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EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR A DIFFERENTIAL INCLUSION PROBLEM INVOLVING THE $p(x)$-LAPLACIAN

GUOWEI DAI

Abstract. In this article we consider the differential inclusion

$$-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \partial F(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

which involves the $p(x)$-Laplacian. By applying the nonsmooth Mountain Pass Theorem, we obtain at least one nontrivial solution; and by applying the symmetric Mountain Pass Theorem, we obtain k-pairs of nontrivial solutions in $W_0^{1,p(x)}(\Omega)$.

1. Introduction

Let Ω be bounded open subset of $\mathbb{R}^N$ with a C^1-boundary $\partial\Omega$. We consider the differential inclusion problem

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) \in \partial F(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $p \in C(\overline{\Omega})$ with $1 < p^- := \inf_\Omega p(x) \le p^+ := \sup_\Omega p(x) < +\infty$, $F(x,u)$ is measurable with respect to x (for every $u \in \mathbb{R}$) and locally Lipschitz with respect to u (for a.e. $x \in \Omega$), and $\partial F(x,u)$ is the Clarke sub-differential of $F(x,\cdot)$.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is said to be $p(x)$-Laplacian, and becomes p-Laplacian when $p(x) \equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated properties than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electro-rheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [25, 27]. Problem with variable exponent growth conditions also appear

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in the mathematical modelling of stationary therm-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal baro-tropic gas through a porous medium [1, 2]. Another field of application of equations with variable exponent growth conditions is image processing [4]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [6, 21, 26, 28, 29] for an overview of and references on this subject, and to [6, 7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 20] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

Since many free boundary problems and obstacle problems may be reduced to partial differential equations with discontinuous nonlinearities, the existence of multiple solutions for Dirichlet boundary value problems with discontinuous nonlinearities has been widely investigated in recent years. In 1981, Chang [3] extended the variational methods to a class of non-differentiable functionals, and directly applied the variational methods for non-differentiable functionals to prove some existence theorems for PDE with discontinuous nonlinearities. Later, in 2000, Kourogenis and Papageorgiou [22] obtained some non-smooth critical point theories and applied these to nonlinear elliptic equations at resonance, involving the p-Laplacian with discontinuous nonlinearities.

Problem (1.1) has been studied extensively when $p(x) \equiv p$ (a constant); see [22, 23]. If f is a Cáratheodory function and $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$, then problem (1.1) becomes

$$\begin{gathered} -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

which has also been studied extensively; see [14, 16]. We emphasize that in our approach, no continuity with respect to the second argument will be required on the function f. So (1.2) need not have a solution. To avoid this situation, we consider functions $f(x,\cdot)$ which are locally essentially bounded and fills the discontinuity gaps of $f(x,\cdot)$, replacing f by an interval $[f_1, f_2]$, where

$$\begin{aligned} f_1(x,s) &= \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-s|<\delta} f(x,t), \\ f_2(x,s) &= \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-s|<\delta} f(x,t). \end{aligned}$$

It is well known that if $F(x,u) = \int_0^u f(x,t)\mathrm{d}t$, then F becomes locally Lipschitz and $\partial F(x,u) = [f_1(x,u), f_2(x,u)]$ (see [24]). This fact motivates the formulation of the *differential inclusion* problem (1.1).

This paper is organized as follows: In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces and generalized gradient of locally Lipschitz function; In Section 3, we give the main results of this paper. In Section 4; we use the nonsmooth Mountain Pass Theorem and symmetric Mountain Pass Theorem to prove our main results.

2. Preliminaries

To discuss problem (1.1), we need some properties of $W_0^{1,p(x)}(\Omega)$ (see [17]) and of the generalized gradient of locally Lipschitz functions, which will be used later. Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element when they are equal almost everywhere.

