Electronic Journal of Differential Equations, Vol. 2010(2010), No. 21, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

ASYMPTOTIC STABILITY OF SWITCHING SYSTEMS

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Abstract. In this article, we study the uniform asymptotic stability of the switched system $u' = f_{\nu(t)}(u)$, $u \in \mathbb{R}^n$, where $\nu : \mathbb{R}_+ \to \{1, 2, \dots, m\}$ is an arbitrary piecewise constant function. We find criteria for the asymptotic stability of nonlinear systems. In particular, for slow and homogeneous systems, we prove that the asymptotic stability of each individual equation $u' = f_p(u)$ ($p \in \{1, 2, \dots, m\}$) implies the uniform asymptotic stability of the system (with respect to switched signals). For linear switched systems (i.e., $f_p(u) = A_p u$, where A_p is a linear mapping acting on E^n) we establish the following result: The linear switched system is uniformly asymptotically stable if it does not admit nontrivial bounded full trajectories and at least one of the equations $x' = A_p x$ is asymptotically stable. We study this problem in the framework of linear non-autonomous dynamical systems (cocyles).

1. Introduction

The aim of this article is studying the uniform asymptotic stability of the switched system

$$x' = f_{\nu(t)}(x), \quad (x \in E^n) \tag{1.1}$$

where $\nu : \mathbb{R}_+ \to \{1, 2, \dots, m\}$ is an arbitrary piecewise constant function, E^n is an n-dimensional euclidian space, and $\mathbb{R}_+ := [0, +\infty)$.

The discrete-time counterpart of (1.1) takes the form

$$x_{k+1} = f_{\nu(k)}(x_k), \tag{1.2}$$

where $\nu : \mathbb{Z}_+ \to \{1, 2, \dots, m\}$ and $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$.

A continuous function $\gamma : \mathbb{R}_+ \to E^n$ (respectively, $\gamma : \mathbb{Z}_+ \to E^n$) is called a solution of (1.1) (respectively, of (1.2)), if $\gamma(t) = \gamma(0) + \int_0^t f_{\nu(s)}(\gamma(s))ds$ (respectively, $\gamma(n+1) = f_{\nu(n)}(\gamma(n))$) for all $t \in \mathbb{R}_+$ (respectively, $n \in \mathbb{Z}_+$).

Denote by $[E^n]$ the space of all linear operators $A : E^n \to E^n$ equipped with the operator norm.

If $f_p(x) = A_p(x)$ ($x \in E^n$ and $p = 1, 2, \dots, m$), where $A_p \in [E^n]$, then (1.1) (respectively, (1.2) is called a linear switched system.

The linear switched system (1.1) (respectively, (1.2)) is called uniformly (with respect to switching signals ν) exponentially stable if there are two positive numbers

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2000 *Mathematics Subject Classification.* 34A37, 34D20, 34D23, 34D45, 37B55, 37C75, 93D20.
Key words and phrases. Uniform asymptotic stability; cocycles; global attractors; uniform exponential stability; switched systems.

Submitted December 21, 2009. Published February 2, 2010.

$\mathcal{N}$ and ν such that $|\gamma(t)| \leq \mathcal{N} e^{-\nu t}|\gamma(0)|$ for all solution γ of (1.1) (respectively, (1.2)) and $t \in \mathbb{R}_+$ (respectively, $t \in \mathbb{Z}_+$).

Remark 1.1. (1) If (1.1) is uniformly exponentially stable, then every equation $x' = A_i x$ ($i = 1, 2, \dots, m$) is also exponentially stable; i.e.,

$$\operatorname{Re} \lambda_j(A_i) < 0 \tag{1.3}$$

for all $j = 1, 2, \dots, n$, where $\sigma(A_i) := \{\lambda_1(A_i), \lambda_2(A_i), \dots, \lambda_n(A_i)\}$ is the spectrum of the linear operator A_i).

(2) From condition (1.3), generally speaking, it does not follow uniformly exponentially stability of linear switched system (1.1) (see, for example, [31] and [33]). However, if in addition the interval between any two consecutive discontinuities of ν if sufficiently large, then the condition (1.3) implies the uniformly exponential stability of (1.1) [36].

The problem of uniform exponential stability for the switched linear systems (both with continuous and discrete time) arises in a number of areas of mathematics: control theory – Cheban [12], Molchanov [34]; linear algebra – Artzrouni [2], Beyn and Elsner [4], Bru, Elsner and Neumann [8], Daubechies and Lagarias [18], Elsner and Friedland [20], Elsner, Koltracht and Neumann [21], Gurvits [25], Vladimirov, Elsner and Beyn [43]; Markov Chains – Gurvits [22], Gurvits and Zaharin [23, 24]; iteration process – Bru, Elsner and Neumann [8], Opoitsev [37] and see also the bibliography therein.

For the discrete linear switched system (1.2), it is established the following result in [15].

Theorem 1.2 (Cheban, Mammana [15])**.** *Let $A_i \in [E^n]$ ($i = 1, 2, \dots, m$). Assume that the following conditions are fulfilled:*

1. *there exists $j \in \{1, 2, \dots, m\}$ such that the operator A_j is asymptotically stable (i.e., $r(A_j) < 1$, where $r(A)$ is the spectral radius of the operator A);*
2. *the discrete linear switched system* (1.2) *has not nontrivial bounded on $\mathbb{Z}$ solutions.*

Then the discrete linear switched system (1.2) *is uniformly exponentially stable*

In this paper we generalize this result for linear switched system (1.1) with continuous time. We present here also some tests of the asymptotic stability of nonlinear switched systems. In particular, for the slow homogeneous switched systems (i.e., $f_p(\lambda u) = \lambda f_p(u)$ for all $u \in \mathbb{R}^n$, $p \in \{1, 2, \dots, m\}$ and $\lambda > 0$) we prove that the asymptotic stability of each individual equation $u' = f_p(u)$ ($p \in \{1, 2, \dots, m\}$) implies the uniform (with respect to switched signals) asymptotic stability of system (1.1). We study this problem in the framework of non-autonomous dynamical systems (cocyles).

This paper is organized as follows: In section 2 we introduce the shift dynamical system on the space of piecewise constant functions which play a very important role in the study of of switched system. We show that every switched system generates a cocycle. This fact allows us to apply the ideas and methods of non-autonomous dynamical systems for studying the switched systems. Here , we present some tests of the asymptotic stability of nonlinear switched systems (1.1) (Theorems 2.20, 2.24 and 3.11).

Section 3 is dedicated to the study of switched homogeneous systems. We give a necessary and sufficient conditions of asymptotic stability of homogeneous systems

(1.1) (Theorem 3.8) and a sufficient condition for the asymptotic stability of slow switched homogeneous systems (1.1) is given (Theorem 3.11).

The main result of Section 4 is Theorem 4.14, which contains a necessary and sufficient conditions for the uniformly exponentially stability of linear switched system (1.1).

2. Asymptotic Stability of Nonlinear Switched Systems

Shift dynamical systems on the space of piecewise constant functions. Let $m \in \mathbb{N} := \{1, 2, \dots\}$ $(m \geq 2)$, $\mathcal{P} := \{1, 2, \dots, m\}$, and $S(\mathbb{R}_+, \mathcal{P})$ be the set of piecewise constant functions $\nu \colon \mathbb{R}_+ \to \mathcal{P}$, i.e., $\nu \in S(\mathbb{R}_+, \mathcal{P})$ if and only if there is a increasing sequence $\{t_k^\nu\}_{k \in \mathbb{Z}_+}$ such that $t_0^\nu := 0$, $t_k \to +\infty$ as $k \to +\infty$, $\nu(t) = p_k \in \mathcal{P}$ for all $t \in [t_k^\nu, t_{k+1}^\nu)$.

Denote by

$$d(\nu_1, \nu_2) := \sum_{k=1}^{+\infty} \frac{1}{2^k} \frac{d_k(\nu_1, \nu_2)}{1 + d_k(\nu_1, \nu_2)} \tag{2.1}$$

for all $\nu_1, \nu_2 \in S(\mathbb{R}_+, \mathcal{P})$, where $d_k(\nu_1, \nu_2) := \int_0^k ||\nu_1(t) - \nu_2(t)||dt$ for all $k \in \mathbb{N}$. By (2.1) is defined a complete metric on the space $S(\mathbb{R}_+, \mathcal{P})$.

Let $\tau > 0$. Denote by $S_\tau(\mathbb{R}_+, \mathcal{P})$ the subset of $S(\mathbb{R}_+, \mathcal{P})$ consisting of functions $\nu \in S(\mathbb{R}_+, \mathcal{P})$ with the sequences $\{t_k^\nu\}_{k \in \mathbb{Z}_+}$ satisfying the condition $t_{k+1}^\nu - t_k^\nu \geq \tau$ for all $k \in \mathbb{Z}_+$.

