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# EXACT MULTIPLICITY OF SOLUTIONS FOR A CLASS OF TWO-POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT. We consider the exact multiplicity of nodal solutions of the boundary value problem

$$u'' + \lambda f(u) = 0, \quad t \in (0,1),$$
$$u'(0) = 0, \quad u(1) = 0,$$

where $\lambda \in \mathbb{R}$ is a positive parameter. $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies $f'(u) > \frac{f(u)}{u}$, if $u \neq 0$. There exist $\theta_1 < s_1 < 0 < s_2 < \theta_2$ such that $f(s_1) = f(0) = f(s_2) = 0$; $uf(u) > 0$, if $u < s_1$ or $u > s_2$; $uf(u) < 0$, if $s_1 < u < s_2$ and $u \neq 0$; $\int_{\theta_1}^0 f(u)du = \int_0^{\theta_2} f(u)du = 0$. The limit $f_\infty = \lim_{s\to\infty} \frac{f(s)}{s} \in (0,\infty)$. Using bifurcation techniques and the Sturm comparison theorem, we obtain curves of solutions which bifurcate from infinity at the eigenvalues of the corresponding linear problem, and obtain the exact multiplicity of solutions to the problem for $\lambda$ lying in some interval in $\mathbb{R}$.

## 1. Introduction

Consider the problem

$$\begin{gathered} u'' + \lambda f(u) = 0, \quad t \in (0,1), \\ u'(0) = 0, \quad u(1) = 0, \end{gathered} \tag{1.1}$$

where $\lambda$ is a positive parameter.

The existence and uniqueness of positive solutions to (1.1) has been extensively studied in the literature, see [3, 4, 6, 12] and references therein. On the other hand, a full description of the positive solution set of (1.1) for most nonlinearities $f$ remains open. Tiancheng Ouyang and Junping Shi [11] determined the exact multiplicity of positive solutions of (1.1) for some special $f$ by applying bifurcation techniques. However, little is known about the whole solution set(including one-sign and sign changing solutions) of (1.1). Junping Shi and Junping Wang [8] considered the whole solution set of (1.1) under the following conditions:

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(C1) $f \in C^1(\mathbb{R},\mathbb{R})$ satisfies $f(0)=0$, $f'(0)>0$;
(C2) $f'(u) > \frac{f(u)}{u}$, if $u \neq 0$;
(C3) The limit $f_\infty = \lim_{s\to\infty} \frac{f(s)}{s} \in (0,\infty)$.

They obtained a full description of the the first $N$ solution curves which bifurcate from the line of trivial solutions. Anuradha and Shivaji [1] gave some similar results where $f$ satisfied $f(0)<0$ and other conditions. Motivated by these works, we will consider the existence and uniqueness of one-sign and sign changing solutions of (1.1) under the following conditions

(H1) $f \in C^1(\mathbb{R},\mathbb{R})$, $f'(u) > \frac{f(u)}{u}$, if $u \neq 0$;
(H2) the limit $f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{s} \in (0,\infty)$;
(H3) There exist $\theta_1 < s_1 < 0 < s_2 < \theta_2$ such that $f(s_1)=f(0)=f(s_2)=0$; $uf(u)>0$, if $u<s_1$ or $u>s_2$; $uf(u)<0$, if $s_1<u<s_2$ and $u\neq 0$; $\int_{\theta_1}^0 f(u)du = \int_0^{\theta_2} f(u)du = 0$.

We obtain curves of one-sign and sign changing solutions of (1.1), bifurcating from $\infty$ at the eigenvalues of the corresponding linear problem of (1.1), and obtain exact multiplicity of one-sign and sign changing solutions of (1.1) for $\lambda$ lying in some interval in $\mathbb{R}$.

**Remark 1.1.** Shi and Wang [8] gave precise global bifurcation structure for the whole solution set of (1.1) when the nonlinearity $f$ satisfying $f'(0)>0$. However, (H3) implies that $f'(0)<0$. Meanwhile, (C1) and (C2) implies that $f(u)u>0$, if $u\neq 0$, but $f(u)u$ has negative parts if $f$ satisfying (H3). So it is interesting to find precise global bifurcation structure for the whole solution set of (1.1) under the conditions (H1)-(H3).

