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GRADIENT ESTIMATION OF A p-HARMONIC MAP

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Abstract. This article presents L^p estimates for the gradient of p-harmonic maps. Since the system satisfies a natural growth condition, it is difficult to use standard elliptic estimates. We use spherical coordinates to convert the system into another system with angle functions. The new system can be estimate by the standard elliptic technique.

1. Results

Let $G \subset \mathbb{R}^n$ $(n \in \{2,3\})$ be a bounded and simply connected domain with smooth boundary ∂G. Denote $S^{n-1} = \{x \in \mathbb{R}^n : x_1^2 + x_2^2 + \dots + x_n^2 = 1\}$. Let g be a smooth map from ∂G into S^{n-1} satisfying $\deg(g, \partial G) = d = 0$. Denote by $\{e_i\}_{i=1}^n$ an orthogonal basis of $\mathbb{R}^n$. We are concerned with the estimate of the gradient of p-harmonic maps on G, where $p > 2$.

We call $u \in W^{1,p}(G, S^{n-1})$ a p-harmonic map on G, if it is a weak solution of (cf. [4])

$$-\operatorname{div} |\nabla u|^{p-2}\nabla u) = u|\nabla u|^p. \tag{1.1}$$

The L^p estimate of the gradient of the weak solutions of p-Laplace system is essential for the better regularity (cf. [3, 4, 5, 6, 7, 11, 12]). Thus, in this paper we prove the following theorem.

Theorem 1.1. *If u is a p-harmonic map on G and $u = g$ on ∂G, then there exists a constant $C > 0$ which only depends on G, g, p, n, such that*

$$\|\nabla u\|_{L^p(G)} \le C.$$

Different from [12], it is not easy to estimate the weak solution since (1.1) satisfies the natural growth condition. In [11], a sharp Gagliardo-Nirenberg inequality is used for obtaining regularity of the $W^{2,p}$-solution. For the $W^{1,p}$ weak solution, this estimate can not be used.

To prove the main theorem, we should list some preliminaries.

Proposition 1.2. *The p-harmonic map u on G satisfies*

$$\int_G |\nabla u|^{p-2}(u \wedge \nabla u)\nabla\zeta dx = 0, \quad \forall \zeta \in W_0^{1,p}(G). \tag{1.2}$$

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On the contrary, the function $u \in W^{1,p}(G, S^{n-1})$ *satisfying* (1.2) *must be a p-harmonic map on* G.

Proof. For simplicity, we only calculate formally. Taking the wedge product (1.1) with u, we have

$$-u \wedge \operatorname{div} |\nabla u|^{p-2}\nabla u) = 0.$$

Noting $\nabla u \wedge \nabla u = 0$, we have

$$-\operatorname{div} |\nabla u|^{p-2} u \wedge \nabla u) = 0.$$

It is easy to see that it satisfies (1.2). On the contrary, if $u \in W^{1,p}(G, S^{n-1})$ satisfies (1.2), namely

$$-\operatorname{div} |\nabla u|^{p-2} u \wedge \nabla u) = 0,$$

which is equivalent to

$$-u \wedge \operatorname{div} |\nabla u|^{p-2}\nabla u) = 0.$$

This means that there exists $\lambda \in \mathbb{R}$ such that

$$-\operatorname{div} |\nabla u|^{p-2}\nabla u) = \lambda u.$$

Taking the inner product with u and noting $|u| = 1$, it is not difficult to deduce that $\lambda = |\nabla u|^p$ a.e. in G. Thus, u is a p-harmonic map on G. □

Proposition 1.3. *If* $n = 2$ *and* u *is a* p*-harmonic map on* G*, and* $u = g$ *on* ∂G*, then*

$$\|\nabla u\|^p_{L^p(G)} = \min\{\int_G |\nabla u|^p dx, u \in W^{1,p}(G, S^{n-1}), u|_{\partial G} = g\}.$$