Theorem 2.1 ([29])**.** *$(S_\tau(\mathbb{R}_+, \mathfrak{C}), d)$ is a compact metric space.*

Let σ be a mapping from $\mathbb{R}_+ \times S_\tau(\mathbb{R}_+, \mathcal{P})$ into $S_\tau(\mathbb{R}_+, \mathcal{P})$ defined by the equality $\sigma(t, \nu) := \nu_t$ for all $t \in \mathbb{R}_+$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$, where ν_t is the t–shift of the function ν, i.e., $\nu_t(s) := \nu(t + s)$ for all $s \in \mathbb{R}_+$. It is easy to verify that:

1. $\sigma(0, \nu) = \nu$ for all $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;
2. $\sigma(t_1 + t_2, \nu) = \sigma(t_2, \sigma(t_1, \nu))$ for all $t_1, t_2 \in \mathbb{R}_+$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;
3. for all $t \in \mathbb{R}_+$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$ there exists $l = l(t, \nu) \in \mathbb{Z}_+$ such that $t \in [t_l^\nu, t_{l+1}^\nu)$, $\{t_k^{\nu_t}\} = \{t_{l+k}^\nu\} - t := \{t_{k+l}^\nu - t : k \in \mathbb{N}\}$ and $t_0^{\nu_t} := 0$.

Theorem 2.2 ([6, 29, 41])**.** *The mapping $\sigma : \mathbb{R}_+ \times S_\tau(\mathbb{R}, \mathcal{P}) \to S_\tau(\mathbb{R}_+, \mathcal{P})$ is continuous and, consequently, $(S_\tau(\mathbb{R}_+, \mathcal{P}), \mathbb{R}_+, \sigma)$ is a dynamical system on $S_\tau(\mathbb{R}_+, \mathcal{P})$.*

Switched Dynamical Systems.

Definition 2.3. A switched dynamical system [31] is a differential equation of the form

$$x' = f_{\nu(t)}(x) \tag{2.2}$$

where $\{f_p : p \in \mathcal{P}\}$ is a family of sufficiently regular functions from E^n on E^n parametrized by some finite index set $\mathcal{P}$, and $\nu : \mathbb{R}_+ \to \mathcal{P}$ is a piecewise constant function of time, called a switching signal.

For example, we can take the function f_p, $p \in \mathcal{P}$, locally Lipschitzian such that the equation $x' = f_p(x)$ generates on E^n a dynamical system $(E^n, \mathbb{R}_+, \pi_p)$.

Remark 2.4. Note that

1. a piecewise constant function $\nu : \mathbb{R}_+ \to E^n$ has at most a countable set $\{t_k^\nu\}$ of discontinuity points;

(2) without loss of the generality we may suppose that $t_k^\nu \to +\infty$ as $k \to +\infty$ (every finite segment $[a,b] \subset \mathbb{R}_+$ contains at most a finite number of points from $\{t_k^\nu\}$).

Definition 2.5. A positive number τ is called a dwell time of (2.2) if for arbitrary switching signal ν the interval between any two consecutive switching times is not smaller then τ, i.e., $t_{k+1}^\nu - t_k^\nu \geq \tau$ for all $k \in \mathbb{N}$.

Definition 2.6. A continuous function $\gamma : \mathbb{R}_+ \to E^n$is called a solution of switched system (2.2), if

$$\gamma(t) = \pi_{\nu(t)}(t - t_k^\nu, \gamma(t_k^\nu)) \tag{2.3}$$

for all $t \in [t_k^\nu, t_{k+1}^\nu)$ and $k \in \{0, 1, 2, \dots\}$.

Denote by $t \to \varphi(t, x, \nu)$ the solution of equation (2.2) with initial condition $\varphi(0, u, \nu) = u$, assuming that an unique solution exists for all $t \in \mathbb{R}_+$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times S_\tau(\mathbb{R}_+, \mathfrak{C})$ possesses the following properties:

(1) $\varphi(0, u, \nu) = u$ for all $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;
(2) $\varphi(t+s, u, \nu) = \varphi(s, \varphi(t, u, \nu), \sigma(t, \nu))$ for all $t, s \in \mathbb{R}_+$, $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$.

Theorem 2.7 ([29])**.** *The mapping $\varphi : \mathbb{R}_+ \times E^n \times S_\tau(\mathbb{R}_+, \mathcal{P}) \to E^n$ is continuous.*

Thus the triple $\langle E^n, \varphi, (S_\tau(\mathbb{R}_+, \mathcal{P}), \mathbb{R}_+, \sigma)\rangle$ is a cocycle under dynamical system $(S_\tau(\mathbb{R}_+, \mathcal{P}), \sigma)$ with the fiber E^n and, consequently, we can study the switched systems (1.1) in the framework of the non-autonomous (cocycle) systems (see, for example, [10, 41]).

Global attractors of dynamical systems. Let (X, ρ) be a complete metric space and $(X, \mathbb{R}_+, \pi)$ be a dynamical system on X.

Definition 2.8 ([10, ChI])**.** A dynamical system $(X, \mathbb{R}_+, \pi)$ is said to be

- pointwise dissipative if there exists a nonempty compact subset $K \subseteq X$ such that
$$\lim_{t\to+\infty} \rho(\pi(t, x), K) = 0 \tag{2.4}$$
for all $x \in X$ (in this case one say that the set K attracts every point x of X);
- compactly dissipative if the equality (2.4) holds uniformly with respect to x on every compact subset M from X; i.e., there exists a nonempty compact subset $K \subseteq X$ attracting every compact subset M in X.

Remark 2.9. It is clear that every compact dissipative dynamical system is pointwise dissipative. The pointwise dissipativity, generally speaking, does not imply the compact dissipativity (see [10, ChI]).

Theorem 2.10 ([10, ChI])**.** *If the metric space (X, ρ) is locally compact, then the dynamical system $(X, \mathbb{R}_+, \pi)$ is compactly dissipative if and only if it is pointwise dissipative.*

Let $M \subseteq X$; denote

$$\Omega(M) := \cap_{t\geq 0}\overline{\cup_{\tau\geq t}\pi(\tau, M)}.$$

If the dynamical system $(X, \mathbb{R}_+, \pi)$ is compactly dissipative and K is a compact subset from M which attracts every compact subset from X, then the set $J := \Omega(K)$

does not depend of the choice of the attracting set K and it is well defined only by dynamical system $(X, \mathbb{R}_+, \pi)$ (see, for example, [10, 26]). The set J is called [10] Levinson center for compactly dissipative dynamical system $(X, \mathbb{R}_+, \pi)$.

Definition 2.11. A subset $M \subseteq X$ is called:

- positively (respectively, negatively) invariant if $\pi(t, M) \subseteq M$ (respectively, $M \subseteq \pi(t, M)$) for all $t \in \mathbb{R}_+$;
- invariant if it is positively and negatively invariant, i.e., $\pi(t, M) = M$ for all $t \in \mathbb{R}_+$;
- orbitally stable if for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ such that $\rho(x, M) < \delta$ implies $\rho(\pi(t, x), M) < \varepsilon$ for all $t \in \mathbb{R}_+$;
- attracting if there exists a positive number α such that $\lim_{t\to+\infty} \rho(\pi(t, x), M) = 0$ for all $x \in B(M, \alpha) := \{x \in X : \rho(x, M) < \alpha\}$;
- asymptotically stable if it is an orbitally stable and attracting set.

Theorem 2.12 ([10, 26])**.** *Let $(X, \mathbb{R}_+, \pi)$ be compactly dissipative dynamical system. Then the following statements hold:*

1. *the Levinson center J of $(X, \mathbb{R}_+, \pi)$ is a nonempty, compact and invariant subset of X;*
2. *J is asymptotically stable;*
3. *J attracts every compact subset M from X;*
4. *J is a maximal compact invariant set in X, i.e., if J' is a compact invariant subset from X, then $J' \subseteq J$.*

Denote

$$\Omega_X := \overline{\cup\{\omega_x : x \in X\}},$$

where $\omega_x := \Omega(\{x\}) = \cap_{t\geq 0}\overline{\cup_{\tau\geq t}\pi(\tau, x)}$.

Theorem 2.13 ([10, ChI])**.** *The pointwise dissipative dynamical system $(X, \mathbb{R}_+, \pi)$ is compact dissipative if and only if there exists a nonempty compact set $M \subseteq X$ possessing the following properties:*

1. *$\Omega_X \subseteq M$;*
2. *M is orbitally stable.*

Remark 2.14. Under the conditions of Theorem 2.13 $J \subseteq M$, where J is the Levinson center of $(X, \mathbb{R}_+, \pi)$.

