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EXISTENCE OF POSITIVE SOLUTIONS TO SOME IMPULSIVE SECOND-ORDER INTEGRODIFFERENTIAL EQUATIONS

RAHIMA ATMANIA

Abstract. In this article, we consider an initial-value problem for second-order nonlinear integrodifferential equations with impulses in a Banach space. By using the monotone iterative technique in a cone together with Arzela-Ascoli theorem and the dominated convergence theorem, we establish the existence of positive solutions of such a problem.

1. Introduction

It is well known that the theory of impulsive differential equations is an important area of research which has been investigated in the last few years by many authors in several directions. So, a great deal of techniques and methods have been used in the study of the second order impulsive differential equations to obtain some quantitative or qualitative results regarding the solutions of such new problems, see for instance [2, 3, 5] . We recall that the impulsive differential equations can model natural phenomena and evolving processes which are subject to abrupt changes such as shocks, food shortenings, natural disasters and so on. Thus, we may treat these short-term perturbations as impulses that affect later on the behavior of the solutions. To learn more about the recent developments of the theory of impulsive equations we refer the reader to the works of Benchohra et al [1] without forgetting to quote the book by Lakshmikantam et al. [4], we recall that the latter is considered as one of the basic references in this domain.

Our contribution in this paper is the investigation of positive solutions to the following second order nonlinear integrodifferential equation

$$x''(t) = F(t, x(\delta_1(t)), x'(\delta_2(t)), Tx(t), Sx(t)), \quad t \in J\backslash\{t_k;\ k = 1, 2\dots\} \tag{1.1}$$

subject to the impulsive conditions

$$\begin{gathered} \Delta x = x(t^+) - x(t^-) = I_k(t, x, x'), \quad t = t_k;\ k = 1, 2\dots, \\ \Delta x' = x'(t^+) - x'(t^-) = \hat{I}_k(t, x, x'), \quad t = t_k;\ k = 1, 2\dots, \end{gathered} \tag{1.2}$$

and the initial conditions

$$x(0) = x_0, \quad x'(0) = x_0^*, \tag{1.3}$$

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where for $x \in X$, a given Banach space, and $t \in J = [0, +\infty)$, the functionals T and S are defined as follows:

$$Tx(t) = \int_0^t g(t, s, x(\delta_3(s)), \int_0^s k(s, \tau, x(\delta_4(\tau)))d\tau)ds,$$

$$Sx(t) = \int_0^{+\infty} h(t, s, x(\delta_5(s)))ds.$$

So, inspired by the results in [2] devoted to the existence of positive solutions to the corresponding problem for $Sx(t) = 0$, $Tx(t) = \int_0^t K(t, s)x(s)ds$ and $\delta_i(t) = t$, $i = 1, \dots 4$, we have established the existence of positive solutions for problem (1.1)-(1.3) by using the monotone iterative technique in a cone of a Banach space X together with Ascoli-Arzela theorem and the dominated convergence theorem on an infinite time interval with the presence of an infinite number of impulses.

2. Preliminaries

We first set the following assumptions:

- (H1) $0 < t_1 < t_2 \dots < t_k \dots$ and $\lim_{k\to\infty} t_k = +\infty$;
- (H2) x_0 and x_0^* are given values in a cone P of a Banach space $(X, \|.\|)$ which defines a partial ordering in X as follows: $x \le y$ if and only if $y - x \in P$. We assume that x_0 is different from the null vector θ of P.
- (H3) $F \in C(J \times P \times P \times P \times P, P)$, $g \in C(J \times J \times P \times P, P)$, $k \in C(J \times J \times P, P)$, $h \in C(J \times J \times P, P)$, I_k, $\hat{I}_k \in C(J \times P \times P, P)$ and $\delta_i \in C(J, J)$ are given functions such that $\delta_i(t) \le t$, $t \in J$, such that $\lim\limits_{t\to\infty} \delta_i(t) = \infty$ for $i = 1, \dots 5$.

