **Remark 2.6.** Let $\lambda_1 = 1/r(T) > 0$. Then (2.5) can be written in the form

$$
\begin{gathered}
(p(t)\varphi'(t))' + \lambda_1 h(t)\varphi(t) = 0, \quad 0 < t < \infty, \\
a\varphi(0) - b\lim_{t\to 0^+} p(t)\varphi'(t) = 0, \qquad (2.8) \\
c\lim_{t\to+\infty} \varphi(t) + d\lim_{t\to+\infty} p(t)\varphi'(t) = 0.
\end{gathered}
$$

eigenvalue

This says that λ_1 is the first eigenvalue of the above eigenvalue problem, with φ being a positive eigenfunction corresponding to λ_1.

The following is a result that is of crucial importance in our proofs and, by Remark 2.3, can be proved as in [17, Lemma 4].

lemma4 **Lemma 2.7.** *Let*

$$
P_0 := \Big\{u \in E : \int_0^\infty \psi(t)u(t)\,\mathrm{d}t \geq \omega\|u\|\Big\},
$$

where $\psi(t)$ is determined by (2.6) and (2.7), and $\omega := \int_0^\infty \gamma(t)\psi(t)\,\mathrm{d}t > 0$. Then $T(P) \subset P_0$ and, in particular, $\varphi \in P_0$.

3. Main results

Let $B_\delta := \{u \in E : \|u\| < \delta\}$ for $\delta > 0$.

thm1 **Theorem 3.1.** *Let* (H1)-(H3) *hold. Suppose*

$$\liminf_{u\to 0^+} \frac{f(t,u)}{u} > \lambda_1, \tag{3.1}$$
sublinear-1

$$\limsup_{u\to \infty} \frac{f(t,u)}{u} < \lambda_1 \tag{3.2}$$
sublinear-2

uniformly for $t \in \mathbb{R}^+$. Then (1.1) *has at least one positive solution.*

Proof. By (3.1), there exist $r > 0$ and $\varepsilon > 0$ such that

$$f(t,u) \geq (\lambda_1 + \varepsilon)u, \quad \forall t \in \mathbb{R}^+,\ u \in [0,r],$$

and thus for any $u \in \overline{B}_r \cap P$, we have

$$(Au)(t) \geq (\lambda_1 + \varepsilon) \int_0^\infty G(t,s)h(s)u(s)\,\mathrm{d}s = (\lambda_1+\varepsilon)(Tu)(t), \quad t \in \mathbb{R}^+. \tag{3.3}$$
Inequ-A-1

Next we shall show that

$$u - Au \neq \mu\varphi, \quad \forall u \in \partial B_r \cap P, \mu \geq 0, \tag{3.4}$$
notequ-1

where φ is defined by (2.5). If the claim is false, then there exist $u_1 \in \partial B_r \cap P, \mu_1 \geq 0$ such that $u_1 - Au_1 = \mu_1\varphi$. Thus $u_1 = Au_1 + \mu_1\varphi \in P_0$ by Lemma 2.7 and $u_1 \geq Au_1$. Combining the preceding inequality with (3.3) (replacing u by u_1) leads to

$$u_1(t) \geq (\lambda_1 + \varepsilon) \int_0^\infty G(t,s)h(s)u_1(s)\,\mathrm{d}s, t \in \mathbb{R}^+. \tag{3.5}$$
Inequ-A-2

Multiply the above by $\psi(t)$ and integrate over $\mathbb{R}^+$ and use (2.6) and (2.7) to obtain

$$\int_0^\infty u_1(t)\psi(t)\,\mathrm{d}t \geq \lambda_1^{-1}(\lambda_1+\varepsilon) \int_0^\infty u_1(t)\psi(t)\,\mathrm{d}t,$$

so that $\int_0^\infty u_1(t)\psi(t)\,\mathrm{d}t = 0$. Recalling $u_1 \in P_0$, we have $u_1 \equiv 0$, a contradiction with $u_1 \in \partial B_r \cap P$. As a result, (3.4) holds. Now Lemma 2.4 implies

$$i(A, B_r \cap P, P) = 0. \tag{3.6}$$
index1-0

On the other hand, by (3.2) and (H3), there exist $0 < \sigma < 1$ and $M > 0$ such that

$$f(t,u) \leq \sigma\lambda_1 u + M, \quad \forall u \geq 0, t \in \mathbb{R}^+. \tag{3.7}$$
Inequ-A-3