Let
$$L^{p(x)}(\Omega)=\{u\in \mathbf{S}(\Omega):\int_\Omega |u(x)|^{p(x)}\mathrm{d}x<+\infty\}$$
with the norm
$$|u|_{L^{p(x)}(\Omega)}=|u|_{p(x)}=\inf\big\{\lambda>0:\int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\mathrm{d}x\le 1\big\},$$
and let
$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega):|\nabla u|\in L^{p(x)}(\Omega)\}$$
with the norm
$$\|u\|_{W^{1,p(x)}(\Omega)}=|u|_{L^{p(x)}(\Omega)}+|\nabla u|_{L^{p(x)}(\Omega)}.$$
Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 ([17]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([17]). *Let $\rho(u)=\int_\Omega |u(x)|^{p(x)}\mathrm{d}x$, for $u\in L^{p(x)}(\Omega)$. Then:*

(1) *For $u\neq 0$, $|u|_{p(x)}=\lambda$ implies $\rho(\frac{u}{\lambda})=1$*
(2) $|u|_{p(x)}<1\ (=1;>1)\Leftrightarrow \rho(u)<1\ (=1;>1)$
(3) *If $|u|_{p(x)}>1$, then $|u|_{p(x)}^{p^-}\le\rho(u)\le|u|_{p(x)}^{p^+}$*
(4) *If $|u|_{p(x)}<1$, then $|u|_{p(x)}^{p^+}\le\rho(u)\le|u|_{p(x)}^{p^-}$*
(5) $\lim_{k\to+\infty}|u_k|_{p(x)}=0$ *if and only if* $\lim_{k\to+\infty}\rho(u_k)=0$
(6) $\lim_{k\to+\infty}|u_k|_{p(x)}=+\infty$ *if and only if* $\lim_{k\to+\infty}\rho(u_k)=+\infty$.

Proposition 2.3 ([17]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds; that is, there exists a positive constant C_0 such that*
$$|u|_{L^{p(x)}(\Omega)}\le C_0|\nabla u|_{L^{p(x)}(\Omega)},\quad \forall u\in W_0^{1,p(x)}(\Omega).$$

So $|\nabla u|_{L^{p(x)}(\Omega)}$ is an equivalent norm in $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\|=|\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.4 ([17]). *(1) Assume that the boundary of Ω possesses the cone property and $p\in C(\overline{\Omega})$. If $q\in C(\overline{\Omega})$ and $1\le q(x)\le p^*(x)$ for $x\in\overline{\Omega}$, then there is a continuous embedding $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$. When $1\le q(x)<p^*(x)$, the embedding is compact, where $p^*(x)=\frac{Np(x)}{N-p(x)}$ if $p(x)<N$, $p^*(x)=\infty$ if $p(x)\ge N$.*

(2) If $p_1(x),p_2(x)\in C(\overline{\Omega})$, and $1<p_1(x)\le p_2(x)$, then $L^{p_2(x)}\hookrightarrow L^{p_1(x)}$, and the embedding is continuous.

Proposition 2.5 ([17]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)}+\frac{1}{p(x)}=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have*
$$|\int_\Omega uv\mathrm{d}x|\le(\frac{1}{p^-}+\frac{1}{q^-})|u|_{p(x)}|v|_{q(x)}.$$

Let $(Y,\|\cdot\|)$ be a real Banach space and Y^* be its topological dual. A function $f:Y\to\mathbb{R}$ is called locally Lipschitz if each point $u\in Y$ possesses a neighborhood Ω_u such that $|f(u_1)-f(u_2)|\le L\|u_1-u_2\|$ for all $u_1,u_2\in\Omega_u$, for a constant $L>0$

depending on Ω_u. The generalized directional derivative of f at the point $u \in Y$ in the direction $v \in X$ is

$$f^0(u,v) = \limsup_{w \to u, t \to 0} \frac{1}{t}(f(w+tv) - f(w)).$$

The generalized gradient of f at $u \in Y$ is

$$\partial f(u) = \big\{u^* \in X^* : \langle u^*, \varphi\rangle \le f^0(u;\varphi) \text{ for all } \varphi \in Y\big\},$$

which is a non-empty, convex and w^*-compact subset of Y^*, where $\langle \cdot, \cdot\rangle$ is the duality pairing between Y^* and Y. We say that $u \in Y$ is a critical point of f if $0 \in \partial f(u)$. For further details, we refer the reader to Chang [3] or Clarke [5].

