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MULTIPLICITY RESULTS FOR p-SUBLINEAR p-LAPLACIAN PROBLEMS INVOLVING INDEFINITE EIGENVALUE PROBLEMS VIA MORSE THEORY

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Abstract. We establish some multiplicity results for a class of p-sublinear p-Laplacian problems involving indefinite eigenvalue problems using Morse theory.

1. Introduction

The purpose of this note is to establish some multiplicity results for a class of p-sublinear p-Laplacian problems involving indefinite eigenvalue problems using Morse theory.

As motivation, we begin by recalling a well-known result for the semilinear elliptic boundary value problem

$$\begin{gathered} -\Delta u = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, f is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying the sublinear growth condition

$$|f(x,t)| \leq C\left(|t|^{r-1} + 1\right) \tag{1.2}$$

for some $r \in (1,2)$, and C denotes a generic positive constant. Weak solutions of (1.1) coincide with the critical points of the C^1-functional

$$\Phi(u) = \int_\Omega \frac{1}{2}\,|\nabla u|^2 - F(x,u), \quad u \in H_0^1(\Omega)$$

where $F(x,t) = \int_0^s f(x,s)\,ds$ is the primitive of f. By (1.2), Φ is bounded from below and satisfies the (PS) condition.

Assume that

$$\lim_{t\to 0} \frac{f(x,t)}{t} = \lambda, \quad \text{uniformly a.e.,} \tag{1.3}$$

which implies $f(x,0) = 0$ a.e. and hence (1.1) has the trivial solution $u(x) \equiv 0$. Let $\lambda_1 < \lambda_2 \leq \dots$ denote the Dirichlet eigenvalues of the negative Laplacian on Ω. If $\lambda > \lambda_1$ and is not an eigenvalue, then (1.1) has at least two nontrivial solutions.

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Indeed, if $\lambda_k < \lambda < \lambda_{k+1}$, then the (cohomological) critical groups of Φ at zero are given by

$$C^q(\Phi, 0) \approx \delta_{qk}\, \mathcal{G}$$

where $\mathcal{G}$ is the coefficient group and $\delta_{\cdot,\cdot}$ denotes the Kronecker delta (see, e.g., Chang [3] or Mawhin and Willem [10]), so Φ has two nontrivial critical points by the following "three critical points theorem" of Chang [2] and Liu and Li [9].

Proposition 1.1. *Let Φ be a C^1-functional defined on a Banach space. If Φ is bounded from below, satisfies* (PS)*, and $C^k(\Phi, 0) \neq 0$ for some $k \geq 1$, then Φ has two nontrivial critical points.*

Remark 1.2. Li and Willem [6] used a local linking to obtain a similar result when λ is an eigenvalue and f satisfies a suitable sign condition near zero.

The above result can be extended to the corresponding p-sublinear p-Laplacian problem

$$\begin{gathered} -\Delta_p\, u = f(x, u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

where $\Delta_p\, u = \operatorname{div}\big(|\nabla u|^{p-2}\, \nabla u\big)$ is the p-Laplacian of u, $p \in (1, \infty)$, and f now satisfies (1.2) with $r \in (1, p)$. Then the associated variational functional

$$\Phi(u) = \int_\Omega \frac{1}{p}\, |\nabla u|^p - F(x, u), \quad u \in W_0^{1,p}(\Omega)$$

is bounded from below and satisfies (PS). Assume that

$$\lim_{t \to 0} \frac{f(x, t)}{|t|^{p-2}\, t} = \lambda, \quad \text{uniformly a.e.} \tag{1.5}$$