In the sequel we shall use the following spaces:
$\mathcal{PC}(J, X) = \{x : J \to X : x(t)$ is continuous at $t \ne t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exists for $k = 1, 2 \dots\}$;
$\mathcal{PC}(J, P) = \{x \in \mathcal{PC}(J, X) : \theta \le x(t);\ t \in J\}$
$\mathcal{PC}^1(J, X) = \{x \in \mathcal{PC}(J, X) : x'(t)$ is continuous at $t \ne t_k$, left continuous at $t = t_k$, and $x'(t_k^+)$ exists for $k = 1, 2 \dots\}$.

Moreover, we introduce the Banach spaces

$$\mathcal{SPC}(J, X) = \{x \in \mathcal{PC}(J, X) : \sup_{t\in J} \frac{\|x(t)\|}{t + 1} < \infty\},$$

with the norm $\|x\|_S = \sup\limits_{t\in J} \frac{\|x(t)\|}{t+1}$, and

$$\mathcal{SPC}^1(J, X) = \{x \in \mathcal{PC}^1(J, X) : \sup_{t\in J} \frac{\|x(t)\|}{t + 1} < \infty \text{ and } \sup_{t\in J} \|x'(t)\| < \infty\}$$

with the norm $\|x\|_D = \max(\|x\|_S, \|x'\|_C)$, where $\|x'\|_C = \sup\limits_{t\in J} \|x'(t)\|$. We note that

$$\mathcal{SPC}(J, P) = \{x \in \mathcal{SPC}(J, X) : \theta \le x(t);\ t \in J\}$$

is a cone in $\mathcal{SPC}(J, X)$, and

$$\mathcal{SPC}^1(J, P) = \{x \in \mathcal{SPC}^1(J, X) : \theta \le x(t) \text{ and } \theta \le x'(t);\ t \in J\}$$

is a cone in $\mathcal{SPC}^1(J, X)$.

We recall that a cone is said to be *normal* if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$, and a cone is said to be *regular* (resp. *fully regular*) if $x_1 \le \dots \le x_n \le \dots \le y$ for some $y \in X$ (resp. $\|x_1\| \le \dots \le \|x_n\| \le$

$\dots \le \sup_n \|x_n\| < \infty$) implies that there is $x_n \in X$, such that $\|x_n - x\| \to 0$ as $n \to \infty$.

Of course, the full regularity of a cone implies its regularity which in turn implies its normality.

By a *positive solution* to the problem (1.1)-(1.3), we mean a function

$$x \in C^2(J \setminus \{t_k\}_{k\ge 1}, X) \cap \mathcal{SPC}^1(J, P)$$

satisfying (1.1)-(1.3) and $x(t) \in P\backslash\{\theta\}$, for every $t \in J$. We need the following lemma whose proof can be handled without any difficulty.

Lemma 2.1. *A function $x \in C^2(J \setminus \{t_k\}_{k\ge 1}, X) \cap \mathcal{SPC}^1(J, P)$ is a solution to the problem* (1.1)-(1.3) *if and only if $x \in \mathcal{PC}(J, P)$ satisfies the impulsive integral equation*

$$\begin{aligned} x(t) = x_0 + t x_0^* + \int_0^t (t-s) F(s, x(\delta_1(s)), x'(\delta_2(s)), Tx(s), Sx(s)) ds \\ + \sum_{0<t_k<t} \big[I_k(t_k, x(t_k), x'(t_k)) + (t - t_k)\hat{I}_k(t_k, x(t_k), x'(t_k)) \big]. \end{aligned} \tag{2.1}$$

Now we are in position to state and prove our main results.