Asymptotic stability of switched systems. We are assuming here that every equation

$$x' = f_p(x) \tag{2.5}$$

$(p \in \mathcal{P})$ has a trivial equilibrium point: $f_p(0) = 0$, $p \in \mathcal{P}$. It is clear that in this case the switched system (2.2) has a trivial solution, i.e., $\varphi(t, 0, \nu) = 0$ for all $t \in \mathbb{R}_+$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$.

Definition 2.15. The trivial solution of the switched system (2.2) is called:

- uniformly (with respect to switching signals ν) stable if for every $\varepsilon > 0$ there exists a positive number $\delta = \delta(\varepsilon)$ (depending only of ε) such that $|u| < \delta$ implies $|\varphi(t, u, \nu)| < \varepsilon$ for all $t \in \mathbb{R}_+$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;
- attracting if there exists a positive number α such that

$$\lim_{t\to+\infty} |\varphi(t, u, \nu)| = 0 \tag{2.6}$$

  for all $|u| < \alpha$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;

- uniformly asymptotically stable if it is uniformly stable and attracting;
- uniformly globally asymptotically stable if it is uniformly asymptotically stable and (2.6) holds for all $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$.

Remark 2.16. Clearly, a necessary condition for uniformly (globally) asymptotically stability under arbitrary switching is that all of the individual subsystems (2.5) are (globally) asymptotically stable. On the other hand, the (global) asymptotic stability of all the individual subsystems (2.5) is not sufficient (see, for example, [31, 33] and also [19] (for the discrete switched systems)).

Definition 2.17. Let $\mathbb{R}_+ \subset \mathbb{T} \subset \mathbb{R}$. A continuous mapping $\gamma_x : \mathbb{T} \to X$ is called a motion of the dynamical system $(X, \mathbb{R}_+, \pi)$ issuing from the point $x \in X$ at the initial moment $t = 0$ and defined on $\mathbb{T}$, if

(a) $\gamma_x(0) = x$;
(b) $\gamma_x(t_2) \in \pi(t_2 - t_1, \gamma_x(t_1))$ for all $t_1, t_2 \in \mathbb{T}$ $(t_2 > t_1)$.

The set of all motions of $(X, \mathbb{R}_+, \pi)$, passing through the point x at the initial moment $t = 0$ is denoted by $\Phi_x(\pi)$ and $\Phi(\pi) := \cup\{\Phi_x(\pi) : x \in X\}$ (or simply Φ).

Definition 2.18. The trajectory $\gamma \in \Phi(\pi)$ defined on $\mathbb{R}$ is called a full (entire) trajectory of the dynamical system $(X, \mathbb{R}_+, \pi)$.

Definition 2.19. A continuous function $\gamma_\nu : \mathbb{R} \to E^n$ is said to be an entire solution of switched system (2.2) if there exists an entire trajectory $\tilde{\gamma} \in \Phi_\nu(\sigma)$ such that $\gamma_\nu(t + s) = \varphi(t, \gamma_\nu(s), \tilde{\gamma}(s))$ for all $t \in \mathbb{R}_+$ and $s \in \mathbb{R}$.

Theorem 2.20. *The trivial solution of switched system* (2.2) *is uniformly globally asymptotically stable if and only if the following conditions are fulfilled:*

(1) *every solution of switched system* (2.2) *is bounded on* $\mathbb{R}_+$*; i.e.,* $\sup_{t \in \mathbb{R}_+} |\varphi(t, u, \nu)| < +\infty$ *for all* $u \in E^n$ *and* $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$*;*
(2) *the switched system does not have nontrivial entire bounded on* $\mathbb{R}$ *solutions.*

Proof. Necessity: Let (2.2) be uniformly asymptotically stable. Let $X := E^n \times S_\tau(\mathbb{R}_+, \mathcal{P})$ and $(X, \mathbb{R}_+, \pi)$ be a skew-product dynamical system generated by cocycle φ; i.e., $\pi(t, (u, \nu)) := (\varphi(t, u, \nu), \sigma(t, \nu))$ for all $t \in \mathbb{R}_+$, $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$. At first we will prove that the skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ is compactly dissipative. To this end, we note that the space X is locally compact because $S_\tau(\mathbb{R}_+, \mathcal{P})$ is compact. According to Theorem 2.10, it is sufficient to establish its pointwise dissipativity. Let $x := (u, \nu) \in E^n \times S_\tau(\mathbb{R}_+, \mathcal{P})$, then the semi-trajectory $\Sigma_x := \cup_{t \geq 0} \pi(t, x)$ is relatively compact because $|\varphi(t, u, \nu)| \to 0$ as $t \to +\infty$. Thus $\omega_x \subseteq \Theta = \{0\} \times S_\tau(\mathbb{R}_+, \mathcal{P})$ for all $x \in X$, where ω_x is the ω–limit set of the point x. This means that $(X, \mathbb{R}_+, \pi)$ is pointwise dissipative and, hence, it is compactly dissipative too.

Now we will establish that under the conditions of Theorem the set Θ is orbitally stable. In fact, if we suppose that this is not true, then there are $\varepsilon_0 > 0$, $\delta_n \to 0$ $(\delta_n > 0)$ and $t_n \to +\infty$ such that

$$\rho(x_n, \Theta) < \delta_n \quad \text{and} \quad \rho(\pi(t_n, x_n), \Theta) \geq \varepsilon_0 \tag{2.7}$$

for all $n \in \mathbb{N}$. Since Θ is a compact set then we may suppose that $\{x_n\} := (u_n, \nu_n)$ is a convergent sequence. Let $\bar{x} := \lim_{n \to +\infty} x_n$, then by (2.7) there exists $\bar{\nu} \in S_\tau(\mathbb{R}_+, \mathcal{P})$ such that $\bar{x} = (0, \bar{\nu})$. On the other hand by compact dissipativity of the dynamical system $(X, \mathbb{R}_+, \pi)$ we may suppose that the sequence $\{\pi(t_n, x_n)\}$ is also convergent. Denote by $\tilde{x}$ its limit; i.e., $\tilde{x} := \lim_{n\to+\infty} \pi(t_n, x_n)$. From (2.7) it follows that $\rho(\tilde{x}, \Theta) \geq \varepsilon_0$ and, consequently, $|\tilde{u}| = \lim_{n\to+\infty} |\varphi(t_n, u_n, \nu_n)| \geq \varepsilon_0$. Thus $\tilde{x} := (\tilde{u}, \tilde{\nu}) \neq (0, \tilde{\nu})$. Now we denote by $\delta_0 = \delta(\varepsilon_0) > 0$ a positive number from the uniform stability of switched system (2.2). Then for a sufficiently large n we will have

$$|\varphi(t_n, x_n, \nu_n)| < \varepsilon_0/2 \tag{2.8}$$

and, consequently, $|\tilde{u}| \leq \varepsilon_0/2$. The obtained contradiction proves our statement.

Taking into account that the set Θ is compact, $\Omega_X \subseteq \Theta$ and Θ is orbitally stable according to Theorem 2.13 (see also Remark 2.14) we obtain $J \subseteq \Theta$. From this inclusion it follows that the switched system does not the nontrivial entire solutions bounded on $\mathbb{R}$. In fact, if $\gamma : \mathbb{R} \to E^n$ is a nontrivial entire solution of (2.2) which is bounded on $\mathbb{R}$, then $\gamma(s) := (\psi(s), \sigma(s, \nu))$ $(s \in \mathbb{R})$ is an entire trajectory of skew-product dynamical system $(X, \mathbb{R}_+, \pi)$ with relatively compact rang $\gamma(\mathbb{R})$ and, consequently, $\gamma(s) \in J$ for all $s \in \mathbb{R}$. Since ψ is a nontrivial solution of (2.2), then $\gamma(\mathbb{R}) \not\subseteq \Theta$. The obtained contradiction proves our statement.

