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EXISTENCE OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS WITH DELAYED ARGUMENTS

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Abstract. In this work we investigate the existence of solutions to differential inclusions with a delayed argument. We use a fixed point theorem to obtain a solution and then provide an estimate of the solution.

1. Introduction

This note concerns the existence of solutions to differential inclusions with delayed argument, $x'(t) \in F(t, x_t)$ for $t \geq t_0$ with the initial condition $x(t) = \varphi(t)$ for $t \leq t_0$.

The first works on differential inclusions were published in 1934-35 by Marchaud [16] and Zaremba [22]. They used the terms contingent and paratingent equations. Later, Wasewski and his collaborators published a series of works and developed the elementary theory of differential inclusions [20, 21]. Within few years after the first publications, the differential inclusions became a basic tool in optimal control theory. Starting from the pioneering work of Myshkis [17], there exists a whole series of papers devoted to paratingent and contingent differential inclusions with delay; see for example Campu [5, 6] and Kryzowa [14]. After this, many works appeared on differential inclusions with delay, for example Anan'ev [1], Deimling [9], Hong [11] and Zygmunt [23]. Recent results about differential inclusions in Banach spaces were obtained by Boudjenah [3], Syam [19] and Castaing-Ibrahim [7]. For more details on differential inclusions see the books by Aubin and Cellina [2], Deimling [9], Smirnov [18], and Kisielewicz [12].

In this work, we study the existence of solutions to differential inclusions with delayed argument, and we extend a result obtained by Anan'ev [1].

2. Preliminaries

Let $\mathbb{R}^n$ denote the n dimensional Euclidean space and $\|\cdot\|$ its norm. Let B be a Banach space with norm $\|\cdot\|_B$. If $x \in B^n = B \times B \times \cdots \times B$, then $x_i \in B$, $i = 1, \dots, n$, and $\|x\|_{B^n} = \big(\sum \|x_i\|_B^2\big)^{1/2}$.

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Let (M,d) be a metric space, $A \subset M$, and ϵ a positive number. We denote by A^ϵ the closed ϵ-neighborhood of A; i.e., $A^\epsilon = \{x \in M : d(x,a) \le \epsilon\}$. Let $\overline{A}$ denote the closure of A and $\operatorname{co} A$ the convex hull of A.

Let $C_{[a,b]}$ be the space of continuous real functions on $[a,b]$ and $L_{p[a,b]}$ the space of real-valued functions whose p-power is integrable on $[a,b]$. For $f \in L_{p[a,b]}$, let $\|f\|_p = (\int_a^b |f(x)|^p dx)^{1/p}$.

Let $\operatorname{Conv}\mathbb{R}^n$ denote the set of all compact, convex and nonempty subsets of $\mathbb{R}^n$.

Fix $t_0 \in \mathbb{R}$, let $h(t)$ be a continuous and positive function for $t \ge t_0$ and F be a set-valued map: $[t_0, +\infty[\times C^n_{[-h(t),0]} \to \operatorname{Conv}\mathbb{R}^n$ such that: $(t,[x]_t) \to F(t,x_t) \in \operatorname{Conv}\mathbb{R}^n$, for $t \ge t_0$ and $x_t \in C^n_{[-h(t),0]}$, where $x_t(\zeta) = x(t+\zeta)$. For $-h(t) \le \zeta \le 0$, $x_t(\cdot)$ represents the history of the state from time $t-h(t)$ to time t.

For fixed t, the map $F(t,.) : C^n_{[-h(t),0]} \to \operatorname{Conv}\mathbb{R}^n$ is called upper semi-continuous, u.s.c for short, if: for all $\epsilon > 0$ there exists $\delta > 0$ such that $\|x_t - y_t\|_{C^n} \le \delta$ implies $F(t,y_t) \subset F^\epsilon(t,x_t)$ where $x_t, y_t \in C^n_{[-h(t),0]}$ (See [2]).

The map $F(.,x_.)$ is called Lebesgue-measurable on $[t_0,\gamma]$, if the set $Z = \{t \in [t_0,\gamma] : F(t,x_t) \cap K \ne \emptyset\}$ is Lebesgue-measurable for any closed set $K \subset \mathbb{R}^n$ (See [2]).

