Electronic Journal of Differential Equations, Vol. 2010(2010), No. 37, pp. 1–5.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

AN INVERSE BOUNDARY-VALUE PROBLEM FOR SEMILINEAR ELLIPTIC EQUATIONS

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Abstract. We show that in dimension two or greater, a certain equivalence class of the scalar coefficient $a(x,u)$ of the semilinear elliptic equation $\Delta u + a(x,u) = 0$ is uniquely determined by the Dirichlet to Neumann map of the equation on a bounded domain with smooth boundary. We also show that the coefficient $a(x,u)$ can be determined by the Dirichlet to Neumann map under some additional hypotheses.

1. Introduction

In this article, we study the inverse boundary-value problem (IBVP) for the semilinear equation

$$\begin{gathered} L_a(u) := \Delta u + a(x,u) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \\ u|_{\partial\Omega} = f, \quad f \in C^{2,\alpha}(\partial\Omega), \end{gathered} \tag{1.1}$$

where $0 < \alpha < 1$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. We assume that the coefficient of the equation satisfies

$$a(x,u), a_u(x,u) \in C^\alpha(\bar{\Omega} \times R), \tag{1.2}$$

$$a_u(x,u) \le 0. \tag{1.3}$$

Then the Dirichlet problem (1.1) has an unique solution $u \in C^{2,\alpha}(\bar{\Omega})$ [2, 8]. We define the nonlinear Dirichlet to Neumann map Λ_a:

$$\Lambda_a(f) = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega},$$

where ν is the unit outer normal on the boundary $\partial\Omega$. The inverse problem is to recover $a(x,u)$ from knowledge of Λ_a.

It was shown in [7] that if $a(x,u)$ satisfies the condition

$$a(x,0) = 0, \tag{1.4}$$

then the uniqueness holds for the above inverse problem.

In this paper we shall study the above inverse problem without the assumption (1.4). We first observe that in the general case, the Dirichlet to Neumann map Λ_a does not determine the coefficient a uniquely.

2000 *Mathematics Subject Classification.* 35R30.
Key words and phrases. Inverse Problem; Dirichlet to Neumann map.

Submitted January 31, 2010. Published March 14, 2010.

To see the nonuniqueness, let a be a coefficient satisfying (1.2) and (1.3) and let ϕ be a function satisfying

$$\phi(x) \in C^{2,\alpha}(\bar{\Omega}), \quad \phi|_{\partial\Omega} = \nabla\phi|_{\partial\Omega} = 0. \tag{1.5}$$

Define the transformation T_ϕ by

$$(T_\phi a)(x,u) = a(x, u + \phi(x)) + \Delta\phi(x) \tag{1.6}$$

Then the new coefficient $T_\phi a$ satisfies the same assumptions (1.2) and (1.3). It is easy to check that $L_{T_\phi a}(u - \phi) = 0$, and the assumption $\phi|_{\partial\Omega} = \nabla\phi|_{\partial\Omega} = 0$ implies

$$(u-\phi)|_{\partial\Omega} = u|_{\partial\Omega}, \quad \frac{\partial(u-\phi)}{\partial\nu}\Big|_{\partial\Omega} = \frac{\partial u}{\partial\nu}\Big|_{\partial\Omega}.$$

Therefore,

$$\Lambda_{T_\phi a} = \Lambda_a. \tag{1.7}$$

We define in the set of coefficients satisfying (1.2) and (1.3) an equivalence relation induced by T_ϕ as follows:

$$a \sim \tilde{a} \quad \text{if} \quad \tilde{a} = T_\phi a. \tag{1.8}$$

Then we see from the above discussion that Λ_a remains the same for any coefficient in the equivalence class $[a]$. Therefore, the correct uniqueness question for (1.1) in the general setting is to ask whether Λ_a determines $[a]$ uniquely.