The associated quasilinear eigenvalue problem

$$\begin{gathered} -\Delta_p\, u = \lambda\, |u|^{p-2}\, u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

is far more complicated. It is known that the first eigenvalue λ_1 is positive, simple, and has an associated eigenfunction φ_1 that is positive in Ω (see Anane [1] and Lindqvist [7, 8]). Moreover, λ_1 is isolated in the spectrum $\sigma(-\Delta_p)$, so the second eigenvalue $\lambda_2 = \inf\, \sigma(-\Delta_p) \cap (\lambda_1, \infty)$ is well-defined. In the ODE case $n = 1$, where Ω is an interval, the spectrum consists of a sequence of simple eigenvalues $\lambda_k \nearrow \infty$, and the eigenfunction φ_k associated with λ_k has exactly $k - 1$ interior zeroes (see, e.g., Drábek [4]). In the PDE case $n \geq 2$, an increasing and unbounded sequence of eigenvalues can be constructed using a standard minimax scheme involving the Krasnoselskii's genus, but it is not known whether this gives a complete list of the eigenvalues.

Perera [11] used a minimax scheme involving the $\mathbb{Z}_2$-cohomological index of Fadell and Rabinowitz [5] to construct a new sequence of eigenvalues $\lambda_k \nearrow \infty$ such that if $\lambda_k < \lambda < \lambda_{k+1}$ in (1.5), then

$$C^k(\Phi, 0) \neq 0$$

and hence Φ has two nontrivial critical points by Proposition 1.1. Thus, problem (1.4) has at least two nontrivial solutions when $\lambda > \lambda_1$ is not an eigenvalue from this particular sequence.

Note that (1.5) implies $t f(x,t) > 0$ for $t \neq 0$ near zero when $\lambda > 0$. Naturally we may ask whether these results hold without such a sign condition. More specifically, can we replace (1.5) with

$$\lim_{t \to 0} \frac{f(x,t)}{|t|^{p-2}\, t} = \lambda\, V(x), \quad \text{uniformly a.e.} \tag{1.6}$$

and let V change sign?

This leads us to the indefinite eigenvalue problem

$$\begin{gathered} -\Delta_p\, u = \lambda\, V(x)\, |u|^{p-2}\, u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.7}$$

We assume that the weight function $V \in L^s(\Omega)$ for some

$$s \begin{cases} > n/p, & p \le n \\ = 1, & p > n. \end{cases} \tag{1.8}$$

Then the smallest positive and largest negative eigenvalues of (1.7) are given by

$$\lambda_1^+ = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_\Omega V(x)\, |u|^p > 0}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega V(x)\, |u|^p}, \quad \lambda_1^- = \sup_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_\Omega V(x)\, |u|^p < 0}} \frac{\int_\Omega |\nabla u|^p}{\int_\Omega V(x)\, |u|^p},$$

respectively. Noting that (1.6) implies

$$F(x,t) = \frac{\lambda}{p}\, V(x)\, |t|^p + o(|t|^p) \quad \text{as } t \to 0, \text{ uniformly a.e.,} \tag{1.9}$$

we shall prove the following result.

Theorem 1.3. *Assume* (1.2) *with* $r \in (1,p)$, $V \in L^s(\Omega)$ *with* s *satisfying* (1.8), *and* (1.9). *If* $\lambda \notin (\lambda_1^-, \lambda_1^+)$ *and is not an eigenvalue of* (1.7), *then problem* (1.4) *has at least two nontrivial solutions.*

Since $\lambda_1^- = -\infty$ when $V \ge 0$ a.e. and $\lambda_1^+ = +\infty$ when $V \le 0$ a.e., this theorem applies in all possible cases:

(i) V changes sign: $\dots < \lambda_1^- < 0 < \lambda_1^+ < \dots$,
(ii) $V \ge 0$ a.e. and $\not\equiv 0$: $-\infty = \lambda_1^- < 0 < \lambda_1^+ < \dots$,
(iii) $V \le 0$ a.e. and $\not\equiv 0$: $\dots < \lambda_1^- < 0 < \lambda_1^+ = +\infty$,
(iv) $V \equiv 0$: $-\infty = \lambda_1^- < \lambda_1^+ = +\infty$ (in this case the theorem is vacuously true).