Let F be a set valued map: $[t_0,+\infty] \times C^n_{[-h(t),0]} \to \operatorname{Conv}\mathbb{R}^n$. A relation of the form

$$x'(t) \in F(t,x_t) \quad \text{for } t \ge t_0. \tag{2.1}$$

is called a differential inclusion with delayed argument.

The generalized Cauchy problem consists of searching a solution of the differential inclusion (2.1) which satisfies the initial condition

$$x(t) = \varphi(t) \quad \text{for } t \le t_0\,. \tag{2.2}$$

A function x is called solution of (2.1)-(2.2) if x is absolutely continuous on $[t_0,\gamma]$ and satisfies the differential inclusion (2.1) a.e, (almost everywhere) on $[t_0,\gamma]$ and the initial condition $x(t) = \varphi(t)$ for $t \le t_0$.

For the proof of our main theorem we need some lemmas including Opial's theorem wich is presented next.

Lemma 2.1 ([15]). *Let $w(t,y)$ be a continuous function from $\mathbb{R}^+ \times \mathbb{R}^+$ to $\mathbb{R}^+$, increasing in y and $M(t)$ a maximal solution of the ordinary differential equation $y' = w(t,y)$, with the initial condition $y(t_0) = y_0$, on the interval $[t_0,T]$, where $T > t_0$ an arbitrary positive number. Let $m(t)$ be a continuous function increasing on $[t_0,T]$ and such that $m'(t) \le w(t,m(t))$ a.e. on $[t_0,T]$. If $m(t_0) \le y_0$, then $m(t) \le M(t)$ for all $t \in [t_0,T]$.*

Lemma 2.2 ([8]). *Let Γ be an upper semicontinuous set-valued map defined on a metric space T with compact and nonempty value in a metric space U and $\{\Theta_n\}$ a sequence of elements of T converging to Θ_0. Then we have*

$$\emptyset \ne \cap_{k=1}^\infty \overline{\operatorname{co}}(\cup_{n=k}^\infty \Gamma(\Theta_n)) \subset \Gamma(\Theta_0).$$

Lemma 2.3 ([10]). *If X is a Banach space and $\{x_n\}$ a sequence of elements of X weakly convergent to x, then there exists a sequence of convex combinations of the elements $\{x_n\}$ which converges strongly to x, in the sense of the norm.*

We will recall the fixed point theorem for multivalued mappings due to Borisovich et al; see [4].

Lemma 2.4 ([4]). *Let X be a normed space, C be a convex subset of X and $\Gamma : C \to 2^C$ be an upper semicontinuous set-valued map. Suppose that for all $x \in C$, $\Gamma(x) \in \operatorname{Conv} C$, then Γ has at least one fixed point in C.*

3. Existence result

First we study the existence of solutions to (2.1)-(2.2) on the interval $[t_0, \gamma]$, where $\gamma > t_0$ (γ an arbitrary fixed reel number). Let us consider the interval $[t_\gamma, \gamma]$, where $t_\gamma = \min\{t - h(t), t \in [t_0,\ \gamma]\}\}$, then x_t denote the restriction of the function $x \in C^n_{[t_0,\gamma]}$ to the interval $[t - h(t), t]$ where $t \in [t_0, \gamma]$. For $x \in C^n_{[t_\gamma,\gamma]}$, we denote the norm of x by

$$\|x\|_{c^n} = \max\{\|x(s)\|_{\mathbb{R}^n}, s \in [t - h(t), t], t \in [t_\gamma, \gamma]\}.$$

We use the following hypotheses:

- (H1) For $t \geq t_0$ the set-valued map $F(t, .) : C^n_{[-h(t),0]} \to \operatorname{Conv} \mathbb{R}^n$ is upper semicontinuous.
- (H2) For each fixed function $x \in C^n_{[t_\gamma,\gamma]}$, the set-valued map $F(., x_.) : [t_0, \gamma] \to \operatorname{Conv} \mathbb{R}^n$, is Lebesgue-measurable on the interval $[t_0, \gamma]$.
- (H3) For any bounded set $Q \subset C^n_{[t_\gamma,\gamma]}$, there exists a function $m : [t_0, \gamma] \to [0, +\infty[$ Lebesgue integrable such that for each measurable function $y : [t_0, \gamma] \to \mathbb{R}^n$ verifying the condition: $y(t) \in F_{Q(t)} = \cup\{F(t, x_t) : x \in Q\}$, almost everywhere on $[t_0, \gamma]$, we have the inequality $\|y(t)\| \leq m(t)$ a.e. on $[t_0, \gamma]$.
- (H4) For each fixed function $x \in C^n_{[t_\gamma,\gamma]}$ and a vector $y \in F(t, x_t)$, we have the inequality: $x'(t).y \leq \Phi(t, \|x_t\|^2_{c^n})$ where $\Phi(t, z)$ is a continuous function on $[t_0, \gamma] \times \mathbb{R}^+ \to \mathbb{R}^+$, positive, increasing in z and such that the ordinary differential equation $z' = 2\Phi(t, z)$ with the initial condition $z(t_0) = A$ (A an arbitrary positive number) has a maximal solution on all $[t_0, \gamma]$.
- (H5) The initial function φ is continuous on $[t_\gamma, t_0]$.

Now we are able to state and prove our existence result.

Theorem 3.1. *Under hypothesis* (H1)–(H5)*, for each $\varphi \in C^n_{[t_\gamma,t_0]}$, problem* (2.1)-(2.2) *has at least one solution on the interval $[t_0, \gamma]$.*

Proof. First we give an estimate of the solution of the differential inclusion (2.1). Suppose that (2.1) has a solution on $[t_0, \gamma]$ and let $g(t) = 1 + \|x(t)\|^2_{\mathbb{R}^n}$. Then

$$g'(t) = 2(x_1(t)x_1'(t) + x_2(t)x_2'(t) + \cdots + x_n(t)x_n'(t)) = 2x(t)x'(t).$$

In view of (H4),

$$\begin{aligned} g'(t) &\leq 2\Phi(t, \|x\|^2_{c^n}) = 2\Phi(t, \max\{\|x(s)\|^2_{\mathbb{R}^n}, t - h(t) \leq s \leq t\}) \\ &= 2\Phi(t, \max\{g(s), s \in [t - h(t), t]\}) \\ &\leq 2\Phi(t, \max\{g(s), \quad s \in [t_\gamma, t]\}). \end{aligned}$$

For $t_0 < u \leq t$ we have

$$\int_{t_0}^{u} g'(\tau)d\tau \leq 2\int_{t_0}^{u} \Phi(\tau, \max\{g(s), s \in [\tau_\gamma, \tau]\})d\tau \leq 2\int_{t_0}^{u} \Phi(\tau, \Lambda(\tau))d\tau,$$

where $\Lambda(\tau) = \max\{g(s), s \in [\tau_\nu, \tau]\}$. This implies

$$g(u) \leq g(t_o) + 2\int_{t_0}^{u} \Phi(\tau, \Lambda(\tau))d\tau.$$

Then

$$\max\{g(u), u \in [\tau_\nu, \tau]\} \le \max\{g(t) + 2\int_{t_0}^{t} \Phi(\tau, \Lambda(\tau))d\tau,\ t \in [t_0, \gamma]\},$$

or just

$$\Lambda(t) \le \max\{1 + \|x(t)\|^2_{\mathbb{R}^n}, t \in [t_0, \gamma]\} + 2\int_{t_0}^{t} \Phi(\tau, \Lambda(\tau))d\tau.$$

Thus

$$\Lambda(t) \le \Lambda(t_0) + 2\int_{t_0}^{t} \Phi(\tau, \Lambda(\tau))d\tau.$$

Using Lemma 2.1, we obtain $\Lambda(t) \le M(t)$, where $M(t)$ is the maximal solution of the ordinary differential equation $z' = 2\Phi(t, z)$ with the initial condition $M(t_0) = \Lambda(t_0)$. Hence we have the inequalities:

$$g(t) \le \Lambda(t) < M(\gamma).$$

It follows that

$$\|x(s)\|^2_{\mathbb{R}^n} \le M(\gamma) - 1 \le M(\gamma).$$

On the interval $[t_0, \gamma]$, we obtain the following estimate for the solution $x(t)$ of (2.1),