**Remark 1.2.** The uniqueness and exact multiplicity of positive solutions have been studied by many authors, see [5, 10] and the references therein. The exact multiplicity results about sign changing solutions have also been researched, see [1, 11] and the references therein. Bari and Rynne [2] consider the global structure of the nodal solutions of the problem

$$
\begin{gathered}
(-1)^m u^{(2m)}(t) = \lambda g(u(t))u(t), \quad t \in (0,1), \\
u^{(i)}(-1) = u^{(i)}(1) = 0, \quad i = 0,\dots,m-1,
\end{gathered}
$$

where $\lambda>0$ is a parameter, the function $g \in C^1(\mathbb{R},\mathbb{R})$ satisfying $\lim_{|\xi|\to\infty} g(\xi) = \infty$, and $g(0)>0$, $\pm g'(\xi)>0$, for all $\pm\xi>0$.

## 2. Preliminary results

Let $Y = C[0,1]$ with the norm $\|y\|_\infty = \max_{t\in[0,1]} |y(t)|$, and let

$$
E = \{y \in C^1[0,1] : y'(0) = y(1) = 0\},
$$

with the norm $\|y\|_E = \max\{\|y\|_\infty, \|y'\|_\infty\}$. Define the operate $L:\ D(L) \subset E \to Y$, by $Lu := -u''$, $u \in D(L)$, where

$$
D(L) = \{u \in C^2[0,1] : u'(0) = u(1) = 0\}.
$$

Then $L^{-1} : Y \to E$ is a completely continuous operator and (1.1) is equivalent to the operator equation

$$
u - \lambda L^{-1}(f(u)) = 0.
$$

We introduce some notation to describe the nodal properties of solutions to (1.1). Firstly, for any $C^1$ function $u$, $x_0$ is a *simple* zero of $u$ if $u(x_0) = 0$ and $u'(x_0) \neq 0$. Now, for any integer $k \geq 1$ and any $\nu \in \{+, -\}$, we define sets $S_k^\nu \subset C^2[0,1]$ as follows: if $u \in S_k^\nu$, then

(i) $u'(0) = 0, \nu u(0) > 0$;
(ii) $u$ has only simple zeros in $[0,1]$ and has exact $k-1$ zeros in $(0,1)$.

The sets $S_k^\nu$ are open in $E$ and disjoint.

Let $\mathbb{E} = \mathbb{R} \times E$, under the product topology. We add the point $\{(\lambda, \infty)_p | \lambda \in \mathbb{R}\}$ into the space $\mathbb{E}$. Put $\Phi_k^\nu = \mathbb{R} \times S_k^\nu$.

Clearly, $u \equiv 0$ is a solution of (1.1) for any $\lambda \in \mathbb{R}$. $(\lambda, 0)$ is called a trivial solution of (1.1). Note that (H1) ensures that the solution of the initial value problem for the differential equation in (1.1) is unique. This fact will be used repeatedly in the following proof so, for brevity, it will be abbreviated to "IVPU".

We first prove the following result about the nodal properties of nontrivial solutions of (1.1).

**Lemma 2.1.** *Suppose* $(\lambda, u)$ *is a nontrivial solution of* (1.1). *Then*

(i) $u \in S_k^\nu$ *for some* $k \in \mathbb{N}$ *and* $\nu \in \{+, -\}$*;*
(ii) $\max_{t\in[0,1]} u(t) > \theta_2$ *and* $\min_{t\in[0,1]} u(t) < \theta_1$ *if* $k \geq 2$*;* $\max_{t\in[0,1]} u(t) > \theta_2$ *if* $u \in S_1^+$*;* $\min_{t\in[0,1]} u(t) < \theta_1$ *if* $u \in S_1^-$*;*
(iii) $u(0) = \max_{t\in[0,1]} u(t)$, *if* $u \in S_k^+$, *and* $u(0) = \min_{t\in[0,1]} u(t)$, *if* $u \in S_k^-$.
(iv) *$u$ has no positive local minimum and/or negative local maximum.*

*Proof.* (i) Since u is nontrivial, "IVPU" implies that all zeros of $u$ are simple. So, (i) is true. In particular, by the boundary condition in (1.1), we have $u(0) \neq 0$ since $u'(0) = 0$. We now describe the qualitative "shape" of the solution $u$.

