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STEADY-STATE THERMAL HERSCHEL-BULKLEY FLOW WITH TRESCA'S FRICTION LAW

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Abstract. We consider a mathematical model which describes the steady-state flow of a Herschel-Bulkley fluid whose the consistency and the yield limit depend on the temperature and with mixed boundary conditions, including a frictional boundary condition. We derive a weak formulation of the coupled system of motion and energy equations which consists of a variational inequality for the velocity field. We prove the existence of weak solutions. In the asymptotic limit case of a high thermal conductivity, the temperature becomes a constant solving an implicit total energy equation involving the consistency function and the yield limit.

1. Introduction

The model of Herschel-Bulkley fluid has been used in various publications to describe the flow of metals, plastic solids and some polymers. The literature concerning this topic is extensive; see e.g. [6, 16] and references therein. The new feature in the model is due to a Fourier type boundary condition, and consists in the appearance of a nonlocal term on the boundary part where Tresca's thermal friction is taken into account.

An intrinsic inclusion leads in a natural way to variational equations which justify the study of problems involving the incompressible, plastic Herschel-Bulkley fluid using arguments of the variational analysis. The paper is organized as follows. In Section 2 we present the mechanical problem of the steady-state Herschel-Bulkley flow where the consistency and the yield limit depend on the temperature and with Tresca's thermal friction law. Moreover, we introduce some notations and preliminaries. In Section 3 we derive the variational formulation of the problem. We prove in Section 4 the existence of weak solutions as well as an existence result to the steady-state Herschel-Bulkley flow with temperature dependent nonlocal consistency, yield limit and tresca's friction, which can be obtained as an asymptotic limit case of a very large thermal conductivity.

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2. Statement of the Problem

We consider a mathematical problem modelling the steady-state flow of a thermal Herschel-Bulkley fluid in a bounded domain $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$, with the boundary Γ of class C^1, partitioned into two disjoint measurable parts Γ_0 and Γ_1 such that $\text{meas}(\Gamma_0) > 0$. The fluid is supposed to be incompressible, the consistency and the yield limit depend on the temperature. The fluid is acted upon by given volume forces of density f. In addition, we admit a possible external heat source proportional to the temperature. On Γ_0 we suppose that the velocity is known. The temperature is given by a homogeneous Neumann boundary condition on Γ_0. We impose on Γ_1 a frictional contact described by a Tresca thermal friction law, as well as a Fourier boundary condition.

We denote by $\mathbb{S}_n$ the space of symmetric tensors on $\mathbb{R}^n$. We define the inner product and the Euclidean norm on $\mathbb{R}^n$ and $\mathbb{S}_n$, respectively, by

$$\mathbf{u} \cdot \mathbf{v} = u_i v_i \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^n \quad \text{and} \quad \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij} \tau_{ij} \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}_n.$$
$$|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2} \quad \forall \mathbf{u} \in \mathbb{R}^n \quad \text{and} \quad |\boldsymbol{\sigma}| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad \forall \boldsymbol{\sigma} \in \mathbb{S}_n.$$

Here and below, the indices i and j run from 1 to n and the summation convention over repeated indices is used. We denote by $\tilde{\boldsymbol{\sigma}}$ the deviator of $\boldsymbol{\sigma} = (\sigma_{ij})$ given by

$$\tilde{\boldsymbol{\sigma}} = (\tilde{\sigma}_{ij}), \quad \tilde{\sigma}_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{n} \delta_{ij},$$

where $\boldsymbol{\delta} = (\delta_{ij})$ denotes the identity tensor.

Let $1 < p < 2$. We consider the rate of deformation operator defined for every $\mathbf{u} \in W^{1,p}(\Omega)^n$ by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))_{1 \le i,j \le n}, \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}).$$

We denote by ν the unit outward normal vector on the boundary Γ. For every vector field $\mathbf{v} \in W^{1,p}(\Omega)^n$ we also write $\mathbf{v}$ for its trace on Γ. The normal and the tangential components of $\mathbf{v}$ on the boundary are

$$\mathbf{v}_\nu = \mathbf{v} \cdot \nu, \quad \mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_\nu \nu.$$

Similarly, for a regular tensor field $\boldsymbol{\sigma}$, we denote by $\boldsymbol{\sigma}_\nu$ and $\boldsymbol{\sigma}_\tau$ the normal and tangential components of $\boldsymbol{\sigma}$ on the boundary given by

$$\boldsymbol{\sigma}_\nu = \boldsymbol{\sigma}\boldsymbol{\nu} \cdot \nu, \quad \boldsymbol{\sigma}_\tau = \sigma\nu - \boldsymbol{\sigma}_\nu \nu.$$

We consider now, the following mechanical problem.

Problem 1. Find a velocity field $\mathbf{u} = (u_i)_{i=\overline{1,n}} : \Omega \to \mathbb{R}^n$, stress field $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=\overline{1,n}} : \Omega \to \mathbb{S}_n$ and a temperature $\theta : \Omega \to \mathbb{R}$ such that

$$\mathbf{u} \cdot \nabla \mathbf{u} = \operatorname{Div}(\boldsymbol{\sigma}) + \mathbf{f} \quad \text{in } \Omega \tag{2.1}$$

$$\left.\begin{aligned} \tilde{\boldsymbol{\sigma}} = \mu(\theta)|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u}) + g(\theta)\frac{\varepsilon(\mathbf{u})}{|\varepsilon(\mathbf{u})|} \quad & \text{if } |\varepsilon(\mathbf{u})| \neq 0 \\ |\tilde{\boldsymbol{\sigma}}| \le g(\theta) \quad \text{if } |\varepsilon(\mathbf{u})| = 0 & \end{aligned}\right\} \quad \text{in } \Omega \tag{2.2}$$

$$\operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega \tag{2.3}$$

$$-k\Delta\theta + \mathbf{u} \cdot \nabla\theta = \boldsymbol{\sigma} \cdot \varepsilon(\mathbf{u}) - \alpha\theta \quad \text{in } \Omega \tag{2.4}$$

$$\mathbf{u} = 0 \quad \text{on } \Gamma_0 \tag{2.5}$$

$$\left.\begin{aligned} \mathbf{u}_\nu = 0, \quad |\boldsymbol{\sigma}_\tau| \le \upsilon(\theta) \\ |\boldsymbol{\sigma}_\tau| < \upsilon(\theta) \implies \mathbf{u}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \upsilon(\theta) \implies \mathbf{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau, \quad \lambda \ge 0 \end{aligned}\right\} \quad \text{on } \Gamma_1 \tag{2.6}$$

$$\frac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \Gamma_0 \tag{2.7}$$

$$k\frac{\partial\theta}{\partial\nu} + \beta\theta = \upsilon(\theta)|\mathbf{u}_\tau| \quad \text{on} \quad \Gamma_1 \tag{2.8}$$