$$\|x(s)\|_{\mathbb{R}^n} < L = M(\gamma)^{1/2} \tag{3.1}$$

Let us set $Q = \{x \in C^n_{[t_\gamma, \gamma]},\ x(t) = \varphi(t)$ for $t \in [t_\gamma, t_0]$ and $\|x(t)\| \le L$ for $t \in [t_0, \gamma]\}$. As the set Q is bounded in space $C^n_{[t_\gamma, \gamma]}$, from (H3), there is a measurable function $m : [t_0, \gamma] \to [0, +\infty[$ and Lebesgue integrable, such that for each measurable function $y : [t_0, \gamma] \to \mathbb{R}^n$ verifying $y(t) \in F_Q(t) = \cup_Q F(t, x_t)$, almost everywhere on $[t_0, \gamma]$, we have the inequality

$$\|y(t)\| \le m(t) \quad \text{a.e. on } [t_0, \gamma].$$

Then by (3.1), we obtain $\|\varphi(t_0)\|_{\mathbb{R}^n} < L$. Thus, we can choose a real number b_1 such that

$$\{x \in \mathbb{R}^n : \|x - \varphi(t_0)\|_{\mathbb{R}^n} \le b_1\} \subset \{x \in \mathbb{R}^n : \|x\|_{\mathbb{R}^n} \le L\}.$$

As the function m is integrable, we can choose $t_1 > 0$ such that

$$\int_{t_0}^{t_1} m(t)dt \le b_1 \tag{3.2}$$

Now we show that (2.1)-(2.2) has at least one solution on $[t_0, t_1]$. Let us consider the set X of functions x of the space $C^n_{[t_0, t_1]}$ satisfying the following three conditions: x is absolutely continuous on $[t_0, t_1]$, $x = \varphi \in C^n_{[t_\gamma, t_0]}$ for $t \in [t_\gamma, t_0]$, and

$$\|x'(t)\| \le m(t) \text{ a.e. on } [t_0,\ t_1] \tag{3.3}$$

We claim that X is compact. For this purpose we show that X is uniformly bounded and equi-continuous. For $x \in X$, we have

$$\|x(t) - \varphi(t_0)\| = \|x(t) - x(t_0)\| = \|\int_{t_0}^{t_1} x'(s)ds\|.$$

Furthermore, from (3.2), we have $\int_{t_0}^{t_1} m(t)dt \le b_1$. Then, using (3.3), we obtain

$$\|\int_{t_0}^{t_1} x'(s)ds\| \le \int_{t_0}^{t_1} \|x'(s)ds\| \le \int_{t_0}^{t_1} m(s)ds \le b_1 \quad \text{a.e. on } [t_0, t_1].$$

This implies

$$\|x(t)-\varphi(t_0)\| \le b_1 \quad \text{a.e. on } [t_0,\ t_1]\,. \tag{3.4}$$

This shows that X is uniformly bounded. Let $t_0 \le t \le t' \le t_1$, then we have

$$\|x(t)-x(t')\| \le \int_t^{t'} \|x'(s)ds\| \le \int_t^{t'} m(s)ds\,.$$

As the Lebesgue integral is absolutely continuous, for each $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|t-t'| < \delta \Rightarrow \|x(t)-x(t')\| < \varepsilon,$$

which shows that X is equi-continuous. Since X is uniformly bounded and equi-continuous, it is compact by Arzela's theorem.

It is easy to show that X is convex. Indeed, let $x, y \in X$ and $\lambda \in [0,1]$, we have

$$\|\frac{d}{dt}(\lambda x(t)+(1-\lambda)y(t))\| \le \lambda\|x'(t)\| + (1-\lambda)\|y'(t)\| \le m(t)$$

a.e. on $[t_0, t_1]$. Then

$$\lambda x + (1-\lambda)y \in X \quad \text{for } \lambda \in [0,1].$$

Let us fix $x \in X$, and with this function we consider a function y such that

$$y'(t) \in F(t, x_t) \quad \text{a.e. on } [t_0,\ t_1] \tag{3.5}$$

We denote by G the set of pairs $(x,y) \in X \times X$ such that (x,y) fulfills the above relation (3.5); i.e.,

$$G = \{(x,y) \in X \times X : y'(t) \in F(t,x_t) \text{ a.e. on } [t_0,t_1]\}\,.$$

Now we show that G is nonempty and closed. Let $x \in X$. In view of Hypothesis (H2) and selector's theorem (see[13]), there is a measurable function $\psi : [t_0, t_1] \to \mathbb{R}$ such that $\psi(t) \in F(t, x_t)$ a.e. on $[t_0, t_1]$. Define the function ξ as

$$\xi(t) = \varphi(t_0) + \int_{t_0}^{t} \psi(s)\,ds \quad \text{for } t \in [t_0, t_1].$$