3. Main results

In this section we give two existence theorems for problem (1.1). For simplicity we write $X = W_0^{1,p(x)}(\Omega)$, denote by c, c_i, l and M the general positive constant (the exact value may change from line to line). The precise hypotheses are the followings:

(HF) $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable locally Lipschitz function with $F(x,0) = 0$ for a.e. $x \in \Omega$ such that
  (i) there exists a constant $c > 0$ such that for a.e. $x \in \Omega$, all $u \in \mathbb{R}$ and all $\xi(u) \in \partial F(x,u)$
$$|\xi(u)| \le c(1 + |u|^{\alpha(x)-1}),$$
  where $\alpha \in C(\overline{\Omega})$ and $p^+ < \alpha^- \le \alpha(x) < p^*(x)$;
  (ii) There exist $M > 0$, $\theta > p^+$ such that
$$0 < \theta F(x,u) \le \langle \xi, u\rangle, \quad \text{a.e. } x \in \Omega, \text{ all } u \in X, \ |u| \ge M, \ \xi \in \partial F(x,u); \tag{3.1}$$
  (iii) $F(x,t) = o(|t|^{p^+}), t \to 0$, uniformly for a.e. $x \in \Omega$.

Because X be a reflexive and separable Banach space, there exist $e_i \in X$ and $e_j^* \in X^*$ such that

$$X = \overline{\operatorname{span}\{e_i : i = 1,2,\dots\}}, \quad X^* = \overline{\operatorname{span}\{e_j^* : j = 1,2,\dots\}},$$
$$\langle e_i, e_j^*\rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j. \end{cases}$$

For convenience, we write $X_i = \operatorname{span}\{e_i\}$, $Y_k = \oplus_{i=1}^k X_i$, $Z_k = \overline{\oplus_{i=k}^\infty X_i}$. In the following we need the nonsmooth version of *Palais-Smale* condition.

Definition 3.1. We say that I satisfies the nonsmooth $(\mathrm{PS})_c$ condition if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$ and $m(u_n) \to 0$, as $n \to +\infty$, has a strongly convergent subsequence, where $m(u_n) = \inf\{\|u^*\|_{X^*} : u^* \in \partial I(u_n)\}$.

In what follows we write the $(\mathrm{PS})_c$-condition as simply the PS-condition if it holds for every level $c \in \mathbb{R}$ for the *Palais-Smale* condition at level c. Let

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} \mathrm{d}x, \quad \Psi(u) = \int_\Omega F(x,u) \mathrm{d}x.$$

By a solution of (1.1), we mean a function $u \in X$ to which there corresponds a mapping $\Omega \ni x \to g(x)$ with $g(x) \in \partial F(x,u)$ for a.e. $x \in \Omega$ having the property that for every $\varphi \in X$, the function $x \to g(x)\varphi(x) \in L^1(\Omega)$ and

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla\varphi \mathrm{d}x = \int_\Omega g(x)\varphi(x)\mathrm{d}x.$$

By standard argument, we show that $u \in X$ is a solution of (1.1) if and only if $0 \in I(u)$, where $I(u) = J(u) - \Psi(u)$. Below we give a proposition that will be used later.

Proposition 3.2 ([16])**.** *The functional $J : X \to \mathbb{R}$ is convex. The mapping $J' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type; namely $u_n \rightharpoonup u$ and $\overline{\lim}_{n\to\infty}(J'(u_n, u_n - u) \le 0$ implies $u_n \to u$.*

Theorem 3.3. *If* (HF) *holds, then* (1.1) *has at least one nontrivial solution.*

Theorem 3.4. *If* (HF) *holds and $F(x,-u) = F(x,u)$ for a.e. $x \in \Omega$ and all $u \in \mathbb{R}$, then* (1.1) *has at least k-pairs of nontrivial solutions.*

To prove Theorems 3.3 and 3.4 we need the following generalizations of the classical Mountain pass Theorem (see [3, 18, 22, 23]) and of the symmetric Mountain pass Theorem [18, 19].