Where $\mathrm{Div}(\boldsymbol{\sigma}) = (\sigma_{ij,j})$ and $\mathrm{div}(\mathbf{u}) = u_{i,i}$. The flow is given by the equation (2.1) where the density is assumed equal to one. Equation (2.2) represents the constitutive law of a Herschel-Bulkley fluid whose the consistency μ and the yield limit g depend on the temperature, $1 < p < 2$ is the power law exponent of the material. (2.3) represents the incompressibility condition. Equation (2.4) represents the energy conservation where the specific heat is assumed equal to one, $k > 0$ is the thermal conductivity and the term $-\alpha\theta$ represents the external heat source with $\alpha > 0$. (2.5) gives the velocity on Γ_0. Condition (2.6) represents a Tresca thermal friction law on Γ_1 where $\upsilon(\theta)$ is the friction yield coefficient for liquid-solid interface. (2.7) is a homogeneous Neumann boundary condition on Γ_0. Finally, (2.8) represents a Fourier boundary condition on Γ_1, where $\beta \ge 0$ represents the Robin coefficient.

Remark 2.1. In the constitutive law (2.2) of the Herschel-Bulkley fluid, the viscosity is given by the formula

$$\eta(\theta) = \mu(\theta)|\varepsilon(\mathbf{u})|^{p-2}. \tag{2.9}$$

We define

$$W = \left\{\mathbf{v} \in W^{1,p}(\Omega)^n : \mathrm{div}(\mathbf{v}) = 0 \text{ in } \Omega,\ \mathbf{v} = 0 \text{ on } \Gamma_0 \text{ and } \mathbf{v}_\nu = 0 \text{ on } \Gamma_1\right\},$$

which is a Banach space equipped with the norm

$$\|\mathbf{v}\|_W = \|\mathbf{v}\|_{W^{1,p}(\Omega)^n}.$$

For the rest of this article, we will denote by c possibly different positive constants depending only on the data of the problem. Denote by p' the conjugate of p and by q' the conjugate of q, $q \in [0, +\infty[$. We introduce the following functionals

$$B : W \times W \times W \to \mathbb{R}, \quad B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{u} \cdot \nabla\mathbf{v} \cdot \mathbf{w}\, dx$$

$$E : W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times W \to \mathbb{R}, \quad E(\theta, \tau, \mathbf{v}) = \int_\Omega \theta\nabla\tau \cdot \mathbf{v}\, dx.$$

We assume

$$\begin{gathered} \forall x \in \Omega, \quad \mu(.,x) \in C^0(\mathbb{R}) \quad \text{and} \\ \exists \mu_1, \mu_2 > 0 : \mu_1 \ \le \mu(y,x) \le \mu_2 \quad \forall y \in \mathbb{R},\ \forall x \in \Omega. \end{gathered} \tag{2.10}$$

$$\begin{gathered} \forall x \in \Omega, \quad g(.,x) \in C^0(\mathbb{R}) \quad \text{and} \\ \exists g_0 > 0 : 0 \le g(y,x) \le g_0 \quad \forall y \in \mathbb{R},\ \forall x \in \Omega. \end{gathered} \tag{2.11}$$

$$\begin{gathered} \forall x \in \Gamma_1, \upsilon(.,x) \in C^0(\mathbb{R}) \quad \text{and} \\ \exists \upsilon_0 > 0 : 0 \le \upsilon(y,x) \le \upsilon_0 \quad \forall y \in \mathbb{R},\ \forall x \in \Gamma_1. \end{gathered} \tag{2.12}$$

Lemma 2.2. *Suppose that*

$$\frac{3n}{n+2} \le p < 2 \quad \text{and} \quad 1 < q < \frac{n}{n-1}. \tag{2.13}$$

Then (1) B is trilinear, continuous on $W \times W \times W$. Moreover, for all $(\mathbf{u}, \mathbf{v}, \mathbf{w}) \in W \times W \times W$ we have $B(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -B(\mathbf{u}, \mathbf{w}, \mathbf{v})$.

(2) E is trilinear, continuous on $W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times W$ and on $H^1(\Omega) \times H^1(\Omega) \times W$. Moreover, $E(\theta, \tau, \mathbf{v}) = -E(\tau, \theta, \mathbf{v})$ for all $(\theta, \tau, \mathbf{v}) \in W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times W$ and for all $(\theta, \tau, \mathbf{v}) \in H^1(\Omega) \times H^1(\Omega) \times W$.

Proof. In these two assertions, the trilinearity is evident.

(1) The Soblov imbedding

$$W^{1,p}(\Omega) \subset L^{\rho}(\Omega) \quad \forall \rho \in [p, \frac{np}{n-p}],$$

combined with (2.13), gives $W^{1,p}(\Omega) \subset L^{\rho}(\Omega)$ for all $\rho \in [p, \frac{2n}{n-2}[$. Particularly,

$$W^{1,p}(\Omega) \subset L^{\frac{3n}{n-1}}(\Omega). \tag{2.14}$$

On the other hand, the use of Hölder's inequality leads to

$$B(\mathbf{u}, \mathbf{v}, \mathbf{w}) \le \|\mathbf{u}\|_{L^{\frac{3n}{n-1}}(\Omega)^n} \|\mathbf{v}\|_{L^p(\Omega)^n} \|\mathbf{w}\|_{L^{\frac{3n}{n-1}}(\Omega)^n}.$$

Consequently, the continuity of B follows from (2.14).

Moreover, the antisymmetry of the convective operator B is valid by the incompressibility condition (2.3) and the boundary conditions given by (2.5), (2.6), using an integration by parts.

(2) The continuity of E on $H^1(\Omega) \times H^1(\Omega) \times W$ is an immediate consequence of the Sobolev imbedding $W \subset L^3(\Omega)^n$ and $H^1(\Omega) \subset L^3(\Omega)$. The proof of the antisymmetry of E is based on the incompressibility condition (2.3) and the boundary conditions given by (2.5), (2.6).

Finally, to prove the continuity of E on $W^{1,q}(\Omega) \times W^{1,q'}(\Omega) \times W$, we proceed as follows. Sobolev's imbedding asserts that

$$\begin{gathered} W^{1,q}(\Omega) \subset L^{\rho}(\Omega) \quad \forall \rho \in]\frac{n}{n-1}, \frac{n}{n-2}[, \\ W \subset L^s(\Omega)^n \quad \forall s \in [n, \frac{2n}{n-2}[. \end{gathered} \tag{2.15}$$

Then, if $\theta \in W^{1,q}(\Omega)$, $\tau \in W^{1,q'}(\Omega)$ and $\mathbf{v} \in W$, the result follows from (2.15), the antisymmetry of E and the continuity of the injection $W^{1,q'}(\Omega) \to C(\bar{\Omega})$ for $q' > n$, that is, $q < \frac{n}{n-1}$, using Hölder's inequality . $\square$

For the rest of this article, we take $\frac{3n}{n+2} \le p < 2$ and $1 < q < \frac{n}{n-1}$.