Sufficiency. Let now all solutions of switched system (2.2) are bounded on $\mathbb{R}_+$ and (2.2) does not have nontrivial bounded on $\mathbb{R}$ entire solutions. Let $x := (u, \nu) \in E^n \times S_\tau(\mathbb{R}_+, \mathcal{P})$, then the semi-trajectory $\Sigma_x := \cup_{t\geq 0}\pi(t, x)$ is relatively compact because $\varphi(\mathbb{R}_+, u, \nu)$ is bounded. Thus the ω–limit set ω_x of the point x is nonempty, compact and invariant. Let $p := (\tilde{u}, \tilde{\nu}) \in \omega_x$, then there exists an entire trajectory $\psi : \mathbb{R} \to \omega_x$ of the dynamical system $(X, \mathbb{R}_+, \pi)$. It is easy to verify that the continuous function $\gamma := pr_2 \circ \psi$ $(pr_2 : X \to S_\tau(\mathbb{R}_+, \mathcal{P}))$ is an entire trajectory of the switched system (2.2) and $\gamma(\mathbb{R})$ is relatively compact. Since trivial solution is an unique bounded on $\mathbb{R}$ entire solution of (2.2) then $\gamma(t) = 0$ for all $t \in \mathbb{R}$. Thus $\omega_x \subseteq \Theta := \{0\} \times S_\tau(\mathbb{R}_+, \mathcal{P})$ for all $x \in X$ and, consequently, Ω_X is a compact subset because $\Omega_X \subseteq \Theta$. This means that $(X, \mathbb{R}_+, \pi)$ is pointwise dissipative and, hence, it is compactly dissipative too.

Let J the Levinson center of $(X, \mathbb{R}_+, \pi)$. We will show that $J = \Theta$. To prove this equality it is sufficient to establish the inclusion $J \subseteq \Theta$ because Θ is a compact and invariant subset of X and, consequently, $\Theta \subseteq J$ (according to Theorem 2.10 J is the maximal compact invariant subset of X). Let $x := (u, \nu) \in J$, then there exists an entire trajectory $\psi : \mathbb{R} \to J$ of dynamical system $(X, \mathbb{R}_+, \pi)$ such that $\psi(0) = x$. Then the function $\gamma : \mathbb{R} \to E^n$ defined by equality $\gamma(s) := pr_2(\psi(s))$ $(s \in \mathbb{R})$ is an entire solution of switched system (2.2) which is bounded on $\mathbb{R}$. Under the conditions of Theorem γ coincides with the trivial solution of (2.2) and, consequently, $x \in \Theta$; i.e., $J \subseteq \Theta$.

To finish the proof it is establish the uniform asymptotic stability of the trivial solution of (2.2). If we suppose that it is not true, then there are $\varepsilon_0 > 0$, $\delta_n \to 0$ $(\delta_n > 0)$ and $t_n \to +\infty$ such that

$$|x_n| < \delta_n \quad \text{and} \quad |\varphi(t_n, u_n, \nu_n)| \geq \varepsilon_0 \tag{2.9}$$

for all $n \in \mathbb{N}$. Since the dynamical system $(X, \mathbb{R}_+, \pi)$ is compactly dissipative, then we may suppose that the sequences $x_n := (u_n, \nu_n)$ and $\pi(t_n, x_n) = (\varphi(t_n, u_n, \nu_n), \sigma(t_n, \nu_n))$ are convergent. Let $\bar{x} := \lim_{n\to+\infty} x_n$ and $\tilde{x} := \lim_{n\to+\infty} \pi(t_n, x_n)$. It is clear that $\tilde{x} \in J \subseteq \Theta$. On the other hand from the inequality (2.9) we obtain $|\tilde{u}| \geq \varepsilon_0$, where $\tilde{x} = (\tilde{u}, \tilde{\nu})$ and, consequently, $\tilde{x} \notin \Theta$. The obtained contradiction proves our statement. $\square$

Corollary 2.21. *The trivial solution of switched system* (2.2) *is uniformly globally asymptotically stable if and only if the following conditions are fulfilled:*

(1) $\lim_{t\to+\infty}|\varphi(t,u,\nu)|=0$ *for all* $u\in E^n$ *and* $\nu\in S_\tau(\mathbb{R}_+,\mathcal{P})$*;*
(2) *the switched system does not have nontrivial entire bounded on* $\mathbb{R}$ *solutions.*

Proof. This statement it follows from Theorem 2.20. In fact, under the conditions of Theorem 2.20 it easy to see that for every bounded on $\mathbb{R}_+$ solution $\varphi(t,u,\nu)$ we have $\sup_{t\to+\infty}|\varphi(t,u,\nu)|=0$. □

Remark 2.22. (1) Note that in the particular case this statement was established by Kloeden [29]. Namely, in [29] it is proved that the trivial (zero) solution of switched system (2.2) is uniformly asymptotically stable (with respect to two-sided switched signals), if the trivial solution of each individual subsystem (2.5) is asymptotically stable, these systems have a common positively invariant absorbing set and the zero solution is the only bounded entire solution of switched system (2.2).

(2) Theorem 2.10 also holds for arbitrary non-autonomous dynamical systems $\langle (X,\mathbb{T}_+,\pi),(Y,\mathbb{T},\sigma),h\rangle$ if the following conditions are fulfilled:

(1) the space Y is compact and invariant (i.e., $\sigma(t,Y)=Y$ for all $t\in\mathbb{T}$);
(2) (X,h,Y) is a finite-dimensional vectorial fibering with the norm $|\cdot|$.

Criteria for asymptotic stability.

Lemma 2.23 ([16])**.** *Let* $f:\mathbb{R}_+\to\mathbb{R}_+$ *be a function satisfying the following conditions:*

(H1) $f(0)=0$*;*
(H2) $f(t)>0$ *for all* $t>0$*;*
(H3) $f:(0,+\infty)\to(0,+\infty)$ *is locally Lipschitz;*
(H4) f *satisfies the condition of Osgoode, i.e.,* $\int_0^\varepsilon\frac{ds}{f(s)}=+\infty$ *for all* $\varepsilon>0$.

Then, the equation

$$u'=-f(u)\tag{2.10}$$

admits an unique solution $\omega(t,r)$ *with initial condition* $\omega(0,r)=r$ *and the mapping* $\omega:\mathbb{R}_+^2\to\mathbb{R}_+$ *possesses the following properties:*

(1) *the mapping* $\omega:\mathbb{R}_+^2\to\mathbb{R}_+$ *is continuous;*
(2) $\omega(t,r)<r$ *for all* $r>0$ *and* $t>0$*;*
(3) *for all* $r>0$ *the mapping* $\omega(\cdot,r):\mathbb{R}_+\to\mathbb{R}_+$ *is decreasing;*
(4) *for all* $t\in\mathbb{R}_+$ *the mapping* $\omega(t,\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *is increasing;*
(5) $\omega(t,0)=0$ *for all* $t\in\mathbb{R}_+$*;*
(6) $\lim_{t\to+\infty}\omega(t,r)=0$ *for all* $r>0$.

Theorem 2.24. *Suppose that there exists a function* $\theta\in C(\mathbb{R},\mathbb{R})$ *satisfying the above properties* (H1)-(H4) *and the inequality*

$$\langle u,f_p(u)\rangle\le-\theta(|u|^2)\tag{2.11}$$

for all $u\in E^n$ *and* $p\in\mathcal{P}$*. Then the following statements hold:*

(1)
$$|\varphi(t,u,\nu)|^2\le\omega(t,|u|^2)$$
for all $t\ge0$ *and* $(u,\nu)\in E^n\times S_\tau(\mathbb{R}_+,\mathcal{P})$*, where* $t\mapsto\omega(t,r)$ *is the solution of equation* $x'=-\theta(x)$ *with the initial condition* $\omega(0,r)=r$*;*
(2) *the trivial solution of switched system* (2.2) *is globally uniformly asymptotically stable.*

Proof. Denote by $(E^n,\mathbb{R}_+,\pi_p)$ $(p\in\mathcal{P})$ the dynamical system generated by equation (2.5). Let now $\nu\in S_\tau(\mathbb{R}_+,\mathcal{P})$, $\{t_k^\nu\}_{k\in\mathbb{Z}+}$ the set of points of discontinuity of u and $t\in\mathbb{R}_+$. Then there exists $k\in\mathbb{Z}_+$ such that $t_k^\nu\le t<t_{k+1}^\nu$ and $\nu(t)=p_k\in\mathcal{P}$ (for all $t\in[t_k^\nu,t_{k+1}^\nu)$). Hence, we have the equality

$$\varphi(t,u,\nu)=\pi_{p_k}(t-t_k^\nu,\varphi(t_k^\nu,u,\nu)). \tag{2.12}$$

According to the condition (2.11) we obtain

$$\frac{d|\pi_{p_k}(t,u)|^2}{dt}\le-\theta(|\pi_{p_k}(t,u)|^2) \tag{2.13}$$

and, consequently,

$$|\pi_{p_k}(t,u)|^2\le\omega(t,|u|^2) \tag{2.14}$$

for all $t\ge 0$ and $u\in E^n$. From (2.12) and (2.14) we have

$$|\varphi(t,u,\nu)|^2=|\pi_{p_k}(t-t_k^\nu,\varphi(t_k^\nu,u,\nu))|^2\le\omega(t-t_k^\nu,|\varphi(t_k^\nu,u,\nu)|^2) \tag{2.15}$$