We shall prove that the set

$$\mathcal{M}_1 := \{u \in P : u = \mu Au, 0 \leq \mu \leq 1\}. \tag{3.8}$$

is bounded. Indeed, for any $u_2 \in \mathcal{M}_1$ we have by (3.7)

$$u_2(t) \leq \sigma\lambda_1 \int_0^\infty G(t,s)h(s)u_2(s)\,\mathrm{d}s + u_0(t) = \sigma\lambda_1(Tu_2)(t) + u_0(t),$$

where $u_0 \in P$ is defined by $u_0(t) = M\int_0^\infty G(t,s)h(s)\,\mathrm{d}s$. Notice $r(\sigma\lambda_1 T) = \lambda_1\sigma r(T) < 1$. This implies $I - \sigma\lambda_1 T$ is invertible and its inverse equals

$$(I - \sigma\lambda_1 T)^{-1} = I + \sigma\lambda_1 T + \sigma^2\lambda_1^2 T^2 + \cdots + \sigma^n\lambda_1^n T^n + \dots.$$

Now we have $(I - \sigma\lambda_1 T)^{-1}(P) \subset P$ and $u_2 \leq (I - \sigma\lambda_1 T)^{-1}u_0$. Therefore, $\mathcal{M}_1$ is bounded. Choosing $R > \max\{r, \sup\{\|u\| : u \in \mathcal{M}_1\}\}$, we have by Lemma 2.5

$$i(A, B_R \cap P, P) = 1. \tag{3.9}$$
index1-1

Now (3.6) and (3.9) imply

$$i(A,(B_R\setminus\overline{B}_r)\cap P,P)=i(A,B_R\cap P,P)-i(A,B_r\cap P,P)=1. \tag{3.10}$$

Thus the operator A has at least one fixed point on $(B_R\setminus\overline{B}_r)\cap P$ and hence (1.1) has at least one positive solution. The proof is completed. □

thm2 **Theorem 3.2.** *Let* (H1)-(H3) *hold. Suppose*

$$\liminf_{u\to\infty}\frac{f(t,u)}{u}>\lambda_1 \tag{3.11}$$

superlinear-1

and

$$\limsup_{u\to 0^+}\frac{f(t,u)}{u}<\lambda_1 \tag{3.12}$$

superlinear-2

uniformly for $t\in\mathbb{R}^+$. Then (1.1) *has at least one positive solution.*

Proof. By (3.11) and (H3), there exist $\varepsilon>0$ and $b>0$ such that

$$f(t,u)\ge(\lambda_1+\varepsilon)u-b,\quad \forall u\ge 0,\ t\in\mathbb{R}^+. \tag{3.13}$$

fsuper

This implies

$$(Au)(t)\ge(\lambda_1+\varepsilon)\int_0^\infty G(t,s)h(s)u(s)\,\mathrm{d}s-b\int_0^\infty G(t,s)h(s)\,\mathrm{d}s \tag{3.14}$$

Ineq-2

for all $u\in P$ and $t\in\mathbb{R}^+$. We shall prove that the set

$$\mathcal{M}_2:=\{u\in P:u=Au+\mu\varphi,\mu\ge 0\}. \tag{3.15}$$

notequ-2

is bounded, where $\varphi\in P$ is given by (2.5). Indeed, if $u\in\mathcal{M}_2$, then we have $u\ge Au$ by definition and $u\in P_0$ by Lemma 2.7. This together with (3.14) leads to

$$u(t)\ge(\lambda_1+\varepsilon)\int_0^\infty G(t,s)h(s)u(s)\,\mathrm{d}s-b\int_0^\infty G(t,s)h(s)\,\mathrm{d}s,\quad t\in\mathbb{R}^+.$$