3. Variational Formulation

The aim of this section is to derive a variational formulation to the problem (P1). To do so we need the following Lemma.

Lemma 3.1. *Assume that $f \in W'$. If $\{\mathbf{u}, \boldsymbol{\sigma}, \theta\}$ are regular functions satisfying* (2.1)-(2.8)*, then*

$$\begin{aligned} &B(\mathbf{u},\mathbf{u},\mathbf{v}-\mathbf{u}) + \int_\Omega \mu(\theta)(|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u}))\cdot(\varepsilon(\mathbf{v})-\varepsilon(\mathbf{u}))\,dx + \phi(\theta,\mathbf{v}) - \phi(\theta,\mathbf{u}) \\ &\geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx \quad \forall \mathbf{v}\in W, \end{aligned} \tag{3.1}$$

$$\begin{aligned} &-E(\theta,\tau,\mathbf{u}) + k\int_\Omega \nabla\theta\cdot\nabla\tau\,dx + \alpha\int_\Omega \theta\tau\,dx + \beta\int_{\Gamma_1}\theta\tau\,ds \\ &= \int_\Omega \mu(\theta)|\varepsilon(\mathbf{u})|^p + g(\theta)|\varepsilon(\mathbf{u})|)\tau\,dx + \int_{\Gamma_1}\upsilon(\theta)|\mathbf{u}_\tau|\tau\,ds \quad \forall \tau\in W^{1,q'}(\Omega), \end{aligned} \tag{3.2}$$

where

$$\phi(\theta,\mathbf{u}) = \int_{\Gamma_1}\upsilon(\theta)|\mathbf{u}_\tau|\,ds + \int_\Omega g(\theta)|\varepsilon(\mathbf{u})|\,dx. \tag{3.3}$$

Proof. Let us start by proving the variational inequality (3.1). Let $\{\mathbf{u}, \boldsymbol{\sigma}, \theta\}$ be regular functions satisfying (2.1)-(2.8) and let $\mathbf{v}\in W$. Using Green's formula and (2.1), (2.2), (2.3), (2.5) and (2.6), we obtain

$$\begin{aligned} &\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot(\mathbf{v}-\mathbf{u})\,dx + \int_\Omega \mu(\theta)(|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u}))\cdot(\varepsilon(\mathbf{v})-\varepsilon(\mathbf{u}))\,dx \\ &+ \int_\Omega g(\theta)|\varepsilon(\mathbf{v})|\,dx - \int_\Omega g(\theta)|\varepsilon(\mathbf{u})|\,dx \\ &\geq \int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})\,dx + \int_{\Gamma_1}\boldsymbol{\sigma\nu}\cdot(\mathbf{v}-\mathbf{u})\,ds. \end{aligned}$$

On the other hand, by (2.6),

$$\int_{\Gamma_1}\boldsymbol{\sigma\nu}\cdot(\mathbf{v}-\mathbf{u})\,ds \geq \int_{\Gamma_1}\upsilon(\theta)|\mathbf{u}_\tau|\,ds - \int_{\Gamma_1}\upsilon(\theta)|\mathbf{v}_\tau|ds.$$

Then (3.1) holds. Now, to prove the variational equation (3.2), we proceed as follows. Applying Green's formula, (2.4), (2.7), (2.8) and Lemma 2.2, we obtain, after a simple calculation,

$$\begin{aligned} &-\int_\Omega \theta(\nabla\tau\cdot\mathbf{v})\,dx + k\int_\Omega \nabla\theta\cdot\nabla\tau\,dx + \alpha\int_\Omega\theta\tau\,dx + \beta\int_{\Gamma_1}\theta\tau\mathbf{ds} \\ &= \int_\Omega \boldsymbol{\sigma}\cdot\varepsilon(\mathbf{u})\tau\,dx + \int_{\Gamma_1}\upsilon(\theta)|\mathbf{u}_\tau|\tau\,ds \quad \forall\tau\in W^{1,q'}(\Omega). \end{aligned}$$

By definition of $\boldsymbol{\sigma}$, using the incompressibility condition (2.3), we can infer

$$\int_\Omega \boldsymbol{\sigma}\cdot\varepsilon(\mathbf{u})\tau\,dx = \int_\Omega(\mu(\theta)|\varepsilon(\mathbf{u})|^p + g(\theta)|\varepsilon(\mathbf{u})|)\tau\,dx,$$

which completes the proof. □

Remark 3.2. In (3.2), the first term on the right hand side has sense, since the injection $W^{1,q'}(\Omega)\to C(\bar{\Omega})$ is continuous for $q' > n$, that is, $q < n/(n-1)$.

Lemma 3.1 leads us to consider the following variational system.

Problem P2. For prescribed data $f \in W'$. Find $\mathbf{u} \in W$ and $\theta \in W^{1,q}(\Omega)$, satisfying the variational system

$$\begin{aligned}
&B(\mathbf{u}, \mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_\Omega \mu(\theta)(|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u})) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))\, dx + \phi(\theta, \mathbf{v}) - \phi(\theta, \mathbf{u}) \\
&\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx \quad \forall \mathbf{v} \in W,
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
&- E(\theta, \tau, \mathbf{u}) + k \int_\Omega \nabla\theta \cdot \nabla\tau\, dx + \alpha \int_\Omega \theta\tau dx + \beta \int_{\Gamma_1} \theta\tau ds \\
&= \int_\Omega (\mu(\theta)|\varepsilon(\mathbf{u})|^p + g(\theta)|\varepsilon(\mathbf{u})|)\tau\, dx + \int_{\Gamma_1} \upsilon(\theta)|\mathbf{u}_\tau|\tau ds \quad \forall \tau \in W^{1,q'}(\Omega).
\end{aligned} \tag{3.5}$$

Now, we consider the weak nonlocal formulation to the mechanical problem (2.1)-(2.3) and (2.5)-(2.6) corresponding formally to the limit model $k = \infty$ (modelling the steady-state Herschel-Bulkley flow with temperature dependent nonlocal consistency, yield limit and friction).