Without loss of generality, assume that $u \in S_k^+$ for some $k \in \mathbb{N}$ in the following proof. When $u \in S_k^-$, the proof is similar. It follows from the fact that $f$ is independent of $t$ and "IVPU" that the graph of $u$ consists of a sequence of positive and negative bumps, together with a half bump at the left end of the interval $[0,1]$, with the following properties (ignoring the half bump):

(a) all the positive (resp. negative) bumps have the same shape (the shapes of the positive and negative bumps may be different);
(b) all the positive (resp. negative) bumps attain the same maximum (resp. minimum) value.
(c) if $\xi \in (\alpha, \beta) \subset (0,1)$ is a critical point of $u$ and $\alpha, \beta$ are two consecutive zeros of $u$, then the graph of $u$ is symmetric about $t = \xi$ on the interval $(\alpha, \beta)$.

Armed with these properties on the shape of $u$ we can continue the proof of the Lemma.

(ii) On the contrary, suppose $\max_{t\in[0,1]} u(t) \leq \theta_2$. Let $u(0) = c$, then $0 < c \leq \theta_2$. Obviously, $u(t) > 0$ when $t > 0$ is small. Suppose $t_1$ is the first zero of $u$, then $u(t) > 0$ on $[0, t_1)$ and $u(t_1) = 0$, $u'(t_1) < 0$. Note that $(\lambda, u)$ satisfies the equation

$$u'' = -\lambda f(u). \tag{2.1}$$

Multiplying both sides of (2.1) by $u'$,

$$u''(t)u'(t) = -\lambda f(u(t))u'(t). \tag{2.2}$$

Integrating (2.2) from 0 to $t_1$,

$$\int_0^{t_1} u''(t)u'(t)dt = -\lambda \int_0^{t_1} f(u(t))u'(t)dt. \tag{2.3}$$

It follows from (2.3) and (H3) that

$$\frac{1}{2}(u'(t_1))^2 = -\lambda \int_0^{t_1} f(u(t))du(t) = -\lambda \int_c^0 f(u)du = \lambda \int_0^c f(u)du \le 0. \tag{2.4}$$

since $c \le \theta_2$. However, the left side of (2.4) is positive. This is a contradiction. If $k = 1$, then the proof is completed. If $k \ge 2$, suppose $\min_{t\in[0,1]} u(t) \ge \theta_1$. Denote $t_2$ is the second zero of $u$, then $u(t) < 0$ on $(t_1, t_2)$ and $u(t_1) = u(t_2) = 0,\ u'(t_1) < 0$. From (a) and (b) in (i), there exists a $\xi_1 \in (t_1, t_2)$ such that $u'(\xi_1) = 0$ and $u(\xi_1) = \min_{t\in[0,1]} u(t) \ge \theta_1$. Integrating (2.2) from $t_1$ to $\xi_1$,

$$\int_{t_1}^{\xi_1} u''(t)u'(t)dt = -\lambda \int_{t_1}^{\xi_1} f(u(t))u'(t)dt. \tag{2.5}$$

It follows from (2.5) and (H3) that

$$-\frac{1}{2}(u'(t_1))^2 = -\lambda \int_{t_1}^{\xi_1} f(u(t))du(t) = -\lambda \int_0^{u(\xi_1)} f(u)du = \lambda \int_{u(\xi_1)}^0 f(u)du \ge 0. \tag{2.6}$$

since $u(\xi_1) \ge \theta_1$. However, the left side of (2.6) is negative. This is a contradiction. Thus, (ii) is true.