Lemma 3.5. *If X is a reflexive Banach space, $I : X \to \mathbb{R}$ is a locally Lipschitz function which satisfies the nonsmooth* (PS)c*-condition, and for some $r > 0$ and $e_1 \in X$ with $\|e_1\| > r$, $\max\{I(0), I(e_1)\} \le \inf\{I(u) : \|u\| = r\}$. Then I has a nontrivial critical $u \in X$ such that the critical value $c = I(u)$ is characterized by the following minimax principle*

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t),$$

where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = e_1\}$.

Lemma 3.6. *If X is a reflexive Banach space and $I : X \to \mathbb{R}$ is even locally Lipschitz functional satisfying the nonsmooth* (PS)c*-condition and*

(i) *$I(0) = 0$;*
(ii) *there exists a subspace $Y \subseteq X$ of finite codimension and number $\beta, \gamma > 0$, such that $\inf\{I(u) : u \in Y \cap \partial B_\gamma(0)\} \ge \beta$, where $B_\gamma = \{u \in X : \|u\| < \gamma\}$ and $\partial B_\gamma = \{u \in X : \|u\| = \gamma\}$;*
(iii) *there is a finite dimensional subspace V of X with $\dim V > \operatorname{codim} Y$, such that $I(v) \to -\infty$ as $\|v\| \to +\infty$ for any $v \in V$.*

Then I has at least $\dim V - \operatorname{codim} Y$ pairs of nontrivial critical points.

4. Proof main results

Let $\widehat{\Psi}$ denote its extension to $L^{\alpha(x)}(\Omega)$. We know that $\widehat{\Psi}$ is locally Lipschitz on $L^{\alpha(x)}(\Omega)$. In fact, by Proposition 2.5, for u, $v \in L^{\alpha(x)}(\Omega)$, we have

$$|\widehat{\Psi}(u) - \widehat{\Psi}(v)| \le \Big(C_1|1|_{\alpha'(x)} + C_2 \max_{w\in U} |w^{\alpha(x)-1}|_{\alpha'(x)}\Big)|u - v|_{\alpha(x)}, \tag{4.1}$$

where U is an open neighborhood involving u and v, w in the open segment joining u and v. However, since $\rho(1) = |\Omega|$, by Proposition 2.2, we have

$$|1|_{\alpha'(x)} < \infty. \tag{4.2}$$

Meanwhile, since

$$
\begin{aligned}
\rho(w^{\alpha(x)-1}) &= \int_\Omega |w^{\alpha(x)-1}|^{\alpha'(x)}\mathrm{d}x \\
&\le \int_\Omega |w|^{\alpha(x)}\mathrm{d}x \\
&\le 2^{\alpha^+}\Big(\int_\Omega |u|^{\alpha(x)}\mathrm{d}x + \int_\Omega |u|^{\alpha(x)}\mathrm{d}x\Big) < \infty,
\end{aligned}
$$

by Proposition 2.2, we also have $|w^{\alpha(x)-1}|_{\alpha'(x)} < \infty$. Then, using Proposition 2.4 and [3, Theorem 2.2], we have that $\Psi = \widehat{\Psi}|_X$ is also locally Lipschitz, and $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)\mathrm{d}x$ (see [24]), where $\widehat{\Psi}|_X$ stands for the restriction of $\widehat{\Psi}$ to X. The interpretation of $\partial\Psi(u) \subseteq \int_\Omega \partial F(x,u)\mathrm{d}x$ is as follows: For every $\xi \in \partial\Psi(u)$ there corresponds a mapping $\xi(x) \in \partial F(x,u)$ for a.e. $x \in \Omega$ having the property that for every $\varphi \in X$ the function $\xi(x)\varphi(x) \in L^1(\Omega)$ and $\langle g, \varphi\rangle = \int_\Omega \xi(x)\varphi(x)\mathrm{d}x$ (see [24]). Therefore, I is a locally Lipschitz functional and we can use the nonsmooth critical point theory.