Problem P3. For prescribed data $\mathbf{f} \in W'$. Find $\mathbf{u} \in W$ and $\Theta \in \mathbb{R}_+$, satisfying the variational inequality

$$\begin{aligned}
&B(\mathbf{u}, \mathbf{u}, \mathbf{v} - \mathbf{u}) + \mu(\Theta) \int_\Omega (|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u})) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))\, dx \\
&+ g(\Theta) \int_\Omega (|\varepsilon(\mathbf{v})| - |\varepsilon(\mathbf{u})|)\, dx + \upsilon(\Theta) \int_{\Gamma_1} (|\mathbf{v}_\tau| - |\mathbf{u}_\tau|)\, ds \\
&\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\, dx \quad \forall \mathbf{v} \in W,
\end{aligned} \tag{3.6}$$

where Θ is a solution to the implicit scalar equation

$$\begin{aligned}
&(\alpha \operatorname{meas}(\Omega) + \beta \operatorname{meas}(\Gamma_1))\Theta \\
&= \mu(\Theta) \int_\Omega |\varepsilon(\mathbf{u})|^p\, dx + g(\Theta) \int_\Omega |\varepsilon(\mathbf{u})|\, dx + \upsilon(\Theta) \int_{\Gamma_1} |\mathbf{u}_\tau|\, ds.
\end{aligned} \tag{3.7}$$

4. Existence Results

In this section we establish two existence theorems for problems (P2) and (P3).

Theorem 4.1. *Problem* (P2) *has a solution* $(\mathbf{u}, \theta)$ *satisfying*

$$\mathbf{u} \in W, \tag{4.1}$$

$$\theta \in W^{1,q}(\Omega). \tag{4.2}$$

Theorem 4.2. *There exists* $(\mathbf{u}, \Theta) \in W \times \mathbb{R}_+$ *a solution to the nonlocal problem* (P3)*, which can be obtained as a limit in* $W \times W^{1,q}(\Omega)$ *as* $k \to \infty$ *of solutions* $(\mathbf{u}_k, \theta_k)$ *of problem* (P2)*.*

The proof of Theorem 4.1 is based on the application of the Kakutani-Glicksberg fixed point theorem for multivalued mappings, using two auxiliary existence results. The first one results from the classical theory for inequalities with monotone operators and convex functionals. The second one results from the theory of elliptic equations and L^1-Data theory. Finally, compactness arguments are used to conclude the proofs. For reader's convenience, let us recall the fixed point theorem [7].

Theorem 4.3 (Kakutani-Glicksberg)**.** *Let X be a locally convex Hausdorff topological vector space and K be a nonempty convex compact. If $L : K \to P(K)$ is an upper semicontinuous mapping and $L(\mathbf{z}) \neq \emptyset$ is a convex and closed subset in K for every $\mathbf{z} \in K$, then there exists at least one fixed point, $\mathbf{z} \in L(\mathbf{z})$.*

The first auxiliary existence result is as follows.

Proposition 4.4. *For every $\mathbf{w} \in W$ and $\lambda \in W^{1,q}(\Omega)$, there exists a unique solution $\mathbf{u} = \mathbf{u}(\mathbf{w}, \lambda) \in W$ to the problem*

$$\begin{aligned} &B(\mathbf{w}, \mathbf{u}, \mathbf{v} - \mathbf{u}) + \int_\Omega (\mu(\lambda)|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u})) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}))\,dx + \phi(\lambda, \mathbf{v}) - \phi(\lambda, \mathbf{u}) \\ &\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u})\,dx \quad \forall \mathbf{v} \in W, \end{aligned} \tag{4.3}$$

and it satisfies the estimate

$$\|\mathbf{u}\|_W \leq c\Big(\frac{\|\mathbf{f}\|_{W'}}{\mu_1}\Big)^{1/(p-1)}. \tag{4.4}$$

Proof. Introducing the functional

$$J : L^p(\Omega)^{n\times n}_s \subset \mathbb{S}_n \to \mathbb{R}, \quad J(\boldsymbol{\sigma}) = \int_\Omega \frac{\mu}{p}|\boldsymbol{\sigma}|^p\,dx.$$

This functional is convex, lower semi-continuous on $L^p(\Omega)^{n\times n}_s$ and Gâteaux differentiable. Its Gâteaux derivate at any point $\boldsymbol{\sigma} \in L^p(\Omega)^{n\times n}_s$ is

$$\langle DJ(\boldsymbol{\sigma}), \boldsymbol{\eta}\rangle_{L^{p'}(\Omega)^{n\times n}_s \times L^p(\Omega)^{n\times n}_s} = \int_\Omega \mu|\sigma|^{p-2}\boldsymbol{\sigma} \cdot \eta\,dx \quad \forall \boldsymbol{\eta} \in L^p(\Omega)^{n\times n}_s.$$

Consequently, DJ is hemi-continuous and monotone. Moreover DJ is strict monotone and bounded. To this aim, we have

$$\begin{aligned} &\langle DJ(\boldsymbol{\sigma}) - DJ(\boldsymbol{\eta}), \boldsymbol{\sigma} - \boldsymbol{\eta}\rangle_{L^{p'}(\Omega)^{n\times n}_s \times L^p(\Omega)^{n\times n}_s} \\ &\geq \int_\Omega \mu(|\boldsymbol{\sigma}| - |\boldsymbol{\eta}|)(|\boldsymbol{\sigma}|^{p-1} - |\boldsymbol{\eta}|^{p-1})\,dx. \end{aligned}$$

Then if $\boldsymbol{\sigma} \neq \boldsymbol{\eta}$, we get $\langle DJ(\boldsymbol{\sigma}) - DJ(\boldsymbol{\eta}), \boldsymbol{\sigma} - \boldsymbol{\eta}\rangle_{L^{p'}(\Omega)^{n\times n}_s \times L^p(\Omega)^{n\times n}_s} > 0$. It means that DJ is strict monotone. On the other hand, for every $\boldsymbol{\sigma} \in L^p(\Omega)^{n\times n}_s$

$$\int_\Omega |\mu|\boldsymbol{\sigma}|^{p-2}\boldsymbol{\sigma}|^{p'}\,dx \leq \mu_2^{p'} \int_\Omega |\boldsymbol{\sigma}|^p\,dx,$$

which proves that DJ is bounded on W. Now, we consider the differential operator

$$\begin{gathered} F_{\mathbf{w}} : W \to W', \quad \mathbf{u} \mapsto F_w\mathbf{u} \quad \forall \mathbf{v} \in W \\ \langle F_{\mathbf{w}}\mathbf{u}, \mathbf{v}\rangle_{W'\times W} = B(\mathbf{w}, \mathbf{u}, \mathbf{v}) + \langle DJ(\varepsilon(\mathbf{u})), \varepsilon(\mathbf{v})\rangle_{L^{p'}(\Omega)^{n\times n}_s \times L^p(\Omega)^{n\times n}_s}. \end{gathered} \tag{4.5}$$