Proof. When $n = 2$, by virtue of $g \in S^{n-1}$ and $\deg(g, \partial G) = 0$, we can write (cf. [1, Eq. (7)])

$$g = \cos\phi_0 e_1 + \sin\phi_0 e_2.$$

Here $\phi_0 \in C^\infty(\partial G, [0, 2\pi])$ is a single-valued function. According to [10, Proposition 2.4], we know that there exists a unique weak solution ϕ of the boundary value problem

$$-\operatorname{div} |\nabla\phi|^{p-2}\nabla\phi) = 0, \quad \text{in } G, \tag{1.3}$$

$$\phi|_{\partial G} = \phi_0. \tag{1.4}$$

Set

$$u = \cos\phi e_1 + \sin\phi e_2. \tag{1.5}$$

It is not difficult to verify by Proposition 1.2 that u is a weak solution of (1.1) with $u|_{\partial G} = g$ if and only if ϕ in (1.5) is a weak solution of (1.3) and (1.4). Therefore, u in (1.5) is the unique weak solution.

In view of $d = 0$, the class $W^{1,p}_g(G, S^{n-1}) = \{v \in W^{1,p}(G, S^{n-1}), u|_{\partial G} = g\}$ is not empty. In fact, the smooth harmonic map with the boundary value g belongs to this class. Consider the minimizing problem

$$\min\{\int_G |\nabla u|^p dx, u \in W^{1,p}_g(G, S^{n-1})\}.$$

Clearly, the minimizer exists, and it is also a p-harmonic map on G. In view of the uniqueness, this minimizer must be u in (1.5). It is easy to see our conclusion. The proof is complete. □

When $n = 3$, we can also convert (1.1) into the form (1.3).

Proposition 1.4. *Let $n = 3$ and u be a p-harmonic map on G. Then there exist single-valued functions $\phi_1(x) \in W^{1,p}(G, [0, \pi])$ and $\phi_2(x) \in W^{1,p}(G, [0, 2\pi])$, such that*

$$\begin{aligned} &\int_G |\nabla u|^{p-2}\big[(\cos\phi_2\nabla\phi_1 - \sin\phi_1\cos\phi_1\sin\phi_2\nabla\phi_2)e_1 \\ &+ (\sin\phi_2\nabla\phi_1 - \sin\phi_1\cos\phi_1\cos\phi_2\nabla\phi_2)e_2 \\ &+ \sin^2\phi_1\nabla\phi_2 e_3]\nabla\zeta dx = 0, \quad \forall\zeta \in W_0^{1,p}(G). \end{aligned} \tag{1.6}$$

Proof. Since G is a simply connected domain and $|u| = 1$, we have the formula of 3-dimension spherical coordinates,

$$u = \cos\phi_1 e_1 + \sin\phi_1\cos\phi_2 e_2 + \sin\phi_1\sin\phi_2 e_3.$$

Here $\phi_j = d\theta + \psi_j$, $j = 1, 2$. Both $\psi_1 \in W^{1,p}(G, [0, \pi])$ and $\psi_2 \in W^{1,p}(G, [0, 2\pi])$ are single-valued functions (cf. [2, 6]). In view of $d = 0$, $\phi_j = \psi_j$ must be single-valued. By calculation,

$$\begin{gathered} \nabla u = -\sin\phi_1\nabla\phi_1 e_1 + (\cos\phi_1\cos\phi_2\nabla\phi_1 - \sin\phi_1\sin\phi_2\nabla\phi_2)e_2 \\ + (\cos\phi_1\sin\phi_2\nabla\phi_1 + \sin\phi_1\cos\phi_2\nabla\phi_2)e_3; \\ |\nabla u|^2 = |\nabla\phi_1|^2 + \sin^2\phi_1|\nabla\phi_2|^2; \\ u \wedge \nabla u = \sin^2\phi_1\nabla\phi_2 e_1 - (\sin\phi_2\nabla\phi_1 + sin\phi_1\cos\phi_1\cos\phi_2\nabla\phi_2)e_2 \\ + (\cos\phi_2\nabla\phi_1 - sin\phi_1\cos\phi_1\sin\phi_2\nabla\phi_2)e_3. \end{gathered} \tag{1.7}$$

Inserting this result into (1.2) yields our conclusion. $\square$

Different from the single equation (1.3), Equation (1.6) is a system when $n = 3$. The uniqueness is not true anymore. The L^p estimate is more complicate than the case $n = 2$. We shall adopt the idea in [8] to establish this estimate.