3. Main results

Besides the mentioned hypotheses (H1) to (H3), we add the following:

(H4) $\|g(t, s, x, y)\| \le m_1(t, s)\|x\| + m_2(s)\|y\|$; $\|k(t, s, x)\| \le m_3(t, s)\|x\|$ and $\|h(t, s, x)\| \le m_4(t, s)\|x\|$, for every $t, s \in J$ and $x, y \in P$, where the functions m_1, m_3, m_4 and m_2 satisfy

$$\sup_{t\in J} \int_0^t m_1(t, s) ds = m_1^* < \infty, \quad \sup_{t\in J} \int_0^{+\infty} m_4(t, s) ds = m_4^* < \infty,$$
$$\sup_{t\in J} \int_0^t m_2(s) \int_0^s m_3(s, \tau) d\tau \, ds = m_2^* < \infty.$$

(H5) $\|F(t, x, y, z, w)\| \le p_1(t)\|x\| + p_2(t)\|y\| + p_3(t)\|z\| + p_4(t)\|w\| + q(t)$, for every $t \in J$ and $x, y, z, w \in P$. The functions p_i, $i = 1, \dots, 4$, and q satisfy

$$\int_0^{+\infty} (s+1) p(s) ds = p^* < \infty, \int_0^{+\infty} q(s) ds = q^* < \infty$$

with $p(t) = \max_{i=1,2,3,4} p_i(t)$.

(H6) $\|I_k(t, x, y)\| \le a_k\|x\| + b_k\|y\| + c_k$, $\|\hat{I}_k(t, x, y)\| \le d_k\|x\| + e_k\|y\| + f_k$, for every $t \in J$ and $x, y \in P$; a_k, b_k, c_k, d_k, e_k, f_k; $k = 1, 2, \dots$, being positive constants such that

$$\sum_{k=1}^{\infty} (t_k + 1) a_k = a < \infty; \quad \sum_{k=1}^{\infty} b_k = b < \infty; \quad \sum_{k=1}^{\infty} c_k = c < \infty;$$
$$\sum_{k=1}^{\infty} (t_k + 1) d_k = d < \infty; \quad \sum_{k=1}^{\infty} e_k = e < \infty; \quad \sum_{k=1}^{\infty} f_k = f < \infty\,.$$

(H7) For each $t \in J$ and x_1, x_2, y_1, y_2, z_1, z_2, w_1 and $w_2 \in P$ such that

$$x_1 \le x_2, \quad y_1 \le y_2, \quad z_1 \le z_2 \quad \text{and } w_1 \le w_2$$

we have $I_k(t,x_1,y_1) \le I_k(t,x_2,y_2)$; $\hat{I}_k(t,x_1,y_1) \le \hat{I}_k(t,x_2,y_2)$; $k=1,2,\dots$ and $F(t,x_1,y_1,z_1,w_1) \le F(t,x_2,y_2,z_2,w_2)$.

Now define the operator B as follows

$$\begin{aligned} Bx(t) &= x_0 + tx_0^* + \int_0^t (t-s)F(s,x(\delta_1(s)),x'(\delta_2(s)),Tx(s),Sx(s))ds \\ &\quad + \sum_{0<t_k<t} [I_k(t_k,x(t_k),x'(t_k)) + (t-t_k)\hat{I}_k(t_k,x(t_k),x'(t_k))], \quad t \in J. \end{aligned} \tag{3.1}$$

Lemma 3.1. *Assume that* (H1)-(H4) *hold. Then T and S are bounded operators from $\mathcal{SPC}(J,X)$ into $\mathcal{SPC}(J,X)$ and $\|T\| \le T^*$; $\|S\| \le S^*$, where $T^* = m_1^* + m_2^*$ and $S^* = m_4^*$. Furthermore, T and S map $\mathcal{SPC}(J,P)$ into itself.*