By Lemma (2.2) and the properties of DJ, we deduce that $F_{\mathbf{w}}$ is hemi-continuous, strict monotone and bounded on W for every $\mathbf{w} \in W$. Therefore, for every $\mathbf{u} \in W$ we have

$$\frac{\langle F_{\mathbf{w}}u, \mathbf{u}\rangle_{W'\times W}}{\|\mathbf{u}\|_W} \geq \mu_1 \frac{\int_\Omega |\varepsilon(\mathbf{u})|^p\,dx}{\|\mathbf{u}\|_W}.$$

Applying the generalized Korn inequality, we obtain

$$\frac{\langle F_{\mathbf{w}}\mathbf{u}, \mathbf{u}\rangle_{W'\times W}}{\|\mathbf{u}\|_W} \geq \mu_1 c\|\mathbf{u}\|_W^{p-1}.$$

It follows that the operator $F_{\mathbf{w}}$ is coercive on W for every $\mathbf{w} \in W$.

Furthermore, the functional $\mathbf{v} \mapsto \phi(\lambda, \mathbf{v})$ is continuous and convex on W, it is then lower semi-continuous on W. Consequently, the existence and uniqueness of the solution result from the classical theorems (see [1]) on variational inequalities with monotone operators and convex functionals.

To prove the estimate (4.4) we proceed as follows, by choosing $\mathbf{v} = 0$ as test function in (4.3), we get

$$\int_\Omega \mu(\lambda)|\varepsilon(\mathbf{u})|^p \, dx \leq \|\mathbf{f}\|_{W'}\|\mathbf{u}\|_W.$$

Hence, Korn's inequality permits to conclude the proof. □

The second auxiliary existence result is as follows.

Proposition 4.5. *Let $\mathbf{u} = \mathbf{u}(\mathbf{w}, \lambda)$ be the solution of problem* (4.3) *given by Proposition 4.4. Then there exists $\theta = \theta(\mathbf{u}, \lambda) \in W^{1,q}(\Omega)$, a solution to the problem*

$$\begin{aligned} &-E(\theta, \tau, \mathbf{u}) + k\int_\Omega \nabla\theta \cdot \nabla\tau \, dx + \alpha\int_\Omega \theta\tau \, dx + \beta\int_{\Gamma_1} \theta\tau \, ds \\ &= \int_\Omega (\mu(\lambda)|\varepsilon(\mathbf{u})|^p + g(\lambda)|\varepsilon(\mathbf{u})|)\tau \, dx + \int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\tau \, ds \quad \forall \tau \in W^{1,q'}(\Omega), \end{aligned} \tag{4.6}$$

and satisfies the estimate

$$\alpha\|\theta\|_{L^q(\Omega)} + \beta\|\theta\|_{L^q(\Gamma)} + \sqrt{k}\|\nabla\theta\|_{L^q(\Omega)^n} \leq \Re(\upsilon_0, \mu_1, \|\mathbf{f}\|_{W'}), \tag{4.7}$$

where $\Re$ is a positive function.

Proof. There is a technical difficulty in the resolution of such problem. To this aim we introduce the following approximate problem

$$\begin{aligned} &-E(\theta_m, \tau, \mathbf{u}) + k\int_\Omega \nabla\theta_m \cdot \nabla\tau \, dx + \alpha\int_\Omega \theta_m\tau \, dx + \beta\int_{\Gamma_1} \theta_m\tau \, ds \\ &= \int_\Omega F_m\tau \, dx + \int_{\mathbf{\Gamma}_1} \upsilon(\lambda)|\mathbf{u}_\tau|\tau \, ds \quad \forall \tau \in H^1(\Omega), \end{aligned} \tag{4.8}$$

where

$$F_m = \frac{m\big[\mu(\lambda)|\varepsilon(\mathbf{u})|^p + g(\lambda)|\varepsilon(\mathbf{u})|\big]}{m + \mu(\lambda)|\varepsilon(\mathbf{u})|^p + g(\lambda)|\varepsilon(\mathbf{u})|} \in L^\infty(\Omega). \tag{4.9}$$

Let us consider for every $\mathbf{u} \in W$ the form $G : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$,

$$G(\theta, \tau) = -E(\theta, \tau, \mathbf{u}) + k\int_\Omega \nabla\theta \cdot \nabla\tau dx + \alpha\int_\Omega \theta\tau dx + \beta\int_{\Gamma_1} \theta\tau ds. \tag{4.10}$$

Lemma 2.2 and the Poincaré type inequality affirm that G is bilinear, continuous and coercive on $H^1(\Omega) \times H^1(\Omega)$ for every $\mathbf{u} \in W$. Furthermore, by Hölder's inequality and Sobolev's trace inequality using the estimate (4.4), we get

$$\Big|\int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\tau \, ds\Big| \leq c\|\tau\|_{H^1(\Omega)}.$$

Consequently, from the Lax-Milgram theorem, there exists a unique solution $\theta_m \in H^1(\Omega)$ to the problem (4.8).

Now, we test the apprixamte equation (4.8) by the function

$$\tau = \operatorname{sign}(\theta_m)[1 - \frac{1}{(1+|\theta_m|)^\xi}] \in H^1(\Omega) \cap L^\infty(\Omega), \quad \xi > 0. \tag{4.11}$$

We find by using some integration by parts (see for instance [4])

$$\xi k \int_\Omega \frac{|\nabla \theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\,dx + \beta C(\xi) \int_{\mathbf{\Gamma}_1} |\theta_m|\,ds \le M, \tag{4.12}$$

where $M = M(\upsilon_0, \mu_1, c, \|f\|_{W'})$ is a positive function. Particularly

$$\int_\Omega \frac{|\nabla \theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\,dx \le \frac{M}{\xi k}. \tag{4.13}$$

Denoting by γ the function

$$\gamma(r) = \int_0^r \frac{dt}{(1+|t|)^{\frac{\xi+1}{2}}}.$$

Then

$$\nabla \gamma(\theta_m) = \frac{\nabla \theta_m}{(1+|\theta_m|)^{(\xi+1)/2}}.$$

We deduce from (4.12) that $\nabla\gamma(\theta_m)$ is bounded in $L^2(\Omega)$, hence $\gamma(\theta_m)$ is bounded in $H^1(\Omega)$. Sobolev's imbedding asserts that $H^1(\Omega) \subset L^\rho(\Omega)$, where $\rho = \frac{2n}{n-2}$ if $n \neq 2$ and $2 < \rho < +\infty$ if $n = 2$.