Multiply the above by $\psi(t)$ and integrate over $\mathbb{R}^+$ and use (2.6) and (2.7) to obtain

$$\int_0^\infty\psi(t)u(t)\,\mathrm{d}t\ge(\lambda_1+\varepsilon)\lambda_1^{-1}\int_0^\infty\psi(t)u(t)\,\mathrm{d}t-b\lambda_1^{-1},$$

so that $\int_0^\infty\psi(t)u(t)\,\mathrm{d}t\le b\varepsilon^{-1}$ for all $u\in\mathcal{M}_2$. Recalling $u\in P_0$, we obtain $\|u\|\le(\varepsilon\omega)^{-1}b$ for all $u\in\mathcal{M}_2$, and thus $\mathcal{M}_2$ is bounded, as required. Taking $R>\sup\{\|u\|:u\in\mathcal{M}_2\}$, we have

$$u-Au\ne\mu\varphi,\quad\forall u\in\partial B_R\cap P,\ \mu\ge 0. \tag{3.16}$$

Now Lemma 2.4 yields

$$i(A,B_R\cap P,P)=0. \tag{3.17}$$

index2-0

By (3.12), there exist $r\in(0,R)$ and $\sigma\in(0,\lambda_1)$ such that

$$f(t,u)\le(\lambda_1-\sigma)u,\quad\forall 0\le u\le r,\ 0\le t<\infty,$$

so that

$$(Au)(t)\le(\lambda_1-\sigma)\int_0^\infty G(t,s)h(s)u(s)\,\mathrm{d}s \tag{3.18}$$

Inequ-1

for all $u\in\overline{B}_r\cap P$ and $t\in\mathbb{R}^+$. We claim that

$$u\ne\mu Au,\quad\forall u\in\partial B_r\cap P,\quad 0\le\mu\le 1. \tag{3.19}$$

hom

Suppose, to the contrary, there exist $u_1 \in \partial B_r \cap P$ and $\mu_1 \in [0,1]$ such that $u_1 = \mu_1 A u_1$. Then we have $u_1 \in P_0$ by Lemma 2.7 and

$$u_1(t) \le (\lambda_1 - \sigma) \int_0^\infty G(t,s)h(s)u_1(s)\,\mathrm{d}s, \quad t \in \mathbb{R}^+$$

by (3.18). Multiply the above by $\psi(t)$ and integrate over $\mathbb{R}^+$ and use (2.6) and (2.7) to obtain

$$\int_0^\infty \psi(t)u_1(t)\,\mathrm{d}t \le \frac{\lambda_1 - \sigma}{\lambda_1} \int_0^\infty \psi(t)u_1(t)\,\mathrm{d}t,$$

so that $\int_0^\infty \psi(t)u_1(t)\,\mathrm{d}t = 0$. Recalling $u_1 \in P_0$, we obtain $u_1 = 0$, a contradiction with $u_1 \in \partial B_r \cap P$. As a result, (3.19) is true. Now Lemma 2.5 yields

$$i(A, B_r \cap P, P) = 1. \tag{3.20}$$

index2-1

Combining (3.17) and (3.20) gives

$$i(A, (B_R \setminus \overline{B}_r) \cap P, P) = i(A, B_R \cap P, P) - i(A, B_r \cap P, P) = -1. \tag{3.21}$$

Consequently the operator A has at least one fixed point on $(B_R \setminus \overline{B}_r) \cap P$, and hence (1.1) has at least one positive solution. The proof is completed. □

thm3 **Theorem 3.3.** *Let* (H1)-(H3) *hold. Suppose that $f(t,u)$ satisfies* (3.1) *and* (3.11). *Moreover, $f(t,u)$ is nondecreasing in u, and that there exists $N > 0$ such that*

$$f(t,N) < \frac{N}{\kappa}, \quad \text{a.e. } t \in \mathbb{R}^+, \tag{3.22}$$

nondecreasing

where $\kappa := \int_0^\infty G(s,s)h(s)\,\mathrm{d}s > 0$. Then (1.1) *has at least two positive solutions.*

