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STOCHASTIC STABILITY OF COHEN-GROSSBERG NEURAL NETWORKS WITH UNBOUNDED DISTRIBUTED DELAYS

PING CHEN, CHUANGXIA HUANG, XIAOLIN LIANG

Abstract. In this article, we consider a model that describes the dynamics of Cohen-Grossberg neural networks with unbounded distributed delays, whose state variable are governed by stochastic non-linear integro-differential equations. Without assuming the smoothness, monotonicity and boundedness of the activation functions, by constructing suitable Lyapunov functional, employing the semi-martingale convergence theorem and some inequality, we obtain some sufficient criteria to check the almost exponential stability of networks.

1. Introduction

Cohen-Grossberg neural networks (CGNN) were first introduced by Cohen and Grossberg [10] in 1983 and soon the class of networks have been the subject of extensive investigation because of their many important applications, such as patter recognition, associative memory and combinatorial optimization, etc. Especially, CGNN with delays have attracted many scientific and technical works due to their applications for solving a number of problems in various scientific disciplines, such application heavily depend on the dynamic behave of networks [29], thus, the analysis of the dynamical behaviors such as stability is a necessary step for practical design of neural networks. To date, many important results on the stability have been reported in the literature, see e.g [2, 3, 4, 6, 12, 14, 18, 19, 23, 24, 26, 27] and reference therein. We refer to Cao and Liang [3] for the mathematical model of CGNN that consists of n ($n > 1$) interconnected neighboring cells whose dynamical behavior are described by

$$\frac{dx_i(t)}{dt} = -h_i(x_i(t))[c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(x_j(t-\tau_{ij}))], \tag{1.1}$$

where $i = 1, 2 \dots, n$; $x_i(t)$ denotes the state variable associated with the ith neuron at time t; $h_i(\cdot)$ represents an amplification function; $c_i(\cdot)$ is an appropriately behaved function; $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n}$ weight the strength of the jth unit

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on the ith unit at time t; $f_j(\cdot)$ denotes a non-linear output function, τ_{ij} corresponds to the time delay required in processing and transmitting a signal from the jth cell to the ith cell.

In formulating the network model (1.1), the delays are assumed to be discrete, however, just as is pointed out in [5], constant fixed delays in the models of delayed feedback systems serve as a good approximation in simple circuits consisting of a small number of cells, but neural networks usually have a spatial extent due to the presence of the presence of an amount of parallel pathways with variety of axon sizes and lengths. Therefore, there will be a distribution of conduction velocities along these pathways and it is of significant importance to consider continuously distributed delays to the neural networks (see [7, 9, 22, 30]). Then model (1.1) can be modified as a system of integro-differential equations of the from

$$\frac{dx_i(t)}{dt} = -h_i(x_i(t))[c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(\int_{-\infty}^{t} K_{ij}(t-s)x_j(s)ds)] \tag{1.2}$$

Which for convenience can be put in the form

$$\frac{dx_i(t)}{dt} = -h_i(x_i(t))[c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij} f_j(\int_{0}^{+\infty} K_{ij}(s)x_j(t-s)ds)] \tag{1.3}$$

With initial values given by $x_i(s) = \phi_i(s)$ for $s \in (-\infty, 0]$, where each $\phi_i(\cdot)$ is bounded and continuous on $(-\infty, 0]$.

Just as is pointed out by Haykin [13], in real nervous systems and in the implementation of artificial neural networks, noise is unavoidable and should be taken into consideration in modelling. Under the effect of the noise, the trajectory of system becomes a stochastic process. Moreover, it was realized that CGNN could be stabilized or destabilized by certain stochastic input [1, 20]. Therefore it is of significant importance to consider stochastic effects to the stability of delayed neural networks, and the existing literature on theoretical studies of stochastic CGNN is predominantly concerned with constant fixed delay, time-varying delays and bounded distributed delays [9, 11, 15, 16, 17, 21, 25, 28]. To the best our knowledge, few authors discuss almost sure exponential stability of stochastic Cohen-Grossberg neural networks with unbounded distributed delays.