for all $t\in(t_k^\nu,t_{k+1}^\nu)$, $u\in E^n$ and $\nu\in S_\tau(\mathbb{R},\mathcal{P})$. Denote by $b_k(u,\nu):=|\varphi(t_k^\nu,u,\nu)|^2$ (for all $k\in\mathbb{Z}_+$), then by inequality (2.15) we obtain

$$b_{k+1}(u,\nu)\le\omega(t_{k+1}^\nu-t_k^\nu,b_k(u,\nu)) \tag{2.16}$$

for all $k\in\mathbb{Z}_+$ and $(u,\nu)\in E^n\times S_\tau(\mathbb{R}_+,\mathcal{P})$. We note that

$$\begin{gathered}
b_0(u,\nu)=|\varphi(t_0^\nu,u,\nu|^2\le\omega(t_0^\nu,|u|^2)\\
b_1(u,\nu)\le\omega(t_1^\nu-t_0^\nu,|\varphi(t_0,u,\nu)|^2)\le\omega(t_1^\nu-t_0^\nu,\omega(t_0^\nu,|u|^2))=\omega(t_1^\nu,|u|^2)\\
\dots\\
b_{k+1}(u,\nu)\le\omega(t_{k+1}^\nu-t_k^\nu,b_k(u,\nu))\le\omega(t_{k+1}^\nu-t_k^\nu,\omega(t_k^\nu,|u|^2))=\omega(t_{k+1}^\nu,|u|^2).
\end{gathered} \tag{2.17}$$

Now, using the (2.12), (2.15) and (2.17), we have

$$\begin{aligned}
|\varphi(t,u,\nu)|^2&\le\omega(t-t_k^\nu,|\varphi(t_k^\nu,u,\nu)-\varphi(t_k^\nu,u,\nu)|^2)\\
&\le\omega(t-t_k^\nu,b_k(u,\nu))\\
&\le\omega(t-t_k^\nu,\omega(t_k^\nu,|u|^2))=\omega(t,|u|^2)
\end{aligned} \tag{2.18}$$

for all $t\ge0$ and $(u,\nu)\in E^n\times S_\tau(\mathbb{R}_+,\mathcal{P})$. From the inequality (2.18) and Lemma 2.23 it follows that $\lim_{t\to+\infty}|\varphi(t,u,\nu|=0$ for all $(u.\nu)\in E^n\times S_\tau(\mathbb{R}_+,\mathcal{P})$. Thus to finish the proof of Theorem it is establish the uniform stability of trivial solution of switched system (2.2). Let ε be an arbitrary positive number and $0<\delta<\varepsilon$, then if $|u|<\delta$ by inequality (2.18) and Lemma 2.23 we have $|\varphi(t,u,\nu)|^2\le\omega(t,|u|^2)<|u^2|<\delta^2<\varepsilon^2$ and, consequently, $|\varphi(t,u,\nu)|<\varepsilon$ for all $t\in\mathbb{R}_+$ and $\nu\in S_\tau(\mathbb{R}_+,\mathcal{P})$. □

Example 2.25. As an illustration of Theorem 2.24 we consider the switched system (2.2) with functions $f_p(x)$ $(p\in\mathcal{P})$ satisfying the condition $\langle u,f_p(u)\rangle\le-\alpha|u|^\beta$ for all $u\in E^n$, where $\alpha>0$ and $\beta\ge2$ (for example $f_p(u):=-\alpha_p u|u|^\beta$ ($\beta\ge0$ and $\alpha_p>0$ for all $p\in\mathcal{P}$), then $\theta(x):=\alpha x^{1+\beta/2}$ and $\alpha:=\min_{p\in\mathcal{P}}\alpha_p$).

Lemma 2.26. *Let $\omega:\mathbb{R}_+^2\to\mathbb{R}_+$ be a continuous function with the properties*

1. $\omega(t+\tau,r)\le\omega(t,\omega(\tau,r))$ *for all* $t,\tau,r\in\mathbb{R}_+$;
2. $\omega(t,r)<r$ *for all* $r>0$ *and* $t>0$;
3. *for all* $t\in\mathbb{R}_+$ *the mapping* $\omega(t,\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *is increasing;*
4. $\omega(t,0)=0$ *for all* $t\in\mathbb{R}_+$.

Then $\lim_{t\to+\infty}\omega(t,r)=0$ *for every* $r>0$.

Proof. Let τ and r be arbitrary positive numbers and $\{x_k\}$ a sequence defined by $x_k:=\omega(k\tau,r)\}$ ($k\in\mathbb{N}$). Under the conditions of Lemma we have $x_{k+1}<x_k$ for all $k\in\mathbb{N}$ and, consequently, $\{x_n\}$ converges. Denote by $c:=\lim_{k\to+\infty}x_n$, then $\omega(\tau,c)=c$. It follows that $c=0$ because, if $c>0$, then $\omega(\tau,c)<c$.

We will prove that $\lim_{t\to+\infty}\omega(t,r)=0$ for every $r>0$. If we suppose that it is not so, then there exist $r_0>0$, $\varepsilon_0>0$ and a sequence $\{t_k\}$ such that $t_k\to+\infty$ and

$$\omega(t_k,r_0)\ge\varepsilon_0. \tag{2.19}$$

Let $\tau>0$, then there exist $n_k\in\mathbb{N}$ and $\tau_k\in[0,\tau)$ such that $t_k=n_k\tau+\tau_k$. From (2.19) we have

$$\varepsilon_0\le\omega(t_k,r_0)=\omega(n_k\tau+\tau_k,r_0)\le\omega(\tau_k,x_k^0), \tag{2.20}$$

where $x_k^0:=\omega(n_k\tau,r_0)$. By reasoning above the sequence $\{x_k^0\}$ converges to 0. Since $\tau_k\in[0,\tau)$ without loss of generality we may suppose that the sequence $\{\tau_k\}$ is convergent. Denote by τ_0 its limit, then $\tau_0\in[0,\tau]$. Passing into limit in (2.20) as $k\to+\infty$ and taking into account the established above facts we will have $\varepsilon_0\le\omega(\tau_0,0)=0$. The obtained contradiction proves our statement. □

Theorem 2.27. *Suppose that there exists a continuous function* $\omega:\mathbb{R}_+^2\to\mathbb{R}_+$ *with the properties*

(1) $\omega(t+\tau,r)\le\omega(t,\omega(\tau,r))$ *for all* $t,\tau,r\in\mathbb{R}_+$*;*
(2) $\omega(t,r)<r$ *for all* $r>0$ *and* $t>0$*;*
(3) *for all* $t\in\mathbb{R}_+$ *the mapping* $\omega(t,\cdot):\mathbb{R}_+\to\mathbb{R}_+$ *is increasing.*

If $|\pi_p(t,x)|\le\omega(t,|x|)$ *for all* $(t,x)\in\mathbb{R}_+\times E^n$ *and* $p=1,\dots,m$*, then the trivial solution of switched system* (2.2) *is globally uniformly asymptotically stable.*

Proof. This statement may be proved using the same reasoning as in the proof of Theorem 2.24 and taking into account Lemma 2.26. □

3. Homogeneous switched systems

Asymptotic stability of homogeneous switched systems.

Definition 3.1. Let X be a linear space. A dynamical system $(X,\mathbb{R}_+,\pi)$ is said to be homogeneous of order k ($k\ge1$) if $\pi(t,\lambda x)=\lambda\pi(\lambda^{k-1}t,x)$ for all $\lambda>0$, $x\in X$ and $t\in\mathbb{R}_+$.

Remark 3.2. Let $f:E^n\to E^n$ be a regular function; i.e., the equation

$$x'=f(x) \tag{3.1}$$

generates on E^n a dynamical system $(X,\mathbb{R}_+,\pi)$, where $t\to\pi(t,x)$ is an unique solution of equation (3.1), defined on $\mathbb{R}_+$ and passing through point $x\in E^n$ at the initial moment $t=0$. If the function f is homogeneous of order k (i.e., $f(\lambda x)=\lambda^kf(x)$ for all $x\in E^n$ and $\lambda>0$), then the dynamical system $(X,\mathbb{R}_+,\pi)$, generated by the equation (3.1), is homogeneous of order k.

Theorem 3.3 ([10, ChII])**.** *Let* X *be a Banach space,* $(X,\mathbb{R}_+,\pi)$ *be an homogeneous of order* k *(*$k\ge1$*) dynamical system and* $\pi(t,0)=0$ *for all* $t\in\mathbb{R}_+$*. Then the following conditions are equivalent:*

(1) *the trivial motion of dynamical system* $(X, \mathbb{R}_+, \pi)$ *is uniformly asymptotically stable;*
(2) *if* $k = 1$ *(respectively,* $k > 1$*), then there exist two positive numbers* $\mathcal{N}$ *and* α *(respectively,* α *and* β*) such that*

$$|\pi(t,x)| \le \mathcal{N} e^{-\alpha t}|x| \quad (\textit{respectively,} \quad |\pi(t,x)| \le (\alpha|x|^{1-k} + \beta t)^{-\frac{1}{k-1}})$$

for all $x \in X$ *and* $t \in \mathbb{R}_+$.