Proof. The monotonicity of f implies that for all $u \in \overline{B}_N \cap P$ and $t \in \mathbb{R}^+$, we have

$$(Au)(t) = \int_0^\infty G(t,s)h(s)f(s,u(s))\,\mathrm{d}s < \int_0^\infty G(s,s)h(s)\frac{N}{\kappa}\,\mathrm{d}s = N, \tag{3.23}$$

so that $\|Au\| < \|u\|$ for all $u \in \partial B_N \cap P$. A consequence of this is

$$u \ne \mu A u, \forall u \in \partial B_N \cap P, 0 \le \mu \le 1.$$

Now Lemma 2.5 implies

$$i(A, B_N \cap P, P) = 1. \tag{3.24}$$

index3-1

On the other hand, in view of (3.1) and (3.11), we may take $R > N$ and $r \in (0,N)$ so that (3.6) and (3.17) hold (see the proofs of Theorems 3.1 and 3.2). Combining (3.6), (3.17) and (3.24), we arrive at

$$i(A, (B_R \backslash \overline{B}_N) \cap P, P) = 0 - 1 = -1, \quad i(A, (B_N \backslash \overline{B}_r) \cap P, P) = 1 - 0 = 1.$$

Consequently the operator A has at least two fixed points, one on $(B_R \backslash \overline{B}_N) \cap P$ and the other on $(B_N \backslash \overline{B}_r) \cap P$. Hence (1.1) has at least two positive solutions. The proof is completed. □

4. Examples

In this section, we provide four examples to illustrate applications of Theorems 3.1–3.3. Let us consider the boundary value problem

$$\begin{gathered} ((1+t^2)u'(t))' + t^{-1/2}e^{-t}f(t,u) = 0, \quad 0 < t < \infty, \\ u(0) = u(\infty) = 0, \end{gathered} \tag{4.1}$$

example

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ satisfies (H3). Now we have $p(t) = 1 + t^2$, $h(t) = t^{-1/2}e^{-t}$, $a = c = 1$, $b = d = 0$,

$$G(t,s) = \frac{2}{\pi}\begin{cases} (\frac{\pi}{2} - \arctan(t))\arctan(s), & 0 \le s \le t < \infty, \\ (\frac{\pi}{2} - \arctan(s))\arctan(t), & 0 \le t \le s < \infty, \end{cases}$$

and

$$\gamma(t) = \frac{2}{\pi}\min\left\{\frac{\pi}{2} - \arctan(t), \arctan(t)\right\}, t \in \mathbb{R}^+.$$

Since

$$\int_0^\infty \frac{\mathrm{d}r}{p(r)} = \frac{\pi}{2} < \infty, \kappa = \int_0^\infty G(s,s)h(s)\,\mathrm{d}s < \frac{\pi^{3/2}}{8} < \infty,$$

conditions (H1)-(H3) hold. By elementary calculus, we have

$$\arctan(s) \ge se^{-2s}, \quad \frac{\pi}{2} - \arctan(s) \ge se^{-2s}, \quad s \in \mathbb{R}^+.$$

The inequalities above, along with Gelfand's theorem, enable us to derive the estimation $8\pi^{-3/2} < \lambda_1 < \frac{1372\sqrt{7}}{15}\pi^{3/2}$, where λ_1 denotes the first eigenvalue of the eigenvalue problem associated with (4.1).

exa4.1 **Example 4.1.** Let $f(t,u) := u^\alpha, t, u \in \mathbb{R}^+$, where $\alpha \in (0,1) \cup (1,\infty)$. If $\alpha \in (0,1)$, then (3.1) and (3.2) are satisfied. If $\alpha \in (1,\infty)$, then (3.11) and (3.12) are satisfied. By Theorems 3.1 and 3.2, Equation (4.1) has at least one positive solution.

exa4.2 **Example 4.2.** Let

$$f(t,u) := \begin{cases} 2\lambda_1 u, & 0 \le u \le 1, \\ \frac{\lambda_1 u}{2} + \frac{3\lambda_1}{2}, & u \ge 1. \end{cases}$$

Now (3.1) and (3.2) are satisfied. By Theorem 3.1, Equation (4.1) has at least one positive solution.

exa4.3 **Example 4.3.** Let

$$f(t,u) := \begin{cases} \frac{\lambda_1 u}{2}, & 0 \le u \le 1, \\ 2\lambda_1 u - \frac{3\lambda_1}{2}, & u \ge 1. \end{cases}$$

Now (3.11) and (3.12) are satisfied. By Theorem 3.2, Equation (4.1) has at least one positive solution.

exa4.4 **Example 4.4.** Let $f(t,u) := \lambda(u^a + u^b)$, where $0 < a < 1 < b$, $0 < \lambda \le 4\pi^{-3/2}$. Now (3.1), (3.11) and (3.22) are satisfied. By Theorem 3.3, Equation (4.1) has at least two positive solutions.