Motivated by the above discussions, in this paper, we investigate almost sure exponential stability of stochastic Cohen-Grossberg neural networks with unbounded distributed delays. By the following stochastic nonlinear integro-differential equations

$$\begin{aligned} dx_i(t) = &-h_i(x_i(t))[c_i(x_i(t)) - \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) \\ &- \sum_{j=1}^{n} b_{ij} f_j(\int_{0}^{+\infty} K_{ij}(s)x_j(t-s)ds)]dt + \sum_{j=1}^{n} \sigma_{ij}(x_j(t))d\omega_j(t), \end{aligned} \tag{1.4}$$

where $t \geq 0$, $\sigma(\cdot) = (\sigma_{ij}(\cdot))_{n\times n}$ is the diffusion coefficient matrix ,and $\omega(t) = (\omega_1(t), \dots, \omega_n(t))^T$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

2. Preliminaries

Let $\mathcal{C} = C((-\infty, 0], \mathbb{R}^n)$ be the Banach space of continuous functions which map into $\mathbb{R}^n$ with the topology of uniform convergence. For $x(t) = (x_1(t), \dots, x_n(t))^T$ in $\mathbb{R}^n$, we define $\|x(t)\| = (\sum_{i=1}^n |x_i(t)|^2)^{1/2}$. For $\varphi \in \mathcal{C}$, define $\|\varphi\| = (\sum_{i=1}^n |\varphi_i|^2)^{1/2}$, where $\|\varphi\| = \sup_{-\infty \le s \le 0}\{\|\varphi(s)\|\}$.

System (1.4) can be rewritten in the vector form

$$\begin{aligned} dx(t) &= -H(x(t))[C(x(t)) - AF(x(t)) \\ &\quad - BF(\int_0^{+\infty} K(s)x(t-s)ds)]dt + \sigma(x(t))d\omega(t) \end{aligned} \tag{2.1}$$

where $x(t) = (x_1(t), \dots, x_n(t))^T$, $H(x(t)) = \operatorname{diag}(h_1(x_1(t)), \dots, h_n(x_n(t)))$,

$$A = (a_{ij})_{n\times n}, \quad B = (b_{ij})_{n\times n}, \quad C(x(t)) = (c_1(x_1(t)), \dots, c_n(x_n(t)))^T,$$

$$F(x(t)) = (f_1(x_1(t)), \dots, f_n(x_n(t)))^T, \quad K(s) = (k_{ij}(s))_{n\times n},$$

$$\sigma(x(t)) = (\sigma_{ij}(x_j(t)))_{n\times n}.$$

The initial conditions for (2.1) are $x(s) = \varphi(s), -\infty \le s \le 0, \varphi \in L^2_{\mathcal{F}_0}((-\infty, 0], \mathbb{R}^n)$, here $L^2_{\mathcal{F}_0}((-\infty, 0], \mathbb{R}^n)$ is $\mathbb{R}^n$-valued stochastic process $\varphi(s), -\infty \le s \le 0$, $\varphi(s)$ is $\mathcal{F}_0$-measurable, $\int_{-\infty}^0 E|\varphi(s)|^2 ds < \infty$.

Let $C^{2,1}(\mathbb{R}^n \times R; R_+)$ denote the family of all nonnegative functions $V(x,t)$ on $\mathbb{R}^n \times R$ which are twice differentiable in x and once differentiable in t. If $V \in C^{2,1}(\mathbb{R}^n \times R; R_+)$, define an operator LV associated with (2.1) as

$$\begin{aligned} LV(x,t) &= V_t(x,t) + V_x(x,t)\{-H(x(t))[C(x(t)) - AF(x(t)) \\ &\quad - BF(\int_0^{+\infty} K(s)x(t-s)ds)]\}dt + \frac{1}{2}\operatorname{trace}[\sigma^T V_{xx}(x,t)\sigma] \end{aligned}$$

where $V_t(x,t) = \frac{\partial V(x,t)}{\partial t}$,

$$V_x(x,t) = \Big(\frac{\partial V(x,t)}{\partial x_1}, \dots, \frac{\partial V(x,t)}{\partial x_n}\Big), \quad V_{xx}(x,t) = \Big(\frac{\partial^2 V(x,t)}{\partial x_i \partial x_j}\Big)_{n\times n}.$$

To establish the main results of the model given in (2.1), some of the following assumptions will apply:

- (H1) Each function $h_i(x)$ is bounded, positive and locally Lipschitz continuous; thus, there exist two positive constants $\underline{h}_i$ and $\overline{h}_i$, such that $0 < \underline{h}_i \le h_i(x) \le \overline{h}_i < +\infty$ for all $x \in R$ and $i = 1, 2, \dots, n$.
- (H2) each $i = 1, 2, \dots, n$, there exist constant $\alpha_i > 0$, such that $x_i(t)c_i(x_i(t)) \ge \alpha_i x_i^2(t)$;
- (H3) Both $f_j(\cdot)$ and $\sigma_{ij}(\cdot)$ are globally Lipschitz, and there exist positive constants $\beta_j, L_{ij}, i, j = 1, 2, \dots, n$, such that

  $$|f_j(u) - f_j(v)| \le \beta_j|u - v| : |\sigma_{ij}(u) - \sigma_{ij}(v)| \le L_{ij}|u - v|,$$

  for any $u, v \in \mathbb{R}$. we also assure that $f_j(0) = \sigma_{ij}(0) = 0$.
- (H4) The delay kernels $K_{ij}, i, j, = 1, 2, \dots, n$ are real-valued nonnegative piecewise continuous defined on $[0, +\infty)$ and satisfy

  $$\int_0^{+\infty} K_{ij}(s)ds = 1 \quad \textit{and} \quad \int_0^{+\infty} K_{ij}(s)e^{\mu s}ds < +\infty$$

  for some positive constant μ.

We notice that the activation functions $f_j(\cdot)$ do not have to be differentiable and monotonically increasing, which including some kinds of typical functions widely used for CGNN designs.This implies that (2.1) has a unique global solution on $t \geq 0$ for the initial conditions [1]. Clearly, (2.1) admits an equilibrium solution $x(t) = 0$.

Definition 2.1 ([20])**.** The trivial solution of (2.1) is said to be almost surely exponentially stable if for almost all sample paths of the solution $x(t)$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \|x(t)\| < 0.$$

Lemma 2.2 (Semi-martingale convergence theorem [20])**.** *Let $A(t)$ and $U(t)$ be two continuous adapted increasing process on $t \geq 0$ with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable. Define*

$$X(t) = \xi + A(t) - U(t) + M(t), \quad \text{for } t \geq 0,$$

If $X(t)$ is nonnegative, then

$$\{\lim_{t\to\infty} A(t) < \infty\} \subset \{\lim_{t\to\infty} X(t) < \infty\} \cap \{\lim_{t\to\infty} U(t) < \infty\} \quad a.s.,$$

where $B \subset D$ a.s. means $P(S \cap D^c) = 0$. In particular, If

$$\lim_{t\to\infty} U(t) < \infty \quad a.s.,$$

then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X(t) < \infty \quad \text{and} \quad \lim_{t\to\infty} U(t) < \infty,$$

that is both $X(t)$ and $U(t)$ converge to finite random variables.

3. Main Results

Theorem 3.1. *Under assumptions* (H1)–(H4)*, if there exist a set of positive constants $q_i, q_{ij}, p_{ij}, r_{ij}, q^*_{ij}, p^*_{ij}, r^*_{ij}$ such that for $i = 1, 2, \dots, n$,*

$$\begin{aligned} &2\underline{h}_i\alpha_i q_i - \sum_{j=1}^{n}[q_i\overline{h}_i^{2p_{ij}}|a_{ij}|^{2q_{ij}}\beta_j^{2r_{ij}} + q_j\overline{h}_j^{2-2p_{ji}}|a_{ji}|^{2-2q_{ji}}\beta_i^{2-2r_{ji}} \\ &+ q_i\overline{h}_i^{2p^*_{ij}}|b_{ij}|^{2q^*_{ij}}\beta_j^{2r^*_{ij}} + q_j\overline{h}_j^{2-2p^*_{ji}}|b_{ji}|^{2-2q^*_{ji}}\beta_i^{2-2r^*_{ji}} + q_jL_{ji}^2] > 0, \end{aligned} \tag{3.1}$$

then the trivial solution of system (2.1) *is almost surely exponentially stable.*