Our proof will be based on an abstract framework for indefinite eigenvalue problems introduced in Perera, Agarwal, and O'Regan [12], which we will recall in the next section.

2. Preliminaries

In this section we recall an abstract framework for indefinite eigenvalue problems introduced in Perera, Agarwal, and O'Regan [12].

Let $(W, \|\cdot\|)$ be a real reflexive Banach space with the dual $(W^*, \|\cdot\|)$ and the duality pairing $(\cdot,\cdot)$. We consider the nonlinear eigenvalue problem

$$A_p\, u = \lambda\, B_p\, u \tag{2.1}$$

in W^*, where $A_p \in C(W, W^*)$ is

(A1) $(p-1)$-homogeneous and odd for some $p \in (1, \infty)$:
$$A_p(\alpha u) = |\alpha|^{p-2}\, \alpha\, A_p\, u \quad \forall u \in W,\, \alpha \in \mathbb{R},$$
(A2) uniformly positive: $\exists\, c_0 > 0$ such that
$$(A_p\, u, u) \geq c_0\, \|u\|^p \quad \forall u \in W,$$
(A3) a potential operator: there is a functional $I_p \in C^1(W, \mathbb{R})$, called a potential for A_p, such that
$$I_p'(u) = A_p\, u \quad \forall u \in W,$$
(A4) of type (S): for any sequence $\{u_j\} \subset W$,
$$u_j \rightharpoonup u, \quad (A_p\, u_j, u_j - u) \to 0 \implies u_j \to u,$$
and $B_p \in C(W, W^*)$ is

(B1) $(p-1)$-homogeneous and odd,
(B2) a compact potential operator.

The following proposition is often useful for verifying (A4).

Proposition 2.1 ([12, Proposition 1.0.3])**.** *If W is uniformly convex and*
$$(A_p\, u, v) \leq r\, \|u\|^{p-1}\, \|v\|, \quad (A_p\, u, u) = r\, \|u\|^p \quad \forall u, v \in W$$
for some $r > 0$, then (A4) holds.

By [12, Proposition 1.0.2], the potentials I_p and J_p of A_p and B_p satisfying $I_p(0) = 0 = J_p(0)$ are given by
$$I_p(u) = \frac{1}{p}\, (A_p\, u, u), \quad J_p(u) = \frac{1}{p}\, (B_p\, u, u),$$
respectively, and are p-homogeneous and even. Let
$$\mathcal{M} = \{u \in W : I_p(u) = 1\}, \quad \mathcal{M}^{\pm} = \{u \in \mathcal{M} : J_p(u) \gtrless 0\}.$$
Then $\mathcal{M} \subset W \setminus \{0\}$ is a bounded complete symmetric C^1-Finsler manifold radially homeomorphic to the unit sphere in W, $\mathcal{M}^{\pm}$ are symmetric open submanifolds of $\mathcal{M}$, and the positive (resp. negative) eigenvalues of (2.1) coincide with the critical values of the even C^1-functionals
$$\Psi^{\pm}(u) = \frac{1}{J_p(u)}, \quad u \in \mathcal{M}^{\pm}$$
(see [12, Sections 9.1 and 9.2]).

Denote by $\mathcal{F}^{\pm}$ the classes of symmetric subsets of $\mathcal{M}^{\pm}$ and by $i(M)$ the Fadell-Rabinowitz cohomological index of $M \in \mathcal{F}^{\pm}$. Then
$$\lambda_k^+ := \inf_{\substack{M \in \mathcal{F}^+ \\ i(M) \geq k}}\ \sup_{u \in M}\ \Psi^+(u), \quad 1 \leq k \leq i(\mathcal{M}^+),$$
$$\lambda_k^- := \sup_{\substack{M \in \mathcal{F}^- \\ i(M) \geq k}}\ \inf_{u \in M}\ \Psi^-(u), \quad 1 \leq k \leq i(\mathcal{M}^-)$$
define nondecreasing (resp. nonincreasing) sequences of positive (resp. negative) eigenvalues of (2.1) that are unbounded when $i(\mathcal{M}^{\pm}) = \infty$ (see [12, Theorems 9.1.2 and 9.2.1]). When $\mathcal{M}^{\pm} = \emptyset$, we set $\lambda_1^{\pm} = \pm\infty$ for convenience.