Proof. First, we note that if $x \in \mathcal{SPC}(J,X)$, then $Tx \in C(J,X)$ and $Sx \in C(J,X)$. On the other hand, from (H4), we have the estimates

$$\begin{aligned} \frac{\|Tx(t)\|}{1+t} &\le \int_0^t m_1(t,s)\frac{1+\delta_3(s)}{1+t}\frac{\|x(\delta_3(s))\|}{1+\delta_3(s)}ds \\ &\quad + \int_0^t m_2(s)\int_0^s m_3(s,\tau)\frac{1+\delta_4(\tau)}{1+t}\frac{\|x(\delta_4(\tau))\|}{1+\delta_4(\tau)}d\tau ds \\ &\le \|x\|_S\big[\int_0^t m_1(t,s)ds + \int_0^t m_2(s)\int_0^s m_3(s,\tau)d\tau ds\big], \end{aligned}$$

so that

$$\|Tx\|_S \le T^*\|x\|_S, \quad t \in J,$$

and

$$\frac{\|Sx(t)\|}{1+t} \le \int_0^{+\infty} m_4(t,s)\frac{1+\delta_5(s)}{1+t}\frac{\|x(\delta_5(s))\|}{1+\delta_5(s)}ds \le \|x\|_S\int_0^{+\infty} m_4(t,s)ds,$$

yielding

$$\|Sx\|_S \le S^*\|x\|_S.$$

Hence Tx and $Sx \in \mathcal{SPC}(J,X)$ from which we get $\|T\| \le T^*$ and $\|S\| \le S^*$. Finally, it is clear that $T : \mathcal{SPC}(J,P) \to \mathcal{SPC}(J,P)$ and $S : \mathcal{SPC}(J,P) \to \mathcal{SPC}(J,P)$, and so, the lemma is proved. □

Lemma 3.2. *If the hypotheses* (H1)-(H6) *are satisfied, then the operator B maps $\mathcal{SPC}^1(J,P)$ into itself and*

$$\|Bx\|_D \le \alpha\|x\|_D + \beta, \quad x \in \mathcal{SPC}^1(J,P), \tag{3.2}$$

where

$$\alpha = (p^*(2+T^*+S^*) + a + b + d + e), \beta = 2\max(\|x_0\|,\|x_0^*\|) + q^* + c + f.$$

Proof. From (H5), (H6) we have, for each $x \in \mathcal{SPC}^1(J,P)$,

$$\int_0^{+\infty} \|F(s,x(\delta_1(s)),x'(\delta_2(s)),Tx(s),Sx(s))\|ds \le p^*(2+T^*+S^*)\|x\|_D + q^* \tag{3.3}$$

and

$$\sum_{k=1}^{\infty} \|I_k(t_k, x(t_k), x'(t_k))\| \leq (a+b)\|x\|_D + c, \tag{3.4}$$

$$\sum_{k=1}^{\infty} \|\hat{I}_k(t_k, x(t_k), x'(t_k))\| \leq (d+e)\|x\|_D + f. \tag{3.5}$$

Next, for each $x \in \mathcal{SPC}^1(J, P)$, we have by virtue of (3.1), lemma 3.1 and (3.3)-(3.5),

$$\begin{aligned}
\frac{\|Bx(t)\|}{1+t} &\leq \frac{1}{1+t}\|x_0\| + \frac{t}{1+t}\|x_0^*\| \\
&\quad + \int_0^t \frac{t-s}{1+t}\|F(s, x(\delta_1(s)), x'(\delta_2(s)), Tx(s), Sx(s))\| ds \\
&\quad + \sum_{0<t_k<t} \frac{1}{1+t}\|I_k(t_k, x(t_k), x'(t_k))\| \\
&\quad + \sum_{0<t_k<t} \frac{t-t_k}{1+t}\|\hat{I}_k(t_k, x(t_k), x'(t_k))\| \\
&\leq (p^*(2+T^*+S^*) + a + b + d + e)\|x\|_D \\
&\quad + (2\max(\|x_0\|, \|x_0^*\|) + q^* + c + f).
\end{aligned}$$

Thus,

$$\|Bx\|_S \leq \alpha\|x\|_D + \beta, \quad x \in \mathcal{SPC}^1(J, P). \tag{3.6}$$