Proof. From assumption (H4), we can choose a constant $0 < \lambda < \mu$, such that

$$\int_0^{+\infty} K_{ji}(s)e^{\lambda s}ds \leq \int_0^{+\infty} K_{ji}(s)e^{\mu s}ds \tag{3.2}$$

and for $\quad i = 1, 2, \dots, n$,

$$\begin{aligned} &2\underline{h}_i\alpha_i q_i - \lambda q_i - \sum_{j=1}^{n}[q_i\overline{h}_i^{2p_{ij}}|a_{ij}|^{2q_{ij}}\beta_j^{2r_{ij}} + q_j\overline{h}_j^{2-2p_{ji}}|a_{ji}|^{2-2q_{ji}}\beta_i^{2-2r_{ji}} \\ &+ q_i\overline{h}_i^{2p^*_{ij}}|b_{ij}|^{2q^*_{ij}}\beta_j^{2r^*_{ij}} + q_j\overline{h}_j^{2-2p^*_{ji}}|b_{ji}|^{2-2q^*_{ji}}\beta_i^{2-2r^*_{ji}}\int_0^{+\infty} K_{ji}(s)e^{\lambda s}ds + q_jL_{ji}^2] > 0, \end{aligned} \tag{3.3}$$

Consider the Lyapunov functional

$$\begin{aligned} V(x(t),t) = e^{\lambda t}\sum_{i=1}^n q_i x_i^2(t) + \sum_{i=1}^n\sum_{j=1}^n q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*} \\ \times \int_0^{+\infty}\big[K_{ij}(s)e^{\lambda s}\int_{t-s}^t e^{\lambda\zeta}x_j^2(\zeta)d\zeta\big]ds. \end{aligned} \tag{3.4}$$

Then the operator $LV(x,t)$ associated with system (2.1) has the form

$$\begin{aligned}
&LV(x(t),t)\\
&= \lambda e^{\lambda t}\sum_{i=1}^n q_i x_i^2(t)\\
&\quad + \sum_{i=1}^n\sum_{j=1}^n q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*}\int_0^{+\infty}[K_{ij}(s)e^{\lambda s}(e^{\lambda t}x_j^2(t)\\
&\quad - e^{\lambda(t-s)}x_j^2(t-s))]ds + 2e^{\lambda t}\sum_{i=1}^n q_i x_i(t)\{-h_i(x_i(t))[c_i(x_i(t))\\
&\quad - \sum_{j=1}^n a_{ij}f_j(x_j(t)) - \sum_{j=1}^n b_{ij}f_j(\int_0^{+\infty}K_{ij}x_j(t-s)ds)]\}\\
&\quad + e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_i\sigma_{ij}^2(x_j(t))\\
&\le \lambda e^{\lambda t}\sum_{i=1}^n q_i x_i^2(t)\\
&\quad + e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_j \overline{h}_j^{2-2p_{ji}^*}|b_{ji}|^{2-2q_{ji}^*}\beta_i^{2-2r_{ji}^*}x_i^2(t)\int_0^{+\infty}K_{ji}(s)e^{\lambda s}ds\\
&\quad - e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*}\int_0^{+\infty}K_{ij}(s)x_j^2(t-s)ds\\
&\quad - 2e^{\lambda t}\sum_{i=1}^n q_i\underline{h}_i|x_i(t)|^2\alpha_i + 2e^{\lambda t}\sum_{i=1}^n q_i[\sum_{j=1}^n \overline{h}_i|a_{ij}|\beta_j|x_j(t)||x_i(t)|\\
&\quad + \sum_{j=1}^n \overline{h}_i|b_{ij}|\beta_j(\int_0^{+\infty}K_{ij}(s)x_j(t-s)x_i(t)ds)]\\
&\quad + e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_j\sigma_{ji}^2(x_i(t))\\
&\le \lambda e^{\lambda t}\sum_{i=1}^n q_i x_i^2(t)\\
&\quad + e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_j \overline{h}_j^{2-2p_{ji}^*}|b_{ji}|^{2-2q_{ji}^*}\beta_i^{2-2r_{ji}^*}x_i^2(t)\int_0^{+\infty}K_{ji}(s)e^{\lambda s}ds
\end{aligned}$$