Statements (iii) and (iv) follow from (c) in (i). □

**Remark 2.2.** *(iv) implies that the zeros of $u$ and the zeros of $u'$ are separated, that is each bump of $u$ contains a single zero of $u'$, and there is exact one zero of $u$ between consecutive zeros of $u'$. Moreover, if $u \in S_k^\nu$, then $u$ has $k-1$ zeros in $(0,1)$ and $u'$ has exact $k-1$ zeros in $(0,1)$.*

For a nontrivial solution of (1.1), $(\lambda, u)$ is *degenerate* if the problem

$$\begin{gathered} w'' + \lambda f'(u)w = 0, \quad t \in (0,1), \\ w'(0) = 0, \quad w(1) = 0 \end{gathered} \tag{2.7}$$

has a nontrivial solution, otherwise it is *nondegenerate.*

Now, we consider the initial value problem

$$\begin{gathered} \phi'' + \lambda f'(u)\phi = 0, \quad t \in (0,\ 1), \\ \phi'(0) = 0, \quad \phi(0) = 1. \end{gathered} \tag{2.8}$$

It plays very important role to study the exact multiplicity of solutions of (1.1). Note that if $\phi$ is the unique solution of (2.8), then any solution of (2.7) can be written $w = c\phi$, where $c \in \mathbb{R}$ is a constant.

**Lemma 2.3.** *If $(\lambda, u) \in \Phi_k^\nu$ is a nontrivial solution of* (1.1)*. Then $(\lambda, u)$ is nondegenerate.*

*Proof.* Suppose $(\lambda, w), (\lambda, \phi)$ is the solutions of (2.7), (2.8), respectively. We claim that

$$\phi(1) \ne 0. \tag{2.9}$$

From this claim, we obtain immediately that (2.7) has only trivial solution since $w(1) = c\phi(1) = 0$ if and only if $c = 0$. So $(\lambda, u)$ is nondegenerate. Therefore, we only need to prove that (2.9) holds.

Since $u \in S_k^\nu$, then all zeros of $u$ are simple. By Lemma 2.1 and Remark 2.2, $u$ has exact $k-1$ zeros in $(0,1)$, and especially, $u'$ has also exact $k-1$ zeros in $(0,1)$. The function $u$ satisfies

$$u'' + \lambda f(u) = 0, \quad t \in (0,1). \tag{2.10}$$

Define the function

$$p(t) = \begin{cases} \frac{f(u(t))}{u(t)}, & u(t) \neq 0, \\ f'(0), & u(t) = 0. \end{cases}$$

Then (2.10) is equivalent to

$$u'' + \lambda p(t) u = 0. \tag{2.11}$$

On the other hand, note that $\phi$ and $u'$ satisfy the following equations respectively:

$$\phi'' + \lambda f'(u)\phi = 0, \tag{2.12}$$

$$(u')'' + \lambda f'(u) u' = 0. \tag{2.13}$$

By (H1), (H2), (H3), we have $p(t) \le f'(u(t))$ for all $t \in (0,1)$. Applying the Sturm comparison lemma to (2.11) and (2.12), we obtain, there exists at least one zero of $\phi$ between any two consecutive zeros of $u$. We extend evenly $u, \phi$ to $[-1,0)$, then $u$ has exact $2(k-1)$ zeros in $(-1,1)$, that is, $u$ has exact $2k$ zeros in $[-1,1]$. This implies that $\phi$ has at least $2k-1$ zeros in $(-1,1)$. Note that $\phi$ is a even function in $[-1,1]$, and $\phi(0) \neq 0$, then $\phi$ has at least $2k$ zeros in $(-1,1)$. Therefore, $\phi$ has at least $k$ zeros in $(0,1)$. On the other hand, between any two consecutive zeros of $\phi$, there exists at least one zero of $u'$. Suppose (2.9) does not hold, i.e., $\phi(1) = 0$. Then $\phi$ has at least $k+1$ zeros in $(0,1]$. Moreover, $u'$ has at least $k$ zeros in $(0,1)$. It is impossible! Thus, $\phi(1) \neq 0$. □

The following Lemma shows that every solution of (1.1) which belongs to $\Phi_k^+$ (resp. $\Phi_k^-$) can be parameterized by its maximum (resp. minimum).