Then $\xi'(t) \in F(t, x_t)$ a.e. on $[t_0, t_1]$. In view of (3.3), we have

$$\|\xi'(t)\| \le m(t) \quad \text{a.e. on } [t_0, t_1],$$

which implies that G is nonempty. G is closed. Indeed, let $(x_k,\ y_k)$ a sequence of elements of G converging to (x,y), we will show that the sequence of derivatives $\{y'_k\}$ is bounded with the norm of $L^n_{1[t_0,t_1]}$. We have:

$$\begin{aligned}
\|y'_k\|^2_{L^n_1} &= \sum_{i=1}^{n} \|[{y'_k}^i]\|^2_{L^n_1} = \sum_{i=1}^{n} \Big(\int_{t_0}^{t_1} |{y'_k}^i(s)|ds\Big)^2 \\
&= \sum_{i=1}^{n} \Big(\int_{t_0}^{t_1} m(s)\Big)^2 d = \sum_{i=1}^{n} b_1^2 \le nb_1.
\end{aligned}$$

We will prove that the sequence $\{y'_k\}$ satisfies the condition: $\lim \int_{E_i} y'_k(s)ds = 0$ uniformly, and for each decreasing sequence $\{E_i\}$ of measurable sets $E_1 \supset E_2 \supset \dots \supset E_n \supset \dots$ such that $\cap_{i=1}^{\infty} E_i = \emptyset$.

We have
$$
\begin{aligned}
|\int_{E_i} y_k'(s)ds| &\le \int_{E_i} |y_k'(s)|ds = \int_{t_0}^{t_1} \chi_{E_i}(s)|y_k'(s)|ds \\
&\le \int_{t_0}^{t_1} \chi_{E_i}(s)m(s)ds = \int_{E_i} m(s)ds,
\end{aligned}
$$
where χ_{E_i} denotes the characteristic function of the set E_i. For $E_i \subset [t_0, t_1]$, the integral exists and
$$
\lim \mu(E_i) = \mu(\cap_{i=1}^{\infty} E_i) = \mu(\emptyset) = 0,
$$
where μ denote the Lesbegue's measure.

From the absolute continuity of the integral, we obtain: For each $\epsilon > 0$, there exists j such that
$$
i > j \Rightarrow \int_{E_i} \chi_{E_i}(s)m(s)ds < \epsilon.
$$
Applying the weak criterion of compactness (see [10]), we show that the sequence $\{ y_k\}$ is weakly compact in the sequential sense. Therefore, there is a subsequence, also denoted by $\{y_k\}$, weakly convergent to a function $z \in L^n_{1[t_0,t_1]}$. Thus, for $t \in [t_0, t_1]$ we have
$$
y(t) = \lim y_k(t) = \lim(\varphi(t_0) + \int_{t_0}^{t} y_k'(s)ds) = \varphi(t_0) + \int_{t_0}^{t} z(s)ds.
$$
This implies $y'(t) = z(t)$.

Weak convergence in $L^n_{1[t_0,t_1]}$ is equivalent to the convergence of the integrals and applying Lemma 2.3, we prove the existence of a sequence of convex combinations $z_j = \{y_j', y_{j+1}', \dots\}$ strongly convergent to $z \in L^n_{1[t_0,t_1]}$.