$$
\begin{aligned}
& - e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*}\int_0^{+\infty} K_{ij}(s)x_j^2(t-s)ds \\
& - 2e^{\lambda t}\sum_{i=1}^{n} q_i \underline{h}_i |x_i(t)|^2\alpha_i \\
& + 2e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{p_{ij}}|a_{ij}|^{q_{ij}}\beta_j^{r_{ij}}|x_i(t)|\overline{h}_i^{1-p_{ij}}|a_{ij}|^{1-q_{ij}}\beta_j^{1-r_{ij}}|x_j(t)| \\
& + 2e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{p_{ij}^*}|b_{ij}|^{q_{ij}^*}\beta_j^{r_{ij}^*}\overline{h}_i^{1-p_{ij}^*}|b_{ij}|^{1-q_{ij}^*}\beta_j^{1-r_{ij}^*}\int_0^{+\infty} K_{ij}(s)x_j(t-s)x_i(t)ds \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_j L_{ji}^2(x_i(t))^2 \\
\le\ & \lambda e^{\lambda t}\sum_{i=1}^{n} q_i x_i^2(t) \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_j \overline{h}_j^{2-2p_{ji}^*}|b_{ji}|^{2-2q_{ji}^*}\beta_i^{2-2r_{ji}^*}x_i^2(t)\int_0^{+\infty} K_{ji}(s)e^{\lambda s}ds \\
& - e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*}\int_0^{+\infty} K_{ij}(s)x_j^2(t-s)ds \\
& - 2e^{\lambda t}\sum_{i=1}^{n} q_i \underline{h}_i |x_i(t)|^2\alpha_i + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{2p_{ij}}|a_{ij}|^{2q_{ij}}\beta_j^{2r_{ij}}|x_i(t)|^2 \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_j \overline{h}_j^{2-2p_{ji}}|a_{ji}|^{2-2q_{ji}}\beta_i^{2-2r_{ji}}|x_j(t)|^2 \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{2p_{ij}^*}|b_{ij}|^{2q_{ij}^*}\beta_j^{2r_{ij}^*}|x_i(t)|^2 \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_i \overline{h}_i^{2-2p_{ij}^*}|b_{ij}|^{2-2q_{ij}^*}\beta_j^{2-2r_{ij}^*}\int_0^{+\infty} K_{ij}(s)x_j^2(t-s)ds \\
& + e^{\lambda t}\sum_{i=1}^{n}\sum_{j=1}^{n} q_j L_{ji}^2(x_i(t))^2 \\
=\ & -e^{\lambda t}\sum_{i=1}^{n}|x_i(t)|^2(-\lambda q_i + 2\underline{h}_i\alpha_i q_i - \sum_{j=1}^{n} q_i \overline{h}_i^{2p_{ij}}|a_{ij}|^{2q_{ij}}\beta_j^{2r_{ij}} \\
& - \sum_{j=1}^{n} q_j \overline{h}_j^{2-2p_{ji}}|a_{ji}|^{2-2q_{ji}}\beta_i^{2-2r_{ji}} - \sum_{j=1}^{n} q_i \overline{h}_i^{2p_{ij}^*}|b_{ij}|^{2q_{ij}^*}\beta_j^{2r_{ij}^*} \\
& - \sum_{j=1}^{n} q_j \overline{h}_j^{2-2p_{ji}^*}|b_{ji}|^{2-2q_{ji}^*}\beta_i^{2-2r_{ji}^*}\int_0^{+\infty} K_{ji}(s)e^{\lambda s}ds - \sum_{j=1}^{n} q_j L_{ji}^2). \qquad (3.5)
\end{aligned}
$$

Using the Itô formula, for $T > 0$, from inequality (3.2), (3.3), (3.4) and (3.5), we have