Keeping in mind that $\gamma(r) \sim r^{\frac{1-\xi}{2}}$ as $r \to +\infty$. Then $|\theta_m|^{\frac{1-\xi}{2}}$ is bounded in $L^\rho(\Omega)$. Consequently

$$|\theta_m|^{\rho(1-\xi)/2} \quad \text{is bounded in } L^1(\Omega). \tag{4.14}$$

Moreover, by Hölder's inequality,

$$\int_\Omega |\nabla\theta_m|^q\,dx \le \Big(\int_\Omega \frac{|\nabla \theta_m|^2}{(1+|\theta_m|)^{\xi+1}}\,dx\Big)^{q/2} \Big(\int_\Omega (1+|\theta_m|)^{(\xi+1)q/(2-q)}\,dx\Big)^{(2-q)/2}.$$

Hence, from (4.13), we obtain

$$\int_\Omega |\nabla\theta_m|^q\,dx \le (\frac{M}{k\xi})^{q/2} \Big(\int_\Omega (1+|\theta_m|)^{(\xi+1)q/(2-q)}\,dx\Big)^{(2-q)/2}. \tag{4.15}$$

Let us choose the couple (ξ, q) such that $\frac{\rho(1-\xi)}{2} = \frac{(\xi+1)q}{2-q}$. It means that $q = \frac{2\rho(1-\xi)}{\rho(1-\xi)+2(1+\xi)}$. Then if $1 < q < \frac{n}{n-1}$, we can choose $0 < \xi < \frac{\rho-2}{\rho+2}$. Consequently, by using (4.14) and (4.15), the following estimate holds

$$\theta_m \quad \text{is bounded in } W^{1,q}(\Omega). \tag{4.16}$$

Combining this with (4.12), we can extract a subsequence $(\theta_\mu)_\mu$, satisfying

$$\theta_\mu \to \theta \quad \text{in } W^{1,q}(\Omega) \text{ weakly}, \tag{4.17}$$

$$\theta_\mu \to \theta \quad \text{in } L^q(\Gamma) \text{ weakly}. \tag{4.18}$$

Recall that Rellich-Kondrachof's theorem affirms the compactness of the imbedding $W^{1,q}(\Omega) \to L^1(\Omega)$. It follows that we can extract a subsequence of θ_μ, still denoted by θ_μ such that

$$\theta_\mu \to \theta \quad \text{in } L^1(\Omega) \text{ strongly}, \tag{4.19}$$

$$\theta_\mu \to \theta \quad \text{in } L^1(\Gamma) \text{ strongly}. \tag{4.20}$$

We conclude that problem (4.6) admits a solution $\theta = \theta(\mathbf{u}, \lambda) \in W^{1,q}(\Omega)$. Using (4.12), (4.14) and (4.15), the estimate (4.7) follows immediately. $\square$

Proof of Theorem 4.1. To apply the Kakutani-Glicksberg fixed point theorem, let us consider the closed convex ball

$$K = \{(\mathbf{w}, \lambda) \in W \times W^{1,p}(\Omega) : \|\mathbf{w}\|_W \leq R_1, \|\lambda\|_{W^{1,q}(\Omega)} \leq R_2\}, \tag{4.21}$$

where $R_1 \geq c(\frac{\|\mathbf{f}\|_{W'}}{\mu_1})^{\frac{1}{p-1}}$ and R_2 is given by the estimate (4.14). The ball K is compact when the topological vector space is provided by the weak topology. Let us built the mapping $L : K \to P(K)$, as follows

$$(\mathbf{w}, \lambda) \mapsto L(\mathbf{w}, \lambda) = \{(\mathbf{u}, \theta)\} \subset K.$$

For every $(\mathbf{w}, \lambda) \in K$, equation (4.6) is linear with respect to θ, and the solution $\mathbf{u}$ is unique. Consequently the set $L(\mathbf{w}, \lambda)$ is convex. To conclude the proof it remains to prove the closeness in $K \times K$ of the graph set

$$G(L) = \{((\mathbf{w}, \lambda), (\mathbf{u}, \theta)) \in K \times K : (\mathbf{u}, \theta) \in L(\mathbf{w}, \lambda)\}.$$

To do so, we consider a sequence $(\mathbf{w}_n, \lambda_n) \in K$, such that $(\mathbf{w}_n, \lambda_n) \to (\mathbf{w}, \lambda)$ in $W \times W^{1,q}(\Omega)$ weakly and $(\mathbf{u}_n, \theta_n) \in L(\mathbf{w}_n, \lambda_n)$. Let us remember that $(\mathbf{u}_n, \theta_n)$ is solution to the problem

$$\begin{aligned}
&B(\mathbf{w}_n, \mathbf{u}_n, \mathbf{v} - \mathbf{u}_n) + \int_\Omega (\mu(\lambda_n)|\varepsilon(\mathbf{u}_n)|^{p-2}\varepsilon(\mathbf{u}_n)) \cdot (\varepsilon(\mathbf{v}) - \varepsilon(\mathbf{u}_n))\, dx \\
&+ \phi(\lambda_n, \mathbf{v}) - \phi(\lambda_n, \mathbf{u}_n) \\
&\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_n)\, dx \quad \forall \mathbf{v} \in W,
\end{aligned} \tag{4.22}$$

$$\begin{aligned}
&- E(\theta_n, \tau, \mathbf{u}_n) + k \int_\Omega \nabla\theta_n \cdot \nabla\tau\, dx + \alpha \int_\Omega \theta_n \tau\, dx + \beta \int_{\Gamma_1} \theta_n \tau\, ds \\
&= \int_\Omega (\mu(\lambda_n)|\varepsilon(\mathbf{u}_n)|^p + g(\lambda_n)|\varepsilon(\mathbf{u}_n)|)\tau\, dx \\
&\quad + \int_{\Gamma_1} \upsilon(\lambda_n)|(\mathbf{u}_n)_\tau|\tau\, ds \quad \forall \tau \in W^{1,q'}(\Omega).
\end{aligned} \tag{4.23}$$

Then, from Propositions 4.4 and 4.5,

$$\|\mathbf{u}_n\|_W \leq R_1 \quad \text{and} \quad \|\theta_n\|_{W^{1,q}(\Omega)} \leq R_2.$$

Thus, we can extract a subsequences $\mathbf{u}_\mu$ and θ_μ such that

$$\mathbf{u}_\mu \to \mathbf{u} \quad \text{in } W \text{ weakly}, \tag{4.24}$$

$$\theta_\mu \to \theta \quad \text{in } W^{1,q}(\Omega) \text{ weakly}. \tag{4.25}$$