**Lemma 2.4.** *Given $k \in \mathbb{N}$ for each $d > 0$(resp. $d < 0$), there exists at most one $\lambda > 0$ such that (1.1) has at most a solution $u$ which belongs to $S_k^+$(resp. $S_k^-$) and satisfies $u(0) = d$.*

The proof of the above lemma can be found in [9].

## 3. The main result and its proof

Our main result reads as follows.

**Theorem 3.1.** *Let (H1)-(H3) hold. Then for every $k \in \mathbb{N}$ and $\nu \in \{+,-\}$, we have:*

(i) *Equation (1.1) has no degenerate solutions. All solutions of (1.1) that belong to $\Phi_k^\nu$ lie on a unique continuous curve $D_k^\nu$. This curve starts from $(\frac{\lambda_k}{f_\infty}, \infty)_p \in \mathbb{E}$, and extends for increasing $\lambda$ such that $\operatorname{Proj}_{\mathbb{R}} D_k^\nu = (\frac{\lambda_k}{f_\infty}, \infty) \subset \mathbb{R}^+$.*

(ii) *For every given parameter* $\lambda \in (\frac{\lambda_k}{f_\infty}, \infty) \subset \mathbb{R}^+$, *there exists exactly one solution of* (1.1) *which belongs to* $S_k^\nu$; *for every given parameter* $\lambda \in (0, \frac{\lambda_k}{f_\infty}]$, *there exists no solution of* (1.1) *which belongs to* $S_k^\nu$, *where* $\lambda_k$ *is the* $k$*th eigenvalue of the linear problem*

$$\begin{gathered} \varphi'' + \lambda\varphi = 0, \quad t \in (0,1), \\ \varphi'(0) = 0, \quad \varphi(1) = 0. \end{gathered} \tag{3.1}$$

**Remark 3.2.** It is well-known that the eigenvalues of (3.1) satisfy

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \lambda_{k+1} < \dots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

for each $\lambda_k$ is simple and the corresponding eigenfunction $\varphi_k$ has exactly $k-1$ zeros in $(0,1)$.

From Theorem 3.1, we obtain immediately the following corollary.

**Corollary 3.3.** *Let* (H1)-(H3) *hold. Then for every* $k \in \mathbb{N}$ *and* $\lambda > 0$: (1.1) *has no nontrivial solution when* $\lambda \in (0, \frac{\lambda_1}{f_\infty}]$; *has exactly two nontrivial solutions, one positive and one negative, when* $\lambda \in (\frac{\lambda_1}{f_\infty}, \frac{\lambda_2}{f_\infty}]$; *has exactly four nontrivial solutions when* $\lambda \in (\frac{\lambda_2}{f_\infty}, \frac{\lambda_3}{f_\infty}]$, *a positive solution, a negative solution, a solution which has one zero on* $(0,1)$ *and* $u(0) > 0$ *and a solution which has one zero on* $(0,1)$ *and* $u(0) < 0$. *In general, when* $\lambda \in (\lambda_k/f_\infty, \lambda_{k+1}/f_\infty]$, (1.1) *has exactly* $2k$ *nontrivial solutions, where*

$$u_1 \in S_1^+, \quad u_2 \in S_1^-, \quad u_3 \in S_2^+, \quad u_4 \in S_2^-, \quad \dots \quad u_{2k-1} \in S_k^+, \quad u_{2k} \in S_k^-.$$

Let $\zeta \in C(\mathbb{R}, \mathbb{R})$ be such that

$$f(u) = f_\infty u + \zeta(u). \tag{3.2}$$

Clearly,

$$\lim_{|u|\to\infty} \frac{\zeta(u)}{u} = 0. \tag{3.3}$$

Let us consider

$$Lu - \lambda f_\infty u = \lambda \zeta(u) \tag{3.4}$$

as a bifurcation problem from infinity. We note that (3.4) is equivalent to (1.1).