$$
\begin{aligned}
&V(x(t),t)\\
&\le \sum_{i=1}^n q_i x_i^2(0) + \sum_{i=1}^n\sum_{j=1}^n q_i \overline{h}_i^{2-2p^*_{ij}} |b_{ij}|^{2-2q^*_{ij}} \beta_j^{2-2r^*_{ij}}\\
&\quad \times \int_0^{+\infty} [K_{ij}(s)e^{\lambda s}\int_{-s}^0 e^{\lambda\zeta} x_j^2(\zeta)d\zeta]ds\\
&\quad - \int_0^T e^{\lambda t}\sum_{i=1}^n |x_i(t)|^2(-\lambda q_i + 2\underline{h}_i\alpha_i q_i - \sum_{j=1}^n q_i\overline{h}_i^{2p_{ij}}|a_{ij}|^{2q_{ij}}\beta_j^{2r_{ij}}\\
&\quad - \sum_{j=1}^n q_j\overline{h}_j^{2-2p_{ji}}|a_{ji}|^{2-2q_{ji}}\beta_i^{2-2r_{ji}} - \sum_{j=1}^n q_i\overline{h}_i^{2p^*_{ij}}|b_{ij}|^{2q^*_{ij}}\beta_j^{2r^*_{ij}}\\
&\quad - \sum_{j=1}^n q_j\overline{h}_j^{2-2p^*_{ji}}|b_{ji}|^{2-2q^*_{ji}}\beta_i^{2-2r^*_{ji}}\int_0^{+\infty} K_{ji}(s)e^{\lambda s}ds - \sum_{j=1}^n q_j L_{ji}^2)\\
&\quad + 2\int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_i|x_i(t)\sigma_{ij}(x_i(t))|d\omega_j(t)\\
&\le \sum_{i=1}^n q_i[1+\sum_{j=1}^n \frac{q_j}{q_i}\overline{h}_j^{2-2p^*_{ji}}|b_{ji}|^{2-2q^*_{ji}}\beta_i^{2-2r^*_{ji}}\int_0^{+\infty}K_{ji}(s)e^{\lambda s}ds]\sup_{-\infty\le s\le 0}\{x_i^2(s)\}\\
&\quad + 2\int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_i|x_i(t)||\sigma_{ij}(x_j(t))|d\omega_j(t).
\end{aligned}
\tag{3.6}
$$

Then right-hand side of (3.6) is a nonnegative martingale and Lemma 2.2 shows

$$
\lim_{T\to 0} X(T) < \infty \quad \text{a.s.}
$$

where

$$
\begin{aligned}
X(T) = \sum_{i=1}^n q_i[1+\sum_{j=1}^n \frac{q_j}{q_i}\overline{h}_j^{2-2p^*_{ji}}|b_{ji}|^{2-2q^*_{ji}}\beta_i^{2-2r^*_{ji}}\int_0^{+\infty}K_{ji}(s)e^{\lambda s}ds\}\\
\times \sup_{-\infty\le s\le 0}\{x_i^2(s)\} + 2\int_0^T e^{\lambda t}\sum_{i=1}^n\sum_{j=1}^n q_i|x_i(t)||\sigma_{ij}(x_j(t))|d\omega_j(t).
\end{aligned}
\tag{3.7}
$$

It follows by Lemma 2.2 that

$$
\lim_{T\to\infty} e^{\lambda t}\sum_{i=1}^n q_i x_i^2(t) < \infty \quad \text{a.s.} \tag{3.8}
$$

which implies

$$
\lim_{T\to\infty} e^{\lambda t}\sum_{i=1}^n x_i^2(t) < \infty \quad \text{a.s.}; \tag{3.9}
$$

that is,

$$
\limsup_{T\to\infty}\frac{1}{T}\log\|x(T)\| \le -\lambda. \tag{3.10}
$$

This completes the proof. □

Corollary 3.2. *Under assumptions* (H1)–(H4)*, if there exist a set of positive constants q_i such that*

$$2\underline{h}_i\alpha_i q_i-\sum_{j=1}^{n}[2q_i+q_j\overline{h}_j^2|a_{ji}|^2\beta_i^2+q_j\overline{h}_j^2|b_{ji}|^2\beta_i^2+q_jL_{ji}^2]>0,\quad i=1,2,\dots,n \tag{3.11}$$

then the trivial solution of (2.1) *is almost sure exponentially stable.*

Corollary 3.3. *Under assumptions* (H1)–(H4)*, if the following inequality holds*

$$2\underline{h}_i\alpha_i-\sum_{j=1}^{n}[(|a_{ij}|+|b_{ij}|)\overline{h}_i\beta_j+(|a_{ji}|+|b_{ji}|)\overline{h}_j\beta_i+L_{ji}^2]>0,\quad i=1,2,\dots,n \tag{3.12}$$

then the trivial solution of (2.1) *is almost sure exponentially stable.*

Proof. Choose $q_i=1$, $q_{ij}=p_{ij}=r_{ij}=q_{ij}^*=p_{ij}^*=r_{ij}^*=\frac{1}{2}$ for $i,j=1,2,\dots,n$ in inequality (3.1). By Theorem 3.1, the proof is complete. □

To the best of our knowledge, few authors have considered the almost sure exponential stability for stochastic CGNN with unbounded distributed delays. We can find the recent papers [2, 3, 7, 14, 30] in this direction, however, all delays in are discrete and the delay functions appearing in them are bounded, obviously, those requirements are relaxed in this paper and as model (2.1) can be viewed as a general case of interval-delayed recurrent neural networks and delayed Hopfiled networks.