It follows from Rellich-Kondrachof's theorem and Sobolev's trace theorem, that we can extract a subsequences of λ_μ, $\mathbf{u}_\mu$ and θ_μ, still denoted by λ_μ, $\mathbf{u}_\mu$ and θ_μ, such that

$$\mathbf{w}_\mu \to \mathbf{w} \quad \text{in } L^s(\Omega)^n \text{ strongly and a.e. in } \Omega, \tag{4.26}$$

$$\lambda_\mu \to \lambda \quad \text{in } \mathbf{L}^1(\Omega) \text{ strongly and a.e. in } \Omega, \tag{4.27}$$

$$\mathbf{u}_\mu \to \mathbf{u} \quad \text{in } L^s(\Omega)^n \text{ strongly and a.e. in } \Omega, \tag{4.28}$$

$$\theta_\mu \to \theta \quad \text{in } L^1(\Omega) \text{ strongly and a.e. in } \Omega, \tag{4.29}$$

$$\mathbf{u}_\mu \to \mathbf{u} \quad \text{in } L^r(\Gamma)^n \text{ strongly and a.e. on } \Gamma, \tag{4.30}$$

$$\theta_\mu \to \theta \quad \text{in } L^1(\Gamma) \text{ strongly and a.e. on } \Gamma, \tag{4.31}$$

where $n \le s < \frac{2n}{n-2}$ and $2 \le r < \frac{2(n-1)}{n-2}$.

Now we prove that $\varepsilon(\mathbf{u}_\mu) \to \varepsilon(\mathbf{u})$ a.e. in Ω. To do so, we proceed as follows. Introducing the positive function

$$\begin{aligned} h_\mu(x) = [&\mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^{p-2}\varepsilon(\mathbf{u}_\mu(x)) \\ &- \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x))] \cdot (\varepsilon(\mathbf{u}_\mu(x)) - \varepsilon(\mathbf{u}(x))). \end{aligned} \tag{4.32}$$

Then

$$\begin{aligned} \int_\Omega h_\mu(x)\,dx \le\ & f(\mathbf{u}_\mu - \mathbf{u}) + B(\mathbf{w}_\mu, \mathbf{u}_\mu, \mathbf{v} - \mathbf{u}_\mu) + \phi(\lambda_\mu, \mathbf{u}) - \phi(\lambda_\mu, \mathbf{u}_\mu) \\ &- \int_\Omega \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x)) \cdot (\varepsilon(\mathbf{u}_\mu(x)) - \varepsilon(\mathbf{u}(x)))\,dx. \end{aligned} \tag{4.33}$$

We know from Lemma 2.2 that

$$B(\mathbf{w}_\mu, \mathbf{u}_\mu, \mathbf{v}) = -B(\mathbf{w}, \mathbf{v}, \mathbf{u}_\mu) + B(\mathbf{w} - \mathbf{w}_\mu, \mathbf{v}, \mathbf{u}_\mu).$$

Then, Lebesgue's dominated convergence theorem applied to the second term on the right-hand side yields convergence

$$B(\mathbf{w}_\mu, \mathbf{u}_\mu, \mathbf{v}) \to B(\mathbf{w}, \mathbf{u}, \mathbf{v}). \tag{4.34}$$

On the other hand, since $\lambda_\mu \to \lambda$ a.e. in Ω and on Γ, the functions g and υ are continuous and due to the weak lower semicontinuity of the continuous and convex functional $\mathbf{v} \mapsto \phi(\lambda, \mathbf{v})$, combined with convergence result (4.24), we deduce from the Lebesgue dominated convergence theorem that

$$\liminf \phi(\lambda_\mu, \mathbf{u}_\mu) \ge \phi(\lambda, \mathbf{u}), \tag{4.35}$$

$$\lim \phi(\lambda_\mu, \mathbf{u}) = \phi(\lambda, \mathbf{u}). \tag{4.36}$$

Since $\lambda_\mu \to \lambda$ a.e. in Ω and on Γ, the function μ is continuous and due to (4.24) and the fact that $|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x))$ is bounded in $L^{p'}(\Omega)_s^{n\times n}$, we obtain by the Lebesgue dominated convergence theorem that

$$\int_\Omega \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x))(\varepsilon(\mathbf{u}_\mu(x)) - \varepsilon(\mathbf{u}(x)))\,dx \to 0. \tag{4.37}$$

Consequently, (4.33), (4.34), (4.35), (4.36) and (4.37) give

$$\lim \|h_\mu\|_{L^1(\Omega)} = 0 \quad \text{and} \quad h_\mu \to 0 \quad \text{a.e.} \tag{4.38}$$

Furthermore, $h_\mu(x)$ can be rewritten as

$$\begin{aligned} h_\mu(x) = &\mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^p - \mu(\lambda_\mu(\mathbf{x}))|\varepsilon(\mathbf{u}_\mu(x))|^{p-2}\varepsilon(\mathbf{u}_\mu(x)) \cdot \varepsilon(\mathbf{u}(x)) \\ &- \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x)) \cdot (\varepsilon(\mathbf{u}_\mu(x)) - \varepsilon(\mathbf{u}(x))), \end{aligned} \tag{4.39}$$

which proves, using the estimate (4.4), that

$$\begin{aligned} \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^p \le\ & h_\mu(x) + c\mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^{p-1} \\ &+ c\mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))| + c. \end{aligned}$$

It follows that $(\varepsilon(\mathbf{u}_\mu(x)))_\mu$ is bounded in $\mathbb{R}^{n\times n}$, we can then extract a subsequence still denoted by $(\varepsilon(\mathbf{u}_\mu(x)))_\mu$, that converges to $\xi \in \mathbb{R}^{n\times n}$. By passage to the limit in h_μ, we deduce that

$$(\mu(\lambda)|\xi|^{p-2}\xi - \mu(\lambda)|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x))) \cdot (\xi - \varepsilon(\mathbf{u}(x))) = 0.$$

Then $\varepsilon(\mathbf{u}(x)) = \xi$. We conclude that

$$\varepsilon(\mathbf{u}_\mu) \to \varepsilon(\mathbf{u}) \quad \text{a.e. in } \Omega. \tag{4.40}$$

Therefore, the sequence $(\mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^{p-2}\varepsilon(\mathbf{u}_\mu(x)))_\mu$ converges a.e. in Ω to $\mu(\lambda(x))|\varepsilon(\mathbf{u}(x))|^{p-2}\varepsilon(\mathbf{u}(x))$. Moreover, this sequence is bounded in $L^{p'}(\Omega)_s^{n\times n}$, then the $L^p - L^q$ compactness theorem (see [4, 11]) implies the convergence in $L^r(\Omega)_s^{n\times n}$ for every $1 < r < p'$. By choosing $\varphi \in D(\Omega)^n$ as test function in inequality (4.22),

$$\begin{aligned} &B(\mathbf{w}, \mathbf{u}, \boldsymbol{\varphi} - \mathbf{u}) + \int_\Omega \mu(\lambda)|\varepsilon(\mathbf{u})|^{p-2}\varepsilon(\mathbf{u})\cdot\varepsilon(\boldsymbol{\varphi})\,dx + \phi(\lambda, \boldsymbol{\varphi}) - \mathbf{f}\cdot(\boldsymbol{\varphi} - \mathbf{u}) \\ &\geq \int_\Omega \mu(\lambda_\mu)|\varepsilon(\mathbf{u}_\mu)|^p\,dx + \phi(\lambda_\mu, \mathbf{u}_\mu). \end{aligned} \tag{4.41}$$

Using (4.35), the fact that $\lambda_\mu \to \lambda$ a.e. in Ω, the continuity of μ and g, the weak lower semicontinuity of the norm $\|\cdot\|_{W^{1,p}(\Omega)^n}$. We conclude that $\mathbf{u}$ is solution to (4.3).