On the other hand, differentiating (3.1) we get for each $t \in J$,

$$\begin{aligned}
(Bx)'(t) &= x_0^* + \int_0^t F(s, x(\delta_1(s)), x'(\delta_2(s)), Tx(s), Sx(s)) ds \\
&\quad + \sum_{0<t_k<t} \hat{I}_k(t_k, x(t_k), x'(t_k)).
\end{aligned}$$

Thanks to (3.3)-(3.5), we have

$$\|(Bx)'(t)\| \leq \|x_0^*\| + p^*(2+T^*+S^*)\|x\|_D + q^* + (d+e)\|x\|_D + f.$$

It follows that

$$\begin{aligned}
\|(Bx)'\|_C &\leq (p^*(2+T^*+S^*) + d + e)\|x\|_D + \|x_0^*\| + q^* + f \\
&\leq \alpha\|x\|_D + \beta, \quad x \in \mathcal{SPC}^1(J, P),
\end{aligned} \tag{3.7}$$

Hence, (3.6) and (3.7) give (3.2); therefore $Bx \in \mathcal{SPC}^1(J, P)$. □

Theorem 3.3. *Assume that P is a fully regular cone and hypotheses* (H1)-(H7) *are satisfied. If $\alpha < 1$, then problem* (1.1)-(1.3) *has at least one positive solution x. In addition, it satisfies*

$$\|x\|_D \leq \frac{\beta}{1-\alpha}. \tag{3.8}$$

Proof. First, we need to prove that the impulsive integral equation has at least one positive solution. Define the sequence $(x_n(t))_{n\geq 1}$ as follows

$$x_0(t) = x_0, \quad x_n(t) = Bx_{n-1}(t); \quad n = 1, 2, \dots \tag{3.9}$$

where B is defined by (3.1) and satisfies (3.2). Then $x_n \in \mathcal{SPC}^1(J,P)$ and

$$\|x_0\|_D \le \beta, \quad \|x_n\|_D \le \alpha\|x_{n-1}\|_D + \beta; \quad n = 1, 2, \dots .$$

We obtain by induction

$$\|x_n\|_D \le \alpha^n\beta + \alpha^{n-1}\beta + \dots + \alpha\beta + \beta = \frac{(1-\alpha^{n+1})}{(1-\alpha)}\beta \le \frac{\beta}{1-\alpha}. \tag{3.10}$$

On the other hand, the sequence $(x_n'(t))_{n\ge1}$ defined by

$$\begin{aligned} x_n'(t) = x_0^* + \int_0^t F(s, x_n(\delta_1(s)), x_n'(\delta_2(s)), Tx_n(s), Sx_n(s))ds \\ + \sum_{0<t_k<t} \hat{I}_k(t_k, x_n(t_k), x_n'(t_k)). \end{aligned}$$

satisfies the estimate

$$\|x_n'\|_C \le \frac{\beta}{1-\alpha}. \tag{3.11}$$

We infer from (H7) that $(x_n(t))_{n\ge1}$ and $(x_n'(t))_{n\ge1}$, for $t \in J$, satisfy

$$\begin{gathered} \theta \le x_n(t) \le x_{n+1}(t); \quad n = 1, 2, \dots, \\ \theta \le x_n'(t) \le x_{n+1}'(t); \quad n = 1, 2, \dots, \end{gathered}$$

As P is a fully regular cone, it follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} x_n(t) = x(t); \quad t \in J, \lim_{n\to\infty} x_n'(t) = y(t); \quad t \in J. \tag{3.12}$$

Now, since $x_n(t)$ is an equicontinuous function on each closed interval J_k; $k = 1, 2, \dots$ such that $J_0 = [0, t_1]$, $J_k =]t_k, t_{k+1}]$; $k = 1, 2, \dots$, then by using Ascoli-Arzela theorem we deduce the existence of a subsequence $(x_{n_i}(t)) \subset (x_n(t))$ such that $x_{n_i}(t)$ converges uniformly to $x(t)$ on each J_k; $k = 1, 2, \dots$, and so, as the cone is normal, the sequence $(x_n(t))$ converges uniformly to $x(t)$ on each J_k; $k = 1, 2, \dots$. Therefore, $x \in \mathcal{PC}(J,P)$, and taking into account (3.10), $x \in \mathcal{SPC}(J,P)$ and $\|x\|_S \le \beta/(1-\alpha)$.