Proposition 1.5. *Let $B(y_0, 4R) \subset\subset G$, then for any $\xi \in C_0^\infty(B(y_0, 3R))$, there holds*

$$\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy \le C\Big(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy\Big)^{1-\frac{2}{p}}.$$

Proof. The equality corresponding with the vector e_1 in the integral system (1.6) is

$$\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1\nabla\phi_2\nabla\zeta dy = 0, \quad \forall\zeta \in W_0^{1,p}(B(y_0, 3R)).$$

Letting $\zeta = \phi_2\xi^p$ where $\xi \in C_0^\infty(B(y_0, 3R))$, we have

$$\begin{aligned} &\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy \\ &\le |\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1(\xi^{p-1}\phi_2)\nabla\phi_2\nabla\xi dy|. \end{aligned}$$

Using Hölder's inequality, we obtain that, for any $\delta \in (0,1)$,

$$\begin{aligned}
&\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy\\
&\le \delta\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy\\
&\quad + C(\delta)\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\xi|^2\xi^{p-2}\phi_2^2 dy.
\end{aligned}$$

Letting δ be sufficiently small, we obtain

$$\begin{aligned}
\int_{B(y_0,3R)} \xi^p|\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2 dy &\le C\int_{B(y_0,3R)} |\nabla u|^{p-2}\xi^{p-2}dy\\
&\le C\Big(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy\Big)^{1-\frac{2}{p}}.
\end{aligned}$$

The proof is complete. □

Proposition 1.6. *Let $B(y_0,4R)\subset\subset G$, then for any $\xi\in C_0^\infty(B(y_0,3R))$, there holds*

$$\int_{B(y_0,3R)} |\nabla u|^{p-2}|\nabla\phi_1|^2\xi^p dy \le C\Big(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy\Big)^{1-\frac{2}{p}}.$$

Proof. The equalities corresponding with e_2 and e_3 in (1.6) are

$$\begin{aligned}
&\int_{B(y_0,3R)} |\nabla u|^{p-2}(\sin\phi_2\nabla\phi_1 - sin\phi_1\cos\phi_1\cos\phi_2\nabla\phi_2)\nabla\zeta dy = 0,\\
&\int_{B(y_0,3R)} |\nabla u|^{p-2}(\cos\phi_2\nabla\phi_1 + sin\phi_1\cos\phi_1\sin\phi_2\nabla\phi_2)\nabla\zeta dy = 0.
\end{aligned}$$

Take $\zeta = \phi_1\xi^p\sin\phi_2$ and $\zeta = \phi_1\xi^p\cos\phi_2$ in two equalities above, respectively. Then, adding one to the other, we obtain

$$\begin{aligned}
\int_{B(y_0,3R)} |\nabla u|^{p-2}|\nabla\phi_1|^2\xi^p dy &\le \Big[\Big|\int_{B(y_0,3R)} |\nabla u|^{p-2}\phi_1\nabla\phi_1\nabla\xi^p dy\Big|\\
&\quad + 2\Big|\int_{B(y_0,3R)} |\nabla u|^{p-2}\sin\phi_1(\nabla\phi_1\nabla\phi_2)\xi^p dy\Big|\\
&\quad + 2\Big|\int_{B(y_0,3R)} |\nabla u|^{p-2}\phi_1\sin\phi_1\nabla\phi_2\nabla\xi^p dy\Big|\Big]\\
&\quad + 2\Big|\int_{B(y_0,3R)} |\nabla u|^{p-2}\phi_1\sin\phi_1\cos\phi_1|\nabla\phi_2|^2\xi^p dy\Big|\\
&:= J_1 + J_2.
\end{aligned}\tag{1.8}$$

Similar to the proof of Proposition 1.5, by applying Hölder's inequality, we also have

$$J_1 \le \delta\int_{B(y_0,3R)} |\nabla u|^{p-2}|\nabla\phi_1|^2\xi^p dy + C(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy)^{1-2/p}. \tag{1.9}$$

To estimate J_2, we firstly consider $\phi_1\in[0,\pi/2]$. Since $\lim_{\phi_1\to 0}\frac{\sin\phi_1}{\phi_1} = 1$, we can find $\delta_0 > 0$ such that as $0<\phi_1<\delta_0$, there holds $1-\frac{\sin\phi_1}{\phi_1}\le 1/2$ which means