As $L^n_{1[t_0,t_1]}$ is a complete space, from any strongly convergent sequence we can extract a subsequence which converges almost everywhere. Then from the sequence $\{z_j\}$ we can extract a subsequence, also denoted by $\{z_j\}$, which converges a.e. to z. Thus we have
$$
\lim z_j(t) = z(t) \quad \text{a.e. on } [t_0, t_1].
$$
We claim that $y'(t) = z(t) \in F(t, x_t)$ a.e. on $[t_0, t_1]$. So, we will show that
$$
z(t) \in \cap_{j=1}^{\infty} \overline{\text{co}}(\cup_{n=j}^{\infty} y_n'(t)).
$$
Let $\{[z_j]\}$ the sequence of the convex combinations of the functions $\{y_j', y_{j+1}', \dots\} = \cup_{n=j}^{\infty} y_n'$. We have
$$
z_j = \sum_{j=1}^{n_j} a_i y_i', \quad i \in \{j, j+1, \dots\}, \quad a_i > 0, \quad \sum_{j=1}^{n_j} a_i = 1.
$$
Then
$$
z_j(t) \in \text{co}(\cup_{n=j}^{\infty} y_n'(t))
$$
for j fixed. As $\lim\ z_j(t) = z(t)$ a.e. on $[t_0, t_1]$, we have that implication: For each neighborhood $U_{z(t)}$ of $z(t)$, there exists N_0 such that $z_j(t) \in U_{z(t)}$ for all $j > N_0$. Therefore
$$
U_{z(t)} \cap \text{co}(\cup_{n=j}^{\infty} y_n'(t)) \neq \emptyset
$$
and hence
$$
z(t) \in \overline{\text{co}}(\cup_{n=j}^{\infty} y_n'(t)).
$$

We have
$$\overline{\mathrm{co}}(\cup_{n=j}^{\infty} y_n'(t)) = \mathrm{co}\,\overline{(\cup_{n=j}^{\infty} y_n'(t))}.$$
Whence we obtain
$$z(t) \in \mathrm{co}\,\overline{(\cup_{n=j}^{\infty} y_n'(t))},$$
so that
$$z(t) \in \cap_{j=1}^{\infty}\overline{\mathrm{co}}(\cup_{n=j}^{\infty} y_n'(t)).$$
From the definition of G, we have $y_n'(t) \in F(t,(x_n)_t)$ a.e. on $[t_0,t_1]$. This implies
$$\cap_{j=1}^{\infty}\overline{\mathrm{co}}(\cup_{n=j}^{\infty} y_n'(t)) \subset \cap_{j=1}^{\infty}\overline{\mathrm{co}}(\cup_{n=jn}^{\infty}(F(t,(x_n)_t)$$
a.e. on $[t_0,t_1]$. Using Lemma 2.2, we obtain
$$\cap_{j=1}^{\infty}\overline{\mathrm{co}}(\cup_{n=jn}^{\infty}(F(t,(x_n)_t) \subset F(t,x_t).$$
From which it follows $z(t) \in F(t,x_t)$; i.e., $y'(t) \in F(t,x_t)$ a.e. on $[t_0,t_1]$. So we conclude that G is closed.

Let us define the set-valued map $\Gamma : X \to 2^X$ such that
$$\Gamma(x) = \{y : y'(t) \in F(t,x_t) \text{ a.e. on } [t_0,t_1]\}.$$
The set $G = \{(x,y)\} \subset X \times X\}$ is the graph of Γ. Since G is closed, the application Γ is upper semi-continuous [2].

Let us show that $\Gamma(x)$ is compact. As $\Gamma(x) \subset X$ and X is compact, then $\Gamma(x)$ is uniformly bounded and we prove the equicontinuity of $\Gamma(x)$ in the same way as we did for X. It is also easy to show that $\Gamma(x)$ is convex.

Using Lemma 2.4, we show that the map Γ has at least one fixed point. Therefore, there is a function $x \in X$ such that $x(t) \in F(t,x_t)$ a.e. on $[t_0,t_1]$, then x is a solution of (2.1)-(2.2) on $[t_0,\, t_1]$.

To complete the proof, we extend the solution on $[t_1,\gamma]$. For $t_1 < \gamma$, we have the implication: $\|x(t_1)\| < L$ implies the existence of $b_2 > 0$ such that
$$\{x \in \mathbb{R}^n : \|x(t) - x(t_1)\| \le b_2\} \subset \{x \in \mathbb{R}^n : \|x\| \le L\}.$$
Thus, there exist $t_2 > t_1$ such that $\int_{t_1}^{t_2} m(t)dt \le b_2$ and we extend the solution on $[t_1,t_2]$.

We can choose all b_i's such that $b_i \ge \epsilon > 0$, hence the sequence $\{b_i\}$ does not converge to 0. After a finite number of steps we can extend the solution to the entire interval $[t_0,\gamma]$. □

Remark. Anan'ev [1] assumed that $y.x'(t) \le K(1 + \|x_t\|_{c^n}^2)$ with $K > 0$. Our hypothesis (H4) is more general than that the one by Anan'ev.

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