Now we consider the operator equation
$$A_p\, u = F'(u) \tag{2.2}$$

where $F \in C^1(W, \mathbb{R})$ with F' compact, whose solutions coincide with the critical points of the functional

$$\Phi(u) = I_p(u) - F(u), \quad u \in W.$$

The following proposition is useful for verifying the (PS) condition for Φ.

Proposition 2.2 ([12, Lemma 3.1.3]). *Every bounded* (PS) *sequence of* Φ *has a convergent subsequence.*

Suppose that $u = 0$ is a solution of (2.2) and the asymptotic behavior of F near zero is given by

$$F(u) = \lambda\, J_p(u) + o(\|u\|^p) \quad \text{as } u \to 0. \tag{2.3}$$

Proposition 2.3 ([12, Proposition 9.4.1]). *Assume (A1) - (A4), (B1), (B2), and* (2.3) *hold,* F' *is compact, and zero is an isolated critical point of* Φ.

(i) *If* $\lambda_1^- < \lambda < \lambda_1^+$, *then* $C^q(\Phi, 0) \approx \delta_{q0}\, \mathbb{Z}_2$.

(ii) *If* $\lambda_{k+1}^- < \lambda < \lambda_k^-$ *or* $\lambda_k^+ < \lambda < \lambda_{k+1}^+$, *then* $C^k(\Phi, 0) \neq 0$.

3. Proof of Theorem 1.3

First let us verify that our problem fits into the abstract framework of the previous section. Let $W = W_0^{1,p}(\Omega)$,

$$(A_p\, u, v) = \int_\Omega |\nabla u|^{p-2}\, \nabla u \cdot \nabla v, \quad (B_p\, u, v) = \int_\Omega V(x)\, |u|^{p-2}\, uv,$$

and

$$F(u) = \int_\Omega F(x, u).$$

Then (A1) and (B1) are clear, $(A_p\, u, u) = \|u\|^p$ in (A2), and (A3) and (B2) hold with

$$I_p(u) = \frac{1}{p} \int_\Omega |\nabla u|^p, \quad J_p(u) = \frac{1}{p} \int_\Omega V(x)\, |u|^p,$$

respectively. By the Hölder inequality,

$$(A_p\, u, v) \le \Big(\int_\Omega |\nabla u|^p \Big)^{1 - \frac{1}{p}} \Big(\int_\Omega |\nabla v|^p \Big)^{1/p} = \|u\|^{p-1}\, \|v\|,$$

so (A4) follows from Proposition 2.1. By (1.2) and (1.9), (2.3) also holds.

Since $\lambda \notin (\lambda_1^-, \lambda_1^+)$ and is not an eigenvalue of (1.7), it now follows from Proposition 2.3 that $C^k(\Phi, 0) \neq 0$ for some $k \ge 1$. By (1.2),

$$|F(x, t)| \le C\, (|t|^r + 1),$$

so by the Sobolev imbedding,

$$\Phi(u) \ge \frac{1}{p}\, \|u\|^p - C\, (\|u\|^r + 1) \quad \forall u \in W_0^{1,p}(\Omega).$$

Since $p > r$, it follows that Φ is bounded from below and coercive. Then every (PS) sequence of Φ is bounded and hence Φ satisfies the (PS) condition by Proposition 2.2. Thus, Φ has two nontrivial critical points by Proposition 1.1.

Remark 3.1. Note that it suffices to assume $\lambda \notin (\lambda_1^-, \lambda_1^+)$ is not an eigenvalue from the particular sequences $(\lambda_k^\pm)$.

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