Lemma 4.1. *If hypotheses* (i) *and* (ii) *hold, then I satisfies the nonsmooth* (PS)-*condition.*

Proof. Let $\{u_n\}_{n\ge1} \subseteq X$ be a sequence such that $|I(u_n)| \le c$ for all $n \ge 1$ and $m(u_n) \to 0$ as $n \to \infty$. Then, from (ii), we have

$$
\begin{aligned}
c \ge I(u_n) &= \int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x \\
&\ge \frac{\|u_n\|^{p^-}}{p^+} - \int_\Omega \frac{1}{\theta}\langle \xi(u_n), u_n\rangle \mathrm{d}x - c_1 \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{\theta}\Big)\|u_n\|^{p^-} + \int_\Omega \frac{1}{\theta}(\|u_n\|^{p^-} - \langle \xi(u_n), u_n\rangle)\mathrm{d}x - c_1 \\
&\ge \Big(\frac{1}{p^+} - \frac{1}{\theta}\Big)\|u_n\|^{p^-} - \frac{1}{\theta}\|\xi\|_{X^*}\|u_n\| - c_1.
\end{aligned}
$$

Hence $\{u_n\}_{n\ge1} \subseteq X$ is bounded.

Thus by passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in X as $n \to \infty$. We have

$$
\langle J'(u_n), u_n - u\rangle - \int_\Omega \xi_n(x)(u_n - u)\mathrm{d}x \le \varepsilon_n\|u_n - u\|
$$

with $\varepsilon_n \downarrow 0$, where $\xi_n \in \partial\Psi(u_n)$. From Chang [3] we know that $\xi_n \in L^{\alpha'(x)}(\Omega)$ ($\alpha'(x) = \frac{\alpha(x)}{\alpha(x)-1}$). Since X is embedded compactly in $L^{\alpha(x)}(\Omega)$, we have that $u_n \to u$ as $n \to \infty$ in $L^{\alpha(x)}(\Omega)$. So using Proposition 2.5, we have

$$
\int_\Omega \xi_n(x)(u_n - u)\,dx \to 0 \quad \text{as } n \to \infty. \tag{4.3}
$$

Therefore we obtain $\limsup_{n\to\infty}\langle J'(u_n), u_n - u\rangle \le 0$. But we know that J' is a mapping of type (S_+). Thus we have

$$
u_n \to u \quad \text{in } X.
$$

$\square$

Lemma 4.2. *If hypotheses* (i), (iii) *hold, then there exist $r > 0$ and $\delta > 0$ such that $I(u) \geq \delta > 0$ for every $u \in X$ and $\|u\| = r$.*

Proof. Let $\varepsilon > 0$ be small enough such that $\varepsilon c_0^{p^+} \leq \frac{1}{2p^+}$, where c_0 is the embedding constant of $X \hookrightarrow L^{p^+}(\Omega)$. From hypothesis (i) and (iii), we have

$$F(x,t) \leq \varepsilon |t|^{p^+} + c(\varepsilon)|t|^{\alpha(x)}. \tag{4.4}$$

Therefore, for every $u \in X$, we have

$$\begin{aligned} I(u) &\geq \frac{1}{p^+}\|u\|^{p^+} - \varepsilon c_0^{p^+}\|u\|^{p^+} - c(\varepsilon)\|u\|^{\alpha^-} \\ &\geq \frac{1}{2p^+}\|u\|^{p^+} - c(\varepsilon)\|u\|^{\alpha^-}, \end{aligned}$$

when $\|u\| \leq 1$. So we can find $r > 0$ small enough and $\delta > 0$ such that $I(u) \geq \delta > 0$ for every $u \in X$ and $\|u\| = r$. □