$\phi_1 \leq 2\sin\phi_1$. When $\delta_0 \leq \phi_1 \leq \pi/2$, there holds $\sin\phi_1 \geq \sin\delta_0 > 0$. Thus, by Proposition 1.5,

$$\begin{aligned} J_2 &\leq 2\int_{B(y_0,3R)\cap[\phi_1<\delta_0]} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy \\ &\quad + \frac{\pi}{2\sin\delta_0}\int_{B(y_0,3R)\cap[\delta_0\leq\phi_1\leq\pi/2]} |\nabla u|^{p-2}\sin^2\phi_1|\nabla\phi_2|^2\xi^p dy \\ &\leq C(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy)^{1-2/p}. \end{aligned}$$

When $\phi_1 \in [\pi/2, \pi]$, we can replace ϕ_1 in the test functions ζ by $\pi - \phi_1$. we can also deduce the same result. Substituting this results and (1.9) into (1.8) and choosing δ sufficiently small, we can complete the proof. $\square$

Proof of Theorem 1.1. Interior estimate. Combining Propositions 1.5 and 1.6, and noting (1.7), we can derive

$$\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy \leq C\Big(\int_{B(y_0,3R)} |\nabla u|^p\xi^p dy\Big)^{1-2/p}.$$

Using Young's inequality, and letting $\xi = 1$ on $B(x, 2R)$, we can deduce that

$$\int_{B(y_0,2R)} |\nabla u|^p dy \leq C. \tag{1.10}$$

The interior estimate is obtained.

In the following, we shall investigate the estimation near the boundary. Let $y_0 \in \partial G$. Since g, G are smooth and $d = 0$, we can find single-valued functions $\Phi_1 \in C^\infty(\partial G, [0, \pi])$ and $\Phi_2 \in C^\infty(\partial G, [0, 2\pi])$, such that

$$g = \cos\Phi_1 e_1 + \sin\Phi_1\cos\Phi_2 e_2 + \sin\Phi_1\sin\Phi_2 e_3.$$

Since ∂G is smooth, Ψ_i is extended into G (a neighborhood of ∂G). Replacing ϕ_i by $\phi_i - \Phi_i$ in the test function ζ as we deal with the interior estimation just now, and arguing as above, we can also deduce that $\int_{G\cap B(y_0,R)} |\nabla u|^p dy \leq C$, where $C > 0$ only depends on n, G, R, p and g. Combining this with (10), we complete the proof. $\square$

Remark. Similar to the argument of $n = 3$, we can generalize Theorem 1.1 to the case $n \geq 4$. In fact, we can write a S^{n-1}-valued map w under the spherical coordinates as

$$\begin{aligned} w &= \cos\theta_1 e_1 + \sin\theta_1\cos\theta_2 e_2 + \sin\theta_1\sin\theta_2\cos\theta_3 e_3 + \dots \\ &\quad + \sin\theta_1 \dots \sin\theta_{n-2}\cos\theta_{n-1}e_{n-1} + \sin\theta_1 \dots \sin\theta_{n-2}\sin\theta_{n-1}e_n, \end{aligned}$$

where $(\theta_1, \dots, \theta_{n-1}) \in [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi]$, and each $\theta_i \in W^{1,p}(G)$. Hence, we have a result as (1.7),

$$\begin{aligned} |\nabla w|^2 &= |\nabla\theta_1|^2 + \sin^2\theta_1|\nabla\theta_2|^2 + \sin^2\theta_1\sin^2\theta_2|\nabla\theta_3|^2 + \dots \\ &\quad + \sin^2\theta_1 \dots \sin^2\theta_{n-2}|\nabla\theta_{n-1}|^2. \end{aligned} \tag{1.11}$$

Thus, (1.2) becomes a system on $\theta_i (i = 1, 2, \dots, n-1)$, which contains $\frac{n(n-1)}{2}$ single equations. Using the idea in [9, §2], we also estimate $L^{p/2}$-norm of each term of the right hand side of (1.11) by choosing some equations from the system properly.

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