The main purpose of this article is to give an affirmative answer to this question. To state the result, let us define for each coefficient a, a set $E_a \in \mathbb{R}^n \times R$ by

$$E_a = ((x,u) \subset \Omega \times R;\, \exists \text{ solution } u \text{ of (1.1) with } u = u(x)), \tag{1.9}$$

and the transformation of E_a by T_ϕ by

$$T_\phi E_a = ((x, u + \phi(x)) \subset \Omega \times R;\, \exists \text{ solution } u \text{ of (1.1) with } u = u(x)). \tag{1.10}$$

Theorem 1.1. *Given $a(x,u)$ and $\tilde{a}(x,u)$ satisfying the conditions* (1.2) *and* (1.3). *If $\Lambda_a = \Lambda_{\tilde{a}}$, then there is a function ϕ satisfying* (1.5) *such that*

$$E_{\tilde{a}} = T_{-\phi} E_a, \tag{1.11}$$

$$\tilde{a}(x,u) = T_\phi a(x,u) \quad \text{on } E_{\tilde{a}}. \tag{1.12}$$

As the example illustrates in [7], in general the set E_a in (1.9) may be a proper subset, and thus (1.12) is the best one can hope for.

Another purpose of this article is to generalize the uniqueness result proven in [7]. The condition (1.4) implies that zero is a constant solution of the equation (1.1). Thus, the equation (1.1) with the coefficient a satisfying (1.4) must carry a common solution $u \equiv 0$. We shall show that the uniqueness holds in the general case as long as a common solution, not necessarily $u \equiv 0$, exists.

Theorem 1.2. *Given $a(x,u)$ and $\tilde{a}(x,u)$ satisfying the conditions* (1.2) *and* (1.3). *Assume that the equation* (1.1) *carries a common solution for both coefficients a and $\tilde{a}$. If $\Lambda_a = \Lambda_{\tilde{a}}$, then*

$$E_a = E_{\tilde{a}}, \tag{1.13}$$

$$a(x,u) = \tilde{a}(x,u) \quad \text{on } E_a. \tag{1.14}$$

Similar problems have been studied for various semilinear and quasilinear elliptic equations and systems [4, 5, 6, 11, 13, 3, 9]. We refer to the survey papers [12, 14] for other recent developments in the field of inverse boundary value problems for semilinear and quasilinear elliptic equations.

The proof of both theorems are based on a linearization argument and the uniqueness result for the linear elliptic equations. In the next section, we give a proof of Theorems 1.1 and 1.2.

2. Proofs of Theorems

Let u_f be the unique solution to (1.1). Using the argument in [12] that is based on Schauder's estimate, we can show that the map $f \to u_f$ is differentiable in the space $C^{2,\delta}(\bar{\Omega})$ for any δ with $0 < \delta < \alpha$.

Let $g \in C^{2,\alpha}(\partial\Omega)$. Denote by u^* the unique solution to the linear problem

$$\Delta u^* + a_u(x, u_f)u^* = 0, \quad u^*|_{\partial\Omega} = g. \tag{2.1}$$

Then for any δ, $0 < \delta < \alpha$,

$$\lim_{t\to 0} \| \frac{u_{f+tg} - u_f}{t} - u^* \|_{C^{2,\delta}(\bar{\Omega})} = 0. \tag{2.2}$$

We denote by $\dot{u}_{f,g}$ the solution u^* in (2.1) as the derivative of u at f in the direction g. Similarly, we have that u_{f+tg} is differentiable in t at any value of t under the $C^{2,\delta}(\bar{\Omega})$ norm, $0 < \delta < \alpha$, and the derivative, denoted by $\dot{u}_{f+tg,g}$, satisfies

$$\Delta \dot{u}_{f+tg} + a_u(x, \nabla u_{f+tg}) \cdot \nabla \dot{u}_{f+tg,g} = 0, \quad \dot{u}_{f+tg,g}|_{\partial\Omega} = g. \tag{2.3}$$

Proof of Theorem 1.1. Given $a(x, u)$ and $\tilde{a}(x, u)$ satisfying the conditions (1.2) and (1.3). We denote by u_f the unique solution of (1.1) and by $\tilde{u}_f$ the unique solution of (1.1) with a replaced by $\tilde{a}$, where a and $\tilde{a}$ are two semilinear coefficients assumed in Theorem 1.1. Under the assumption that $\Lambda_a = \Lambda_{\tilde{a}}$, we have that

$$\frac{\partial u_f}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial \, \tilde{u}_f}{\partial \nu}\Big|_{\partial\Omega} \tag{2.4}$$

for each $f \in C^{2,\alpha}(\partial\Omega)$. Then for any $g \in C^{2,\alpha}(\partial\Omega)$,

$$\frac{\partial u_{f+tg}}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial \tilde{u}_{f+tg}}{\partial \nu}\Big|_{\partial\Omega}, \ \forall t \in \mathbb{R}. \tag{2.5}$$