Next, by a double differentiation of $x_n(t)$; $n = 1, 2, \dots$, we obtain

$$\begin{aligned} \|x_n''(t)\| &\le p(t)(1+t)[2 + T^* + S^*]\|x_{n-1}\|_D + q(t); t \ne t_k, \quad k = 1, 2, \dots, \\ &\le p(t)(1+t)[2 + T^* + S^*]\frac{\beta}{1-\alpha} + q(t) = f(t). \end{aligned} \tag{3.13}$$

We note that the function $f(t)$ is bounded on any finite interval. We observe from (3.13) that $x_n'(t)$, $n \ge 1$ are equicontinuous functions on each J_k; $k = 1, 2, \dots$. We conclude that the sequence $(x_n'(t))$ converges uniformly on each J_k; $k = 1, 2, \dots$, to $y(t)$. Hence, $x'(t)$ exists and $x'(t) = y(t)$ on each J_k; $k = 1, 2, \dots$, $x' \in \mathcal{PC}(J,P)$ and $\|x'\|_C \le \beta/(1-\alpha)$. Consequently, $x \in \mathcal{SPC}^1(J,P)$ and $\|x\|_D \le \beta/(1-\alpha)$. On the other hand, since

$$\|k(s,\tau,x_{n-1}(\delta_4(\tau))) - k(s,\tau,x(\delta_4(\tau)))\| \le 2m_3(t,s)\frac{\beta}{1-\alpha},$$

by applying the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_0^s k(s,\tau,x_{n-1}(\delta_4(\tau)))d\tau = \int_0^s k(s,\tau,x(\delta_4(\tau)))d\tau, \quad s \in J.$$

By the same reasoning for g and h we obtain

$$\lim_{n\to\infty} Tx_{n-1}(t) = Tx(t), \quad \lim_{n\to\infty} Sx_{n-1}(t) = Sx(t), \quad t \in J$$

and once again, for F,

$$\begin{aligned}
&\lim_{n\to\infty}\int_0^t F(s,x_{n-1}(\delta_1(s)),x'_{n-1}(\delta_2(s)),Tx_{n-1}(s),Sx_{n-1}(s))\\
&=\int_0^t F(s,x(\delta_1(s)),x'(\delta_2(s)),Tx(s),Sx(s)),\ t\in J.
\end{aligned}\tag{3.14}$$

Likewise, I_k and $\hat{I}_k$, $k=1,2,\dots$ satisfy the inequalities

$$\begin{gathered}
\|I_k(t_k,x_n(t_k),x'_n(t_k))-I_k(t_k,x(t_k),x'(t_k))\|\le 2(a_k+b_k)\frac{\beta}{1-\alpha}+2c_k\\
\|\hat{I}_k(t_k,x_n(t_k),x'_n(t_k))-\hat{I}_k(t_k,x(t_k),x'(t_k))\|\le 2(d_k+e_k)\frac{\beta}{1-\alpha}+2f_k;
\end{gathered}$$

which give at once the limits

$$\begin{gathered}
\lim_{n\to\infty} I_k(t_k,x_n(t_k),x'_n(t_k))=I_k(t_k,x(t_k),x'(t_k))\\
\lim_{n\to\infty} \hat{I}_k(t_k,x_n(t_k),x'_n(t_k))=\hat{I}_k(t_k,x(t_k),x'(t_k)).
\end{gathered}$$

Now, since the series $\sum_{0<t_k<t} I_k$ and $\sum_{0<t_k<t} \hat{I}_k$ converge, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{0<t_k<t}[I_k(t_k,x_n(t_k),x'_n(t_k))+(t-t_k)\hat{I}_k(t_k,x_n(t_k),x'_n(t_k))]\\
&=\sum_{0<t_k<t}[I_k(t_k,x(t_k),x'(t_k))+(t-t_k)\hat{I}_k(t_k,x(t_k),x'(t_k))].
\end{aligned}\tag{3.15}$$