References

Agarwal [1] R. P. Agarwal, O. G. Mustafa, Yu. V. Rogovchenko; *Existence and asymptotic behavior of solutions of a boundary value problem on an infinite interval*, Math. Comput. Modelling 41 (2005) 135-157.

Baxley [2] V. Baxley; *Existence and uniqueness for nonlinear boundary value problems on infinite intervals*, J. Math. Anal. Appl. 147 (1990) 127-133.

Chen [3] S. Chen, T. Zhang; *Singular boundary value problems on a half line*, J. Math. Anal. Appl. 195 (1995) 149-168.

Guo_1 [4] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, Orlando, 1988.

Hai [5] D. Hai, K. Schmitt; *On radial solutions of quasilinear boundary value problems*, Progr. Nonlinear Differential Equations Appl. 35 (1999) 349-361.

Krein [6] M. G. Krein, M. A. Rutman; *Linear operators leaving invariant a cone in a Banach space*, Transl. Amer. Math. Soc. 10 (1962) 199-325.

Liu1 [7] Y. Liu; *Boundary value problems for second order differential equations on unbounded domains in a Banach space*, Applied Math. Comput. 135 (2003) 569-583.

Liu2 [8] Y. Liu; *Existence and unboundedness of positive solutions for singular boundary value problems on half line*, Applied Math. Comput. 144 (2003) 543-556.

Liu3 [9] L. Liu, Z. Wang, Y. Wu; *Multiple positive solutions of the singular boundary value problems for second-order differential equations on the half line*, Nonlinear Anal. 71 (2009) 2564-2575.

Lian1 [10] H. Lian, P. Wang, W. Ge; *Unbounded upper and lower solutions method for Sturm-Liouville boundary value problem on infinite intervals*, Nonlinear Anal. 70 (2009) 2627-2633.

Lian2 [11] H. Lian, W. Ge; *Existence of positive solutions for Sturm-Liouville boundary value problems on the half line*, J. Math. Anal. Appl. 321 (2006) 781-792.

Ma3 [12] R. Ma; *Existence of positive solutions for second-order boundary value problems on infinity intervals*, Applied Math. Lett. 16 (2003) 33-39.

Sun [13] Y. Sun, Y. Sun, L. Debnath; *On the existence of positive solutions for singular boundary value problems on the half line*, Applied Math. Lett. 22 (2009) 806-812.

Tan [14] J. Tan; *The radial solutions of second-order semilinear elliptic equations*, Acta Math. Appl. Sinica 19 (1996) 57-64.

Tian1 [15] Y. Tian, W. Ge, W. Shan; *Positive solution for three-point boundary-value problem on the half line*, Comput. Math. Appl. 53 (2007) 1029-1039.

Tian2 [16] Y. Tian, W. Ge; *Positive solutions for multi-point boundary value problem on the half line*, J. Math. Anal. Appl. 325 (2007) 1339-1349.

Yang [17] Z. Yang, D. O'Regan; *Positive solvability of systems of nonlinear Hammerstein integral equations*, J. Math. Anal. Appl. 311 (2005) 600-614.

Yan [18] B. Yan, D. O'Regan, R. P. Agarwal; *Unbounded solutions for singular boundary value problems on the semi-infinite interval: Upper and lower solutions and multiplicity*, J. Comput. Appl. Math. 197 (2006) 365-386.

Zhang [19] X. Zhang, L. Liu, Y. Wu; *Existence of positive solutions for second-order semipositone differential equations on the half line*, Applied Math. Comput. 185 (2007) 628-635.

Zhang1 [20] X. Zhang, L. Liu; *Positive solutions of superlinear semipositone singular Dirichlet boundary value problems*, J. Math. Anal. Appl. 316 (2006) 525-537.

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