Our final goal is to show that θ is solution of (4.23). To do so, we proceed as follows. Introducing the function

$$\chi_\mu(x) = \mu(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|^p + g(\lambda_\mu(x))|\varepsilon(\mathbf{u}_\mu(x))|. \tag{4.42}$$

From (4.40), we remark that $\chi_\mu \to \chi$ a.e. in Ω, where

$$\chi(x) = \mu(\lambda(x))|\varepsilon(\mathbf{u}(x))|^p + g(\lambda(x))|\varepsilon(\mathbf{u}(x))|. \tag{4.43}$$

Substituting in (4.22) and taking $\mathbf{v} = \mathbf{u}$ as test function, the passage to limit, using Lebesgue's dominated convergence theorem, gives

$$\lim\Big[\int_\Omega \chi_\mu(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda_\mu)|(\mathbf{u}_\mu)_\tau|\,ds\Big] \leq \int_\Omega \chi(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\,ds. \tag{4.44}$$

On the other hand, we know from the weak lower semicontinuity of the norm $\|\cdot\|_{W^{1,p}(\Omega)^n}$ and the functional $\mathbf{v} \mapsto \phi(\lambda, \mathbf{v})$, that

$$\liminf\Big[\int_\Omega \chi_\mu(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda_\mu)|(\mathbf{u}_\mu)_\tau|\,ds\Big] \geq \int_\Omega \chi(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\,ds. \tag{4.45}$$

We deduce from (4.44) and (4.45) that

$$\lim\Big[\int_\Omega \chi_\mu(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda_\mu)|(\mathbf{u}_\mu)_\tau|\,ds\Big] = \int_\Omega \chi(x)\,dx + \int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\,ds,$$

which implies, using the continuity of the injection $W^{1,q'}(\Omega) \to C(\bar{\Omega})$ and the Lebesgue dominated convergence theorem, that for every $\tau \in W^{1,q'}(\Omega)$,

$$\begin{aligned} &\lim\Big[\int_\Omega (\mu(\lambda_\mu)|\varepsilon(\mathbf{u}_\mu)|^p + g(\lambda_\mu)|\varepsilon(\mathbf{u}_\mu)|)\tau\,dx + \int_{\Gamma_1} \upsilon(\lambda_\mu)|(\mathbf{u}_\mu)_\tau|\tau\,ds\Big] \\ &= \int_\Omega (\mu(\lambda)|\varepsilon(\mathbf{u})|^p + g(\lambda)|\varepsilon(\mathbf{u})|)\tau\,dx + \int_{\Gamma_1} \upsilon(\lambda)|\mathbf{u}_\tau|\tau\,ds. \end{aligned} \tag{4.46}$$

Thus, we conclude that θ is solution to (4.23). Hence, $(\mathbf{u}_n, \theta_n) \to (\mathbf{u}, \theta) \in L(\mathbf{w}, \lambda)$ in $W \times W^{1,q}(\Omega)$ weakly. By virtue of Kakutani-Glicksberg's fixed point theorem, the mapping L admits a fixed point $(\mathbf{u}, \theta) \in L(\mathbf{u}, \theta)$, which solves problem (P2). $\square$

Remark 4.6. This proof permits also to verify the continuous dependence of the solution $(\mathbf{u}(\mathbf{w}, \lambda), \theta(\mathbf{u}, \lambda)) \in W \times W^{1,q}(\Omega)$ of problem (4.3), (4.6) with respect to the function $(\mathbf{w}, \lambda) \in W \times W^{1,q}(\Omega)$.

Proof of Theorem 4.2. Let $(\mathbf{u}_k, \theta_k)$ be a solution to the problem (P2), corresponding to each $k > 0$ and let $k \to +\infty$. From the estimates (4.4), (4.7) and using Rellich-Kondrachof's theorem, we can extract a subsequence of $(\mathbf{u}_k, \theta_k)$, still denoted by $(\mathbf{u}_k, \theta_k)$, satisfying

$$\begin{gathered}
\mathbf{u}_k \to \mathbf{u} \quad \text{in } W \text{ weakly}, \\
\mathbf{u}_k \to \mathbf{u} \quad \text{in } L^s(\Omega)^n \text{ strongly}, \\
\nabla\theta_k \to 0 \quad \text{in } L^1(\Omega)^n \text{ strongly}, \\
\theta_k \to \Theta = \text{a constant} \quad \text{in } L^1(\Omega) \text{ strongly},
\end{gathered}$$

where $n \le s < 2n/(n-2)$. We can proceed as in the proof of Theorem 4.1 to get the convergence

$$\begin{aligned}
&\lim \Big[\int_\Omega \mu(\theta_k)|\varepsilon(\mathbf{u}_k)|^p \tau\, dx + \int_\Omega g(\theta_k)|\varepsilon(\mathbf{u}_k)|\tau\, dx + \int_{\Gamma_1} \upsilon(\theta_k)|(\mathbf{u}_k)_\tau|\tau\, ds\Big] \\
&= \mu(\Theta)\int_\Omega |\varepsilon(\mathbf{u})|^p \tau \mathbf{dx} + g(\Theta)\int_\Omega |\varepsilon(\mathbf{u})|\tau\, dx + \upsilon(\Theta)\int_{\Gamma_1} |\mathbf{u}_\tau|\tau\, ds.
\end{aligned} \tag{4.47}$$

Then, we can pass to the limit $k \to +\infty$ in (3.5) and taking $\tau = 1$ to obtain the implicit scalar equation (3.7). Now, taking the limit $k \to +\infty$ in (3.4), it follows that $\mathbf{u}$ solves the nonlocal inequality (3.6). Moreover, the scalar equation (3.7) asserts that $\Theta \ge 0$. □

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