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OSCILLATION CRITERIA FOR SEMILINEAR ELLIPTIC EQUATIONS WITH A DAMPING TERM IN $\mathbb{R}^n$

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Abstract. We use a method based on Picone-type identities to find oscillation conditions for the equation

$$\sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u,\nabla u) + c(x)u = 0\,,$$

with Dirichlet boundary conditions on bounded and unbounded domains. In this article, the above method substitudes the traditional Riccati techniques [3, 8] used for unbounded domains.

1. Introduction

We consider semilinear Dirichlet problems associated with the elliptic equation

$$\ell u := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u,\nabla u) + c(x)u = 0 \tag{1.1}$$

in a smooth, open and bounded (or unbounded) domain $G \subset \mathbb{R}^n$, $n \geq 3$.

Oscillation conditions for (1.1) when f does not depend on ∇u are shown in [7]. Inspired by those results, we find conditions on f, a_{ij} and c for (1.1) to be oscillatory in $\mathbb{R}^n$. We recall that (1.1) is said to be oscillatory in $\mathbb{R}^n$ if for all $R > 0$, any of its (classical) solutions (extended to the whole space) has a simple zero in $\Omega_R := \{x \in \mathbb{R}^n : \|x\| > R\}$.

In this article, we use the notation:

$$D_i\{.\} := \frac{\partial}{\partial x_i}\{.\} := \{.\}_{,i}\,;$$

$$a(Y,W) := \sum_{i,j=1}^{n} a_{ij}Y^iW^j, \quad \text{for } Y, W \in \mathbb{R}^n,\ a \in M_{n\times n}\,,$$

where $M_{n\times n}$ denotes the space of $n \times n$-matrices. The function $f(x,u,\nabla u)$ plays the role of the damping term in (1.1). We use the hypotheses:

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(H1) The functions $a_{ij} \in C^1(\overline{G};\mathbb{R}_+)$ are symmetric and continuous with
$$\sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \geq 0 \quad \forall (x,\xi) \in G \times \mathbb{R}^n \quad (> 0 \text{ if } \xi \neq 0).$$
(H2) The function $c \in C(\overline{G};\mathbb{R})$; $f \in C(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n;\mathbb{R})$ is non constant; $\mathbb{R}_+ := (0,\infty)$ and $\bar{\mathbb{R}}_+ := [0,\infty)$. The (classical) solutions for (1.1) are those which belong to the space $C^1(\overline{G}) \cap C^2(G)$.

(H3) Teh function f satisfies: for each $t \in \mathbb{R}$, $\xi \in \mathbb{R}^n$,
 (i) $tf(x,t,\xi) > 0$ or
 ii) $tf(x,t,\xi) < 0$
 for all $x \in G$.

Oscillatory solutions will be extended to the whole space, if they were expressed only in a bounded set G. When the domain is the whole space $\mathbb{R}^n$, Hypotheses (H1)–(H3) need to hold outside G, for the oscillatory results to be true.

2. Preliminaries

For (smooth) functions u and w, as in [1], from the expressions $D_i\{ua_{ij}D_ju - (u^2/w)a_{ij}D_jw\}$ and $u\ell u$ satisfies the property that if $w \neq 0$, then
$$\begin{aligned} &\sum_{i,j=1}^n D_i\{ua_{ij}(x)D_ju - \frac{u^2}{w}\, a_{ij}D_jw\} \\ &= w^2 a\Big(\nabla[\frac{u}{w}],\nabla[\frac{u}{w}]\Big) + u\ell u - \frac{u^2}{w}\ell w + u^2\{\frac{f(x,w,\nabla w)}{w} - \frac{f(x,u,\nabla u)}{u}\} \end{aligned} \tag{2.1}$$
and if $u \neq 0$, then
$$\begin{aligned} &\sum_{i,j=1}^n D_i\Big\{wa_{ij}(x)D_jw - \frac{w^2}{u}\, a_{ij}D_ju\Big\} \\ &= u^2 a\Big(\nabla[\frac{w}{u}],\nabla[\frac{w}{u}]\Big) + w\ell w - \frac{w^2}{u}\ell u + w^2\{\frac{f(x,u,\nabla u)}{u} - \frac{f(x,w,\nabla w)}{w}\}\,. \end{aligned} \tag{2.2}$$