Remark 3.4. For system (2.1), when $h_i(x_i(t))=1$, then it turns out to be following stochastic cellular neural networks with unbounded distributed delays

$$dx(t)=[-C(x(t))+AF(x(t))+BF(\int_0^{+\infty}K(s)x(t-s)ds)]dt+\sigma(x(t))d\omega(t),$$

using Theorem 3.1, one can easy to get a set of similar corollary for checking the almost sure exponential stability for the trivial solution of this system.

From the results in this paper, it is easy to see that our development results are more general than those reported in [1, 29]. Moreover, we conclude the stability of system (2.1) is dependent of the magnitude of noise, and therefore, noisy fluctuations should be regarded adequately.

4. Example

In this section, we present an example to demonstrate the correctness and effectiveness of the main results. Consider the stochastic Cohen-Grossberg neural networks with unbounded distributed delays

$$\begin{aligned}dx_1(t)=-(1+\sin x_1(t))[10x_1-\sum_{j=1}^{2}a_{1j}f_j(x_j(t))\\ -\sum_{j=1}^{2}b_{1j}f_j(\int_0^{+\infty}K_{1j}(s)x_j(t-s)ds)]dt+\sum_{j=1}^{2}\sigma_{1j}(x_j(t))d\omega_j(t),\end{aligned}$$

$$
\begin{aligned}
dx_2(t) = &-(1+\cos x_2(t))[4x_2 - \sum_{j=1}^{2} a_{2j} f_j(x_j(t)) \\
&- \sum_{j=1}^{2} b_{2j} f_j(\int_0^{+\infty} K_{2j}(s)x_j(t-s)ds)]dt + \sum_{j=1}^{2} \sigma_{2j}(x_j(t))d\omega_j(t).
\end{aligned} \tag{4.1}
$$

This system satisfies all assumptions in this paper with $f_1(x) = \tanh x, f_2(x) = \frac{1}{2}(|x-1| - |x+1|)$, by taking $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $b_{11} = b_{12} = b_{21} = b_{22} = 1$, and $q_1 = 1$, $a_1 = 18$, $a_2 = 10$, $\underline{h}_1 = 1$, $\overline{h}_1 = 2$, $\underline{h}_2 = 3$, $\overline{h}_2 = 8$, $\beta_1 = \beta_2 = 1$, $L_{ij} = 1$, $i, j = 1, 2$,

$$
K(s) = \begin{pmatrix} 2e^{-2s} & 4e^{-4s} \\ 3e^{-3s} & 5e^{-5s} \end{pmatrix}.
$$

By simple a computation, one can easily show that

$$
2\underline{h}_1\alpha_1 - \sum_{j=1}^{2}[(|a_{1j}| + |b_{1j}|)\overline{h}_1\beta_j + (|a_{j1}| + |b_{j1}|)\overline{h}_j\beta_1 + L_{j1}^2] = 6 > 0, \tag{4.2}
$$

and

$$
2\underline{h}_2\alpha_2 - \sum_{j=1}^{2}[(|a_{2j}| + |b_{2j}|)\overline{h}_2\beta_j + (|a_{j2}| + |b_{j2}|)\overline{h}_j\beta_2 + L_{j2}^2] = 6 > 0. \tag{4.3}
$$

from Corollary 3.3 we know that (4.1) is almost surely exponentially stable.

Conclusions. In this paper, we have investigated a stochastic Cohen-Grossberg neural networks with unbounded distributed delays, whose state variable are governed by stochastic non-linear integro-differential equations, which is more general than the previous published papers. By constructing suitable Lyapunov functional, employing the semi-martingale convergence theorem and some inequality, we obtain some sufficient criteria ensuring the almost exponential stability of the networks, and the stability of this system is dependent of the magnitude of noise. Furthermore, the derived conditions for stability of stochastic cellular neural networks with unbounded distributed can be viewed as byproducts of our results.

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