Definition 3.4. A switched system (2.2) is said to be homogeneous of order k ($k \ge 1$), if every function f_p ($p \in \mathcal{P}$) is homogeneous of order k.

Definition 3.5. A non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ is said to be homogeneous [10] (of order $k = 1$) if the following conditions are fulfilled:
(1) (X, h, Y) is a vectorial bundle fiber;
(2) $\pi(t, \lambda x) = \lambda \pi(t, x)$ for all $t \in \mathbb{R}_+$, $x \in X$, and $\lambda > 0$.

Let (X, h, Y) be a vector bundle, $X_y := \{x \in X : h(x) = y\}$ and θ_y be the null element of X_y . Denote by $\Theta := \cup\{\theta_y : y \in Y\}$ the null (trivial) section of (X, h, Y).

Definition 3.6. The trivial section Θ of non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ is said to be globally asymptotically stable if the following conditions hold:

- $\lim_{t\to+\infty} |\pi(t, x)| = 0$ for all $x \in X$;
- for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(t, x)| < \varepsilon$ for all $t \in \mathbb{R}_+$.

Theorem 3.7 ([9],[10, ChII])**.** *Let* (X, h, Y) *be a finite-dimensional vectorial fiber bundle,* $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ *be an homogeneous non-autonomous dynamical system and* Y *be compact and invariant (i.e.,* $\sigma(t, Y) = Y$ *for all* $t \in \mathbb{R}_+$*), then the following statements are equivalent:*
(1) $\liminf_{t\to+\infty} |\pi(t, x)| = 0$ *for every* $x \in X$*;*
(2) $\lim_{t\to+\infty} |\pi(t, x)| = 0$ *for every* $x \in X$*;*
(3) *the trivial section* Θ *is globally asymptotically stable;*
(4) $\lim_{t\to+\infty} \sup_{|x|\le r} |\pi(t, x)| = 0$ *for every* $r > 0$*;*
(5) *there are two positive numbers* $\mathcal{N}$ *and* α *such that* $|\pi(t, x)| \le \mathcal{N} e^{-\alpha t}|x|$ *for all* $x \in X$ *and* $t \in \mathbb{R}+$.

Theorem 3.8. *Let* (2.2) *be an homogeneous switched system of order* $k = 1$*. Then the following statements are equivalent:*
(1) $\liminf_{t\to+\infty} |\varphi(t, u, \nu)| = 0$ *for all* $u \in E^n$ *and* $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$*;*
(2) $\lim_{t\to+\infty} |\varphi(t, u, \nu)| = 0$ *for all* $u \in E^n$ *and* $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$*;*
(3) *the trivial solution of* (2.2) *is globally asymptotically stable;*
(4) $\lim_{t\to+\infty} \sup_{|u|\le r,\ \nu\in S_\tau(\mathbb{R}_+,\mathcal{P})} |\varphi(t, u, \nu)| = 0$ *for every* $r > 0$*;*
(5) *there are two positive numbers* $\mathcal{N}$ *and* α *such that* $|\varphi(t, u, \nu)| \le \mathcal{N} e^{-\alpha t}|u|$ *for all* $u \in E^n$*,* $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$ *and* $t \in \mathbb{R}+$.

Proof. Let (2.2) be an homogeneous switched system of order $k = 1$, $X := E^n \times S_\tau(\mathbb{R}_+, \mathcal{P})$ and $(X, \mathbb{R}_+, \pi)$ be a skew-product dynamical system generated by cocycle φ; i.e., $\pi(t, (u, \nu)) := (\varphi(t, u, \nu), \sigma(t, \nu))$ for all $t \in \mathbb{R}_+$, $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$. Then it is easy to show that the non-autonomous dynamical system

$\langle (X,\mathbb{R}_+,\pi),(S_\tau(\mathbb{R}_+,\mathcal{P}),\mathbb{R}_+,\sigma),h\rangle$, where $h := pr_2 : X \to S_\tau(\mathbb{R}_+,\mathcal{P})$, generated by (2.2) is homogeneous of order $k = 1$. Since the space $S_\tau(\mathbb{R}_+,\mathcal{P})$ is compact and invariant with respect to translations (i.e., $\sigma(t, S_\tau(\mathbb{R}_+,\mathcal{P})) = S_\tau(\mathbb{R}_+,\mathcal{P})$ for all $t \in \mathbb{R}_+$), then to finish the proof of Theorem 3.8 it is sufficient to apply Theorem 3.7. □

Remark 3.9. (a) Theorem 3.8 (equivalence of the conditions (2) and (3)) refines Theorem 2.10 (see also Corollary 2.21) for homogeneous switched systems.

(b) The equivalence of the conditions (1), (2), (3) and (5). was established by Angeli D. [1].

Slow homogeneous switched systems.

Definition 3.10. A switched system (2.2) is said to be slow if its dwell time τ is large enough.

Theorem 3.11. *Suppose that the following conditions are fulfilled:*

1. *every individual equation* (2.5) *admits a trivial asymptotically stable solution;*
2. *the switched system* (2.2) *is homogeneous of order* $k = 1$*;*
3. *the switched system* (2.2) *is slow.*

Then the switched system (2.2) *is globally asymptotically stable.*

Proof. Suppose that (2.2) has order of the homogeneity $k = 1$. Then by Theorem 3.3, there are positive numbers $\mathcal{N}_p$ and α_p ($p \in \mathcal{P}$) such that

$$|\pi_p(t,u)| \le \mathcal{N}_p e^{-\alpha_p t}|u| \tag{3.2}$$

for all $t \in \mathbb{R}_+$, $u \in E^n$, and $p \in \mathcal{P}$. Denote by $\mathcal{N} := \max_{p\in\mathcal{P}} N_p$ and $\alpha := \min_{p\in\mathcal{P}} \alpha_p$. Now we will choose the number τ such that $\delta := \mathcal{N}e^{-\alpha\tau} < 1$, then from (3.2) we have

$$\begin{aligned}
|\varphi(t,u,\nu)| &= |\pi_{p_k}(t-t_k^\nu, \varphi(t_k^\nu,u,\nu))| \\
&\le \mathcal{N}e^{-\alpha(t-t_k^\nu)}|\varphi(t_k^\nu,u,\nu)| \\
&= \mathcal{N}|\pi_{p_{k-1}}(t_k^\nu - t_{k-1}^\nu, \varphi(t_{k-1}^\nu,u,\nu))| \\
&\le \mathcal{N}\delta|\varphi(t_{k-1}^\nu,u,\nu))| \le \dots \\
&\le \mathcal{N}\delta^k|\varphi(t_0^\nu,u,\nu))| \qquad\qquad \le \mathcal{N}^2\delta^k|u|
\end{aligned} \tag{3.3}$$

for all $t \in [t_{k+1}^\nu, t_k^\nu)$. Since $k \to +\infty$ as $t \to +\infty$, then from (3.3) we obtain

$$\lim_{t\to+\infty} \sup_{|u|\le r,\ \nu\in S_\tau(\mathbb{R}_+,\mathcal{P})} |\varphi(t,u,\nu)| = 0$$

for every $r > 0$. Now to finish the proof, it is sufficient to apply Theorem 3.7. □

Remark 3.12. (1) For the linear switched systems (i.e., $f_p(u) = A_p u$ for all $u \in E^n$ and $p \in \mathcal{P}$, where A_p ($p \in \mathcal{P}$) is a linear operator acting on E^n) Theorem 3.11 was established by Morse [36] (see also [27]).

(2) Theorem 3.11 also holds for the infinite-dimensional switched systems (2.2).

Theorem 3.13. *Suppose that there exists a continuous function* $\omega : \mathbb{R}_+^2 \to \mathbb{R}_+$ *with the properties*

1. *there exists a positive number* τ_0 *such that* $\omega(\tau_0, r) < r$ *for all* $r > 0$*;*
2. *for all* $t \in \mathbb{R}_+$ *the mapping* $\omega(t,\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing;*

(3) $\omega(t,0)=0$ *for all* $t\in\mathbb{R}_+$*;*
(4) $\lim_{t\to+\infty}\omega(t,r)=0$ *for every* $r>0$.