Lemma 4.3. *If hypotheses* (ii) *holds, then there exists $u_1 \in X$ such that $I(u_1) \leq 0$.*

Proof. From (ii), there exist $M > 0$, $c_2 > 0$ such that (see [18, p. 298])

$$F(x,u) \geq c_2|u|^\theta$$

for all $|u| > M$ and a.e. $x \in \Omega$. Thus for $1 < t \in \mathbb{R}$, we have

$$\begin{aligned} \int_\Omega F(x,tu)\mathrm{d}x &= \int_{\{t|u|>M\}} F(x,tu)\mathrm{d}x + \int_{\{t|u|\leq M\}} F(x,tu)\mathrm{d}x \\ &\geq c_2 t^\theta \int_{\{t|u|>M\}} |u|^\theta \mathrm{d}x - c_3. \end{aligned}$$

Therefore, for $t > 1$, we have

$$\begin{aligned} I(tu) &\leq \frac{1}{p^-} t^{p^+} \int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - c_2 t^\theta \int_{\{t|u|>M\}} |u|^\theta \mathrm{d}x + c_3 \\ &= \frac{1}{p^-} t^{p^+} \int_\Omega |\nabla u|^{p(x)}\mathrm{d}x - c_2 t^\theta \int_\Omega |u|^\theta \mathrm{d}x + c_2 t^\theta \int_{\{t|u|\leq M\}} |u|^\theta \mathrm{d}x + c_3. \end{aligned} \tag{4.5}$$

Noting that $c_2 t^\theta \int_{\{t|u|\leq M_3\}} |u|^\theta$ is bounded, it follows that

$$I(tu) \to -\infty \quad \text{as } t \to +\infty.$$

□

Proof of Theorem 3.3. Using Lemma 3.5 and Lemmas 4.1-4.3, we can find an $u \in X$ such that $I(u) > 0$ (hence $u \neq 0$) and $0 \in \partial I(u)$. Hence $u \in X$ is a nontrivial solution of (1.1). □

Proof of Theorem 3.4. Firstly, we can easily see that I is even functional on X. We claim that $I(u) \to -\infty$ as $\|u\| \to +\infty$, for any $u \in Y_k$. We assume $\|u\| \geq 1$. From (4.4), we have

$$I(u) \leq \frac{1}{p^-}\|u\|^{p^+} - c_4|u|_\theta^\theta + c_4 \int_{\{|u|\leq M\}} |u|^\theta \mathrm{d}x + c_5.$$

Since Y_k is finite dimensional, all norms of Y_k are equivalent. For $p^+ < \theta$, we get $I(u) \to -\infty$ as $\|u\| \to +\infty$. We can apply Lemma 3.6 with $V = Y_k$ and $Y = X$. From Lemma 4.1 and Lemma 4.2, we get k-pairs of nontrivial critical points, which are solutions of (1.1). □

We remark that using the same method as in hte proof of Theorems 3.3 and 3.4, we can obtain the same results for the corresponding differential inclusion problems with Neumann boundary data.

As an example of a nonsmooth potential function $F(x,u)$ satisfying (HF), we have

$$F(x,u)=\frac{1}{p^+}|u|^{p^+}+\frac{1}{\alpha(x)}|u|^{\alpha(x)}.$$

Then we can check that it satisfies all hypotheses of Theorem 3.3. Note that in this case. $\partial F(x,u)=|u|^{p^+-1}\operatorname{sgn}(u)+|u|^{\alpha(x)-1}\operatorname{sgn}(u)$, where

$$\operatorname{sgn}(u)=\begin{cases}1, & \text{if } u>0,\\ [-1,1] & \text{if } u=0,\\ -1 & \text{if } u<0.\end{cases}$$

Moreover, it is obvious that $F(x,-u)=F(x,u)$. So F satisfies all the hypotheses in Theorem 3.4.

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Guowei Dai
Department of Mathematics, Northwest Normal University, Lanzhou, 730070, China
E-mail address: daigw06@lzu.cn