Differentiating (2.5) in t at $t = 0$, we get

$$\frac{\partial \dot{u}_{f,g}}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial \, \dot{\tilde{u}}_{f,g}}{\partial \nu}\Big|_{\partial\Omega}, \tag{2.6}$$

where $\dot{u}_{f,g}$ and $\dot{\tilde{u}}_{f,g}$ satisfy

$$\Delta \dot{u}_{f,g} + a_u(x, u_f)\dot{u}_{f,g} = 0, \quad \dot{u}_{f,g}|_{\partial\Omega} = g, \tag{2.7}$$

$$\Delta \dot{\tilde{u}}_{f,g} + \tilde{a}_u(x, \tilde{u}_f)\dot{\tilde{u}}_{f,g} = 0, \quad \dot{\tilde{u}}_{f,g}|_{\partial\Omega} = g. \tag{2.8}$$

Since for a fixed $f \in C^{2,\alpha}(\partial\Omega)$, (2.6) holds for all $g \in C^{2,\alpha}(\partial\Omega)$, we have that the Dirichlet to Neumann maps of (2.7) and (2.8) must be equal; i.e.

$$\Lambda^*_{a_u(x,u_f)} = \Lambda^*_{\tilde{a}_u(x,\tilde{u}_f)}. \tag{2.9}$$

Then the uniqueness results established in [10] can be applied to obtain

$$a_u(x, u_f) = \tilde{a}_u(x, \tilde{u}_f) \quad \text{on } \Omega, \tag{2.10}$$

and consequently,

$$\dot{u}_{f,g}(x) = \dot{\tilde{u}}_{f,g}(x) \text{ on } \Omega. \tag{2.11}$$

Replacing f by tf and g by f in (2.11) we get

$$\dot{u}_{tf,f}(x) = \dot{\tilde{u}}_{tf,f}(x) \quad \text{on } \Omega,\ \forall t \in \mathbb{R}. \tag{2.12}$$

In other words,

$$d/dt(u_{tf}(x)) = d/dt(\tilde{u}_{tf}(x)) \quad \text{on } \Omega,\ \forall t \in \mathbb{R}.$$

Thus, there is a function $\phi \in C^{2,\alpha}(\bar{\Omega})$, independent of t, such that

$$u_f(x) = \tilde{u}_f(x) + \phi(x), \quad x \in \Omega. \tag{2.13}$$

Clearly, the function ϕ is independent of f, since by (2.12), each f carries the same ϕ as $f = 0$ does.

Since (2.13) holds for all f, we have that (2.13) implies (1.11). Also, combining (2.4) with (2.13), we see that ϕ satisfies the boundary condition in (1.5).

Substituting the right hand side of (2.13) in (1.1), we obtain

$$\Delta(\tilde{u}_f + \phi) + a(x, \tilde{u}_f + \phi) = 0. \tag{2.14}$$

Since

$$\Delta\tilde{u}_f + \tilde{a}(x, \tilde{u}_f) = 0, \tag{2.15}$$

combining (2.14) with (2.15) yields

$$\tilde{a}(x, \tilde{u}_f) = a(x, \tilde{u}_f + \phi) + \Delta\phi,$$

which implies (1.12). This completes the proof. □

Proof of Theorem 1.2. Repeating the argument used in the proof of Theorem 1.1, yields that (2.13) holds for all f. Since there is a common solution, we have that the function ϕ must be the zero function. Thus, for all f,

$$u_f(x) = \tilde{u}_f(x), \quad x \in \Omega. \tag{2.16}$$

This shows that $E_a = E_{\tilde{a}}$, which is (1.13). Substituting (2.16) in (1.1), we obtain that for all f,

$$a(x, u_f) = \tilde{a}(x, \tilde{u}_f), \quad x \in \Omega.$$

Therefore,

$$a(x, u) = \tilde{a}(x, u), \quad (x, u) \in E_a.$$

This completest the proof. □

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