If $|\pi_i(t,x)|\le\omega(t,|x|)$ *for all* $(t,x)\in\mathbb{R}_+\times E^n$ *and* $\tau\ge\tau_0$*, then the trivial solution of switched system* (2.2) *is globally uniformly asymptotically stable.*

Proof. Let $t\in\mathbb{R}_+$, $x\in E^n$, $\nu\in S_\tau(\mathbb{R}_+,\mathcal{P})$ and $\tau\ge\tau_0$, then there exists an unique $k\in\mathbb{N}$ such that $t\in[t_k^\nu,t_{k+1}^\nu)$. Then we have

$$|\varphi(t,u,\nu)|=|\pi_{p_k}(t-t_k^\nu,\varphi(t_k^\nu,u,\nu))|\le\omega(t-t_k^\nu,|\varphi(t_k^\nu,u,\nu)|)\le\omega(\tau_0,|\varphi(t_k^\nu,u,\nu)|) \tag{3.4}$$

and, consequently,

$$c_{k+1}(u,\nu)\le\omega(\tau_0,c_k(u,\nu)), \tag{3.5}$$

where $c_k(u,\nu):=|\varphi(t_k^\nu,u,\nu)|$. From the inequality (3.5) we have

$$c_k(u,\nu)\le\omega(k\tau_0,|u|)\le\omega(k\tau_0,r) \tag{3.6}$$

for all $k\in\mathbb{N}$, $\nu\in S_\tau(\mathbb{R}_+,\mathcal{P})$ and $u\in E^n$ with condition $|u|\le r$. Using the same reasoning as in the proof of Lemma 2.26 we may prove that the sequence $\{\omega(k\tau_0,r)\}$ is convergent and its limit is equal to zero. Since $k\to+\infty$ as $t\to+\infty$, then from (3.3) we obtain

$$\lim_{t\to+\infty}\sup_{|u|\le r,\ \nu\in S_\tau(\mathbb{R}_+,\mathcal{P})}|\varphi(t,u,\nu)|=0 \tag{3.7}$$

for every $r>0$. Now to complete the proof, it is sufficient to show that from the condition (3.7) it follows the uniform stability of trivial solution of switched system (2.2). If we suppose that it is not so, then there are $\varepsilon_0>0$, $\delta_k\to0$, $u_k\in E^n$ with $|u_k|<\delta_k$, $t_k\to+\infty$, and $\nu_k\in S_\tau(\mathbb{R}_+,\mathcal{P})$ such that

$$|\varphi(t_k,u_k,\nu_k)|\ge\varepsilon_0 \tag{3.8}$$

for all $k\in\mathbb{N}$. On the other hand we have

$$\varepsilon\le|\varphi(t_k,u_k,\nu_k)|\le\sup_{|u|\le r_0,\ \nu\in S_\tau(\mathbb{R}_+,\mathcal{P})}|\varphi(t,u,\nu)|, \tag{3.9}$$

where $r_0:=\sup\{\delta_k|\ k\in\mathbb{N}\}$. Passing into limit in (3.8) as $k\to+\infty$ and taking into account (3.7) we get $\varepsilon_0\le0$. The obtained contradiction ends the proof of Theorem. ☐

Remark 3.14. (1) If $\omega(t,r)=\mathcal{N}e^{-\alpha t}r$, then from Theorem 2.27 we obtain Theorem 3.11.

(2) Note that the problem of asymptotic stability of slow homogeneous switched system is solved by Theorem 3.11 (in the case when $k=1$) and it is open in the general case (i.e., in the case when $k>1$).

(3) Suppose that that every individual system (2.5) is asymptotically stable and homogeneous of order $k>1$. Then by Theorem 3.3 there are positive numbers $a_p\le1$ and b_p such that

$$|\pi_p(t,u)|\le\frac{|u|}{(a_p+b_p|u|^{k-1}t)^{1/(k-1)}} \tag{3.10}$$

for all $t\in\mathbb{R}_+$, $u\in E^n$ and $p=1,\dots,m$. Denote by

$$\omega(t,r):=\frac{r}{(a+br^{k-1}t)^{1/(k-1)}},$$

where $a := \min\{a_p|\ p = 1,\dots,m\}$ and $b := \min\{b_p : p = 1,\dots,m\}$. Then under the assumptions above we have $|\pi_p(t,u)| \le \omega(t,|u|)$ for all $t \in \mathbb{R}_+$, $u \in E^n$ and $p = 1,\dots,m$. According to the definition of the number a, we have $0 < a \le 1$.

(3.1) If $a = 1$, then by Theorem 2.27 the switched system (2.2) will be globally uniformly asymptotically stable.

(3.2) If $a < 1$ the problem of asymptotic stability of switched system (2.2) remains open.

4. Asymptotic stability of linear switched systems

Linear Non-autonomous Dynamical Systems. Let W, Y be two complete metric spaces and $(Y,\mathbb{R}_+,\sigma)$ be a semi-group dynamical system on Y.

Definition 4.1 ([41]). Recall that the triplet $\langle W,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ (or shortly φ) is called a cocycle over $(Y,\mathbb{R}_+,\sigma)$ with fiber W if φ is a continuous mapping from $\mathbb{R}_+ \times W \times Y$ to W satisfying the following conditions:

(a) $\varphi(0,x,y) = x$ for all $(x,y) \in W \times Y$;

(b) $\varphi(t+\tau,x,y) = \varphi(t,\varphi(\tau,x,y),\sigma(\tau,y))$ for all $t,\tau \in \mathbb{R}_+$ and $(x,y) \in W \times Y$.

If W is a Banach space and

(c) $\varphi(t,\lambda x_1+\mu x_2,y) = \lambda\varphi(t,x_1,y)+\mu\varphi(t,x_2,y)$ for all $\lambda,\mu \in \mathbb{R}$ (or $\mathbb{C}$), $x_1,x_2 \in W$ and $y \in Y$,

then the cocycle φ is called linear.

Definition 4.2. Let $\langle W,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ be a cocycle (respectively, linear cocycle) over $(Y,\mathbb{R}_+,\sigma)$ with the fiber W (or shortly φ). If $X := W \times Y, \pi := (\varphi,\sigma)$, i.e., $\pi((u,y),t) := (\varphi(t,x,y),\sigma(t,\omega))$ for all $(u,\omega) \in W \times \Omega$ and $t \in \mathbb{R}_+$, then the dynamical system $(X,\mathbb{R}_+,\pi)$ is called [41] a skew product dynamical system over $(Y,\mathbb{R}_+,\sigma)$ with the fiber W.

Definition 4.3. Let $(X,\mathbb{R}_+,\pi)$ and $(Y,\mathbb{R}_+,\sigma)$ be two dynamical systems and $h : X \to Y$ be a homomorphism from $(X,\mathbb{R}_+,\pi)$ onto $(Y,\mathbb{R}_+,\sigma)$. A triplet $\langle(X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R}_+,\sigma)$, $h\rangle$ is called a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ over the dynamical system $(Y,\mathbb{R}_+,\sigma)$ with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle(X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R}_+,\sigma),h\rangle$ $(X := W\times Y)$, which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W,\varphi,(Y,\mathbb{R}_+,\sigma)\rangle$ over $(Y$, $\mathbb{R}_+$, $\sigma)$.–

Let $\langle(X,\mathbb{R}_+,\pi)$, $(Y,\mathbb{R}_+,\sigma)$, $h\rangle$ be a non-autonomous dynamical system. Denote by $X^s := \{x \in X :\ \lim_{t\to+\infty}|\pi(t,x)| = 0\}$, $X^s_y := X^s \cap X_y$, and $X_y := h^{-1}(y)$ $(y \in Y)$.

Let (X,h,Y) be a locally trivial vectorial fiber bundle [5, 28].

Definition 4.4. A non-autonomous dynamical system $\langle(X,R_+,\pi),(Y,R_+,\sigma),h\rangle$ is said to be linear, if the map $\pi^t : X_y \to X_{\sigma(t,y)}$ is linear for every $t \in R_+$ and $y \in Y$, where $\pi^t := \pi(t,\cdot)$.

Definition 4.5. The entire trajectory of the semigroup dynamical system $(X,\mathbb{T},\pi)$ passing through the point $x \in X$ at $t = 0$ is defined as the continuous map $\gamma : \mathbb{R} \to X$ that satisfies the conditions $\gamma(0) = x$ and $\pi^t\gamma(s) = \gamma(s+t)$ for all $t \in \mathbb{R}_+$ and $s \in \mathbb{R}$, where $\pi^t := \pi(t,\cdot)$.

Let $\Phi_x(\pi)$ be the set of all entire trajectories of $(X, \mathbb{R}_+, \pi)$ passing through x at $t = 0$ and $\Phi(\pi) = \cup\{\Phi_x(\pi) : x \in X\}$.