Taking the limit of $x_n(t)=Bx_{n-1}(t)$, when $n\to\infty$, we get from (3.14) and (3.15), that $x(t)$ is a solution of (2.1). Consequently, $x(t)$ is a solution of (1.1)-(1.3). Finally, as $x_0\in P\backslash\{\theta\}$, then $x(t)$ is positive, which completes the proof. □

Example 3.4. Let us consider the initial-value problem

$$\begin{gathered}
x''(t)=\frac{1}{(t+1)^{102}}\Big(e^{-t}(t+1)^{102}+x(t)+\frac{x'(t)}{2}+Tx(t)+Sx(t)\Big),\quad t\neq k;\ k\ge 1,\\
\Delta x=I_k(k,x,x')=\frac{1}{1000^k}\Big(x(k)+\sqrt{x'(k)}+\cos k\Big),\quad k=1,2\dots,\\
\Delta x'=\hat{I}_k(k,x,x')=\frac{1}{1000^k}\Big(\sqrt{x(k)}+x'(k)+\cos k\Big),\quad k=1,2\dots,\\
x(0)=\frac{1}{2};\quad x'(0)=\frac{1}{2},
\end{gathered}\tag{3.16}$$

where

$$\begin{gathered}
Tx(t)=\int_0^t\Big(\frac{1}{(t+1+s)^{3/2}}x(s)+\frac{e^{-s}}{\sqrt{s}}\int_0^s\frac{1}{(s+\tau)^{1/2}}d\tau\Big)ds,\\
Sx(t)=\int_0^{+\infty}\frac{t-s}{t+1}e^{-s}\sqrt{x(s)}ds.
\end{gathered}$$

It is easy to check that all the hypotheses cited in the preliminaries and lemmas are satisfied. Hence Theorem 3.3 ensures the existence of at least one positive solution to (3.16) whenever

$$\alpha=(p^*(2+T^*+S^*)+a+b+d+e)<1.$$

This solution satisfies the impulsive integral equation (2.1) with $\|x\|_D \le \frac{\beta}{1-\alpha}$; where $\beta = 1+q^*+c+f$. According to the above notations a straightforward computation gives the following:

$$
\begin{gathered}
p^* = \int_0^{+\infty} \frac{(s+1)}{(s+1)^{102}} ds = \frac{1}{100}; \quad q^* = \int_0^{+\infty} e^{-s} ds = 1 \\
T^* = m_1^* + m_2^*, \quad m_1^* = \sup_{t\in J} \int_0^t \frac{1}{(t+1+s)^{3/2}} ds < 2, \\
m_2^* = \sup_{t\in J} \int_0^t \frac{e^{-s}}{\sqrt{s}} \int_0^s \frac{1}{(s+\tau)^{1/2}} d\tau ds = 2(\sqrt{2}-1), \\
S^* = m_4^* = \sup_{t\in J} \int_0^{+\infty} \frac{t-s}{t+1} e^{-s} ds = 1, \\
a = d = \sum_{k=1}^{\infty} (k+1) \frac{1}{1000^k} < \frac{1}{999} + \frac{1}{1000}, \\
b = e = c = f = \sum_{k=1}^{\infty} \frac{1}{1000^k} = \frac{1}{999}.
\end{gathered}
$$

Therefore, $\alpha = \frac{1}{100}(2\sqrt{2} + m_1^* + 2) + \frac{2}{999} + 2a < 1$ and $\beta = 2(1 + \frac{1}{999})$.

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Rahima Atmania

Laboratory of Applied Mathematics (LMA), Department of Mathematics, University of Badji Mokhtar Annaba, P.O. Box 12, Annaba 23000, Algeria

E-mail address: atmanira@yahoo.fr