The results from Rabinowitz [7] for (3.4) can be stated as follows:

**Lemma 3.4.** *For each integer* $k \geq 1$, $\nu \in \{+, -\}$, *all nontrivial solutions of* (1.1) *near* $\big(\frac{\lambda_k}{f_\infty}, \infty\big)_p$ *lie on a smooth local curve* $\mathcal{D}_k^\nu$, *and* $\mathcal{D}_k^\nu \setminus \{\big(\frac{\lambda_k}{f_\infty}, \infty\big)_p\} \subset \Phi_k^\nu$.

*Proof of Theorem 3.1.* (i) From Lemma 2.3, (1.1) has no degenerate solution. We give the proof only for $u(0) > 0$. When $u(0) < 0$, the proof is similar.

By Lemma 3.4, all solutions of (1.1) near the point $(\frac{\lambda_k}{f_\infty}, \infty)_P$ and $u(0) > 0$ lie on a unique continuous local curve $\mathcal{D}_k^+$ which bifurcating from $\big(\frac{\lambda_k}{f_\infty}, \infty\big)_p$, and $\mathcal{D}_k^+ \setminus \{\big(\frac{\lambda_k}{f_\infty}, \infty\big)_p\} \subset \Phi_k^+$. By Lemma 2.3 and the implicit function theorem, we can continue this local curve to a maximal interval of definition over the $\lambda$-axis. We still denote the curve $\mathcal{D}_k^+$. If we extend $\mathcal{D}_k^+$ for decreasing $\lambda$, then this curve will intersect with the hyperplane $\{0\} \times E$ at some point $(\tilde{u}, 0)$ with $\tilde{u}(0) > \theta_2$. This contradicts $u \equiv 0$ if $\lambda = 0$, since $f(0) = 0$. So, we must extend $\mathcal{D}_k^+$ for increasing $\lambda$. By Lemma 2.3 and the implicit function theorem, it cannot stop at a point such as $(\lambda_0, u_0)$ where $\frac{\lambda_k}{f_\infty} < \lambda_0 < \infty$ and $u_0(0) < \infty$. On the other hand, by Lemma

2.4, it also can not blow up at some point $(\lambda_*, \infty)_p$ with $\frac{\lambda_k}{f_\infty} < \lambda_* < \infty$. Therefore, this curve must continue for increasing $\lambda$ such that $\mathrm{Proj}_{\mathbb{R}} D_k^+ = (\frac{\lambda_k}{f_\infty}, \infty) \subset \mathbb{R}$. Moreover, if $(\lambda, u) \in \mathcal{D}_k^+$ and $\lambda \to \infty$, then there must be a constant $M \geq \theta_2$ such that $\|u\|_\infty \to M$.

Finally, we claim that all solutions of (1.1) which belong to $\Phi_k^+$ must lie on $D_k^+$.

If $M = \theta_2$, by Lemma 2.4, the above claim is naturally right. If $M > \theta_2$, on the contrary, we suppose there is a solution $(\lambda_0, u_0)$ of (1.1) and $(\lambda_0, u_0) \in \Phi_k^+$, but $(\lambda_0, u_0) \notin \mathcal{D}_k^+$. By Lemma 2.3 and the implicit function theorem, all solutions of (1.1) near $(\lambda_0, u_0)$ must lie on a unique local curve which through $(\lambda_0, u_0)$. We denote this local curve $\Gamma_0$. Then for any $(\lambda, u) \in \Gamma_0$, we have $\theta_2 < \|u\|_\infty < M$ from Lemma 2.1. By Lemma 2.3 and the implicit function theorem, $\Gamma_0$ must continue with decreasing $\lambda$. However, (1.1) has only trivial solution if $\lambda = 0$. Thus, when $\Gamma_0$ continues with decreasing $\lambda$, it will have no place to go. Therefore, the above claim is correct.

Statement (ii) is a direct consequence of (i). The proof is complete. □

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