Definition 4.6. Let (X, h, Y) be a finite-dimensional vectorial fiber bundle with the norm $|\cdot|$. The non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}_+, \sigma), h\rangle$ is said to be non-critical [40] (satisfying Favard's condition) if $B(\pi) = \Theta$, where $B(\pi) := \{\gamma \in \Phi(\pi) : \ \sup_{s\in\mathbb{R}} |\gamma(s)| < +\infty\}$ and $\Theta := \{\theta_\omega : \theta_y \in X_y, |\theta_y| = 0, y \in Y\}$.

Definition 4.7. The linear non-autonomous dynamical system $\langle (X, \mathbb{R}_+, \pi)$, $(Y, \mathbb{R}_+, \sigma)$, $h\rangle$ is said to be:

- convergent, if$\lim_{t\to\infty} |\pi(t, x)| = 0$ for all $x \in X$;
- uniformly stable, if for all $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $|x| < \delta$ implies $|\pi(t, x)| < \varepsilon$ for all $t \ge 0$;
- uniformly asymptotically stable, if it is uniformly stable and convergent;
- uniformly exponentially stable, if there are two positive numbers $\mathcal{N}$ and ν such that
$$|\pi(t, x)| \le \mathcal{N} e^{-\nu t}|x| \tag{4.1}$$
for all $x \in X$ and $t \ge 0$.

Theorem 4.8 ([10])**.** *Let Y be a compact space and*
$$\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle \tag{4.2}$$
be a linear non-autonomous dynamical system. Then the following conditions are equivalent:

(1) *the non-autonomous dynamical system (4.2) is convergent;*
(2) *the non-autonomous dynamical system (4.2) is uniformly asymptotically stable;*
(3) *the non-autonomous dynamical system (4.2) is uniformly exponentially stable.*

Definition 4.9. A point $y \in Y$ is said to be Poisson stable if $y \in \omega_y$; i.e., there exists a sequence $\{t_k\} \subseteq \mathbb{R}_+$ such that $t_k \to +\infty$ and $\sigma(t_k, y) \to y$ as $k \to +\infty$.

Theorem 4.10 ([15])**.** *Let $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ be a linear non-autonomous dynamical system and the following conditions be fulfilled:*

(1) *$\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ is non-critical* [40]*;*
(2) *Y is compact and invariant ($\pi^t Y = Y$ for all $t \in \mathbb{R}_+$);*
(3) *there exists a Poisson stable point $y \in Y$ such that $H^+(y) = Y$, where $H^+(y) := \overline{\{\sigma(t, y)|\ t \in \mathbb{R}_+\}}$ and by bar is denoted the closure in Y;*
(4) *there exists at least one asymptotical stable fiber X_{y_0} (i.e., $X^s_{y_0} = X_{y_0}$).*

Then $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ is asymptotically stable; i.e., $X = X^s$.

Linear Switched Systems. Let $\varphi(t, x, \omega)$ be the solution of the linear switched system
$$x' = A_{\nu(t)}(x) \tag{4.3}$$
with initial condition $\varphi(0, x, \omega) = x$, assuming that a unique solution exists for all $t \in \mathbb{R}_+$. Then the mapping $\varphi : \mathbb{R}_+ \times E^n \times S_\tau(\mathbb{R}_+, \mathcal{P})$ possesses the following properties:

(1) $\varphi(0, u, \nu) = u$ for all $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;

(2) $\varphi(t+s, u, \nu) = \varphi(s, \varphi(t, u, \nu), \sigma(t, \nu))$ for all $t, s \in \mathbb{R}_+$, $u \in E^n$ and $\nu \in S_\tau(\mathbb{R}_+, \mathcal{P})$;
(3) the mapping φ is continuous;
(4) $\varphi(t, \lambda_1 u_1 + \lambda_2 u_2, \nu) = \lambda_1 \varphi(t, u_1, \nu) + \lambda_2 \varphi(t, u_2, \nu)$ for all $\lambda_1, \lambda_2 \in \mathbb{R}$ (or $\mathbb{C}$), $u_1, u_2 \in E^n$, and $\omega \in S_\tau(\mathbb{R}_+, \mathcal{P})$.

Thus the triplete $\langle E^n, \varphi, (S_\tau(\mathbb{R}_+, \mathcal{P}), \mathbb{R}_+, \sigma)\rangle$ is a linear cocycle under dynamical system $(S_\tau(\mathbb{R}_+, \mathcal{P}), \sigma)$ with the fiber E^n and, consequently, we can study the linear switched systems (4.3) in the framework of the linear non-autonomous (cocycle) systems (see, for example, [10, 41]).

Theorem 4.11. *Let $\langle E^n, \varphi, (S_\tau(\mathbb{R}_+, \mathcal{P}), \mathbb{R}_+, \sigma)\rangle$ be a linear cocycle generated by linear switched system* (4.3)*, then the following conditions are equivalent:*
(1) *the linear switched system* (4.3) *is convergent;*
(2) *the linear switched system* (4.3) *is uniformly asymptotically stable;*
(3) *the linear switched system* (4.3) *is uniformly exponentially stable.*

This above statement follows directly from Theorem 4.8.

Remark 4.12. The equivalence of the second and third statements this is a well known fact (see, for example, [35]). The equivalence of the first and third statements is a new result for the linear switched systems (4.3).

Theorem 4.13 ([11])**.** *The following statements hold:*
(1) $S_\tau(\mathbb{R}_+, \mathcal{P}) = \overline{\mathrm{Per}(\sigma)}$*, where* $\mathrm{Per}(\sigma)$ *is the set of all periodic points of* $(S_\tau(\mathbb{R}_+, \mathcal{P}), \mathbb{R}_+, \sigma)$ *(i.e., $\varphi \in \mathrm{Per}(\sigma)$, if there exists $h > 0$ such that $\sigma(h+t, \varphi) = \sigma(t, \varphi)$ for all $t \in \mathbb{R}_+$);*
(2) $S_\tau(\mathbb{R}_+, \mathcal{P})$ *is invariant, i.e., $\sigma^t S_\tau(\mathbb{R}_+, \mathcal{P}) = S_\tau(\mathbb{R}_+, \mathcal{P})$ for all $t \in \mathbb{R}_+$.*

Theorem 4.14. *Let $A_i \in [E^n]$ $(i = 1, 2, \dots, m)$. Assume that the following conditions are fulfilled:*
(1) *there exists $j \in \{1, 2, \dots, m\}$ such that the equation $x' = A_j x$ is exponentially stable;*
(2) *the linear switched system* (4.3) *has not nontrivial bounded on $\mathbb{R}$ solutions.*

Then the linear switched system (4.3) *is uniformly exponentially stable (with respect to switching signal ν).*

Proof. Let $\mathcal{P} := \{1, 2, \dots, m\}$, $Y := S_\tau(\mathbb{R}_+, \mathcal{P})$ and $(Y, \mathbb{R}_+, \sigma)$ be a semi-group dynamical system of shifts on Y. According to Theorem 4.13 the shift dynamical system $(Y, \mathbb{R}_+, \sigma)$ possesses the following properties:
(1) Y is compact;
(2) $Y = \overline{\mathrm{Per}(\sigma)}$, where $\mathrm{Per}(\pi)$ the set of all periodic points of dynamical system $(Y, \mathbb{R}_+, \sigma)$;
(3) there exists a Poisson stable point $y \in Y$ such that $Y = H^+(y)$.

Let $\langle E^n, \varphi, (Y, \mathbb{R}_+, \sigma)\rangle$ be a cocycle, generated by linear switched system (4.2), $(X, \mathbb{R}_+, \pi)$ be a skew-product system associated by cocycle φ (i.e., $X := E^n \times Y$ and $\pi := (\varphi, \sigma)$) and $\langle (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}_+, \sigma), h\rangle$ $(h := pr_2 : X \to Y)$ be a linear non-autonomous dynamical system, generated by cocycle φ. Denote by $\nu_0 : \mathbb{R}_+ \to \mathcal{P}$ the mapping defined by equality $\nu_0(t) = j$ for all $t \in \mathbb{R}_+$. Since the equation $x' = A_j x$ is exponentially stable, then the fiber X_{ν_0} is asymptotically stable. Now to finish the proof of Theorem it is sufficient to refer Theorem 4.10. $\square$

Remark 4.15. It is easy to see that this statement is reversable; i.e., Theorem 1.2 gives necessary and sufficient conditions for the uniformly exponentially stability of linear switched system (1.1).

Acknowledgments. This paper was written while the second author was visiting the Department of Mathematics, University of Limoges (French), in the winter of 2008. He would like to thank people of this university for their very kind hospitality. He also gratefully acknowledges the financial support of the University of Limoges.

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