Lemma 2.1. *Assume* (H1)–(H3) *hold. Let u and v be solutions of*
$$\ell v := \sum_{ij=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)v + c(x)v + f(x,v,\nabla v) = 0 \quad \textit{in } G; \tag{2.3}$$
$$Lu := \sum_{ij=1}^n \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + c(x)u = 0 \quad \textit{in } G; \tag{2.4}$$
$$u\big|_{\partial G} = 0 \quad \textit{or} \quad v\big|_{\partial G} = 0. \tag{2.5}$$
Then as in (2.1),
$$\sum_{i,j=1}^n D_i\{va_{ij}(x)D_jv - \frac{v^2}{u}\, a_{ij}D_ju\} = u^2 a\Big(\nabla[\frac{v}{u}],\nabla[\frac{v}{u}]\Big) - vf(x,v,\nabla v)$$
if $u \neq 0$ in G and if $v \neq 0$ in G
$$\sum_{i,j=1}^n D_i\Big\{ua_{ij}(x)D_ju - \frac{u^2}{v}\, a_{ij}D_jv\Big\} = v^2 a\Big(\nabla[\frac{u}{v}],\nabla[\frac{u}{v}]\Big) + u^2\frac{f(x,v,\nabla v)}{v}\,.$$

Then the two solutions cannot be simultaneously non zero throughout G. Consequently

- (i) *there is no non negligible domain $\Omega \subset G$ in which the solutions u and v satisfy $uv > 0$ and $u|_{\partial\Omega} = v|_{\partial\Omega} = 0$;*
- (ii) *in between two consecutive zeroes of each one lies one zero of the other.*

Proof. Assume that in G two solutions u and v are of the same sign and have value zero on ∂G. Assume that (H3i) holds. Then integration over G of (2.1) where v replaces w, gives

$$0 = \int_G \Big[v^2 a\Big(\nabla[\frac{u}{v}], \nabla[\frac{u}{v}]\Big) + u^2 \frac{f(x,v,\nabla v)}{v}\Big] dx \tag{2.6}$$

which cannot hold as the second member is strictly positive. Assume that (H3ii) holds . Then integration over G of (2.2) with v replacing w gives

$$0 = \int_G \Big\{u^2 a\Big(\nabla[\frac{v}{u}], \nabla[\frac{v}{u}]\Big) - v f(x,v,\nabla v)\Big\} dx \tag{2.7}$$

and we get the same conclusion as the second member of the equation is strictly positive. □

Remark 2.2. Among the admissible functions f we have:

(A1). Define $f(x,u,\nabla u) := g_1(x,u) + g_2(u,\nabla u)$ for all $t \neq 0$, $\xi \in \mathbb{R}^n$, $x \in \mathbb{R}^n$. In the case (H3i), $tg_1(x,t)$ and $tg_2(t,\xi)$ are strictly positive functions. In the case (H3ii), $tg_1(x,t)$ and $tg_2(t,\xi)$ are strictly negative functions. In either case

$$\int_G u^2 \frac{f(x,v,\nabla v)}{v}\, dx \geq 0\,.$$

(A2). Define $f(x,u,\nabla u) := g_1(x,u) + \overrightarrow{B}.\nabla\zeta(u)$, where

$$tg_1(x,t) \leq 0 \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{2.8}$$

$\overrightarrow{B} = (b_1(x), b_2(x), \dots, b_n(x))$ is a vector field, $u\nabla\zeta(u) \equiv \nabla\psi(u)$ for some $\psi \in C^1(\mathbb{R})$ which keeps the same sign in $\mathbb{R}$ and either

$$\frac{\partial b_i}{\partial x_i} \geq 0 \quad \text{for } i = 1,2,\dots,n, \text{ if } \psi \text{ is a non negative function,} \tag{2.9}$$

$$\frac{\partial b_i}{\partial x_i} \leq 0 \quad \text{for } i = 1,2,\dots,n, \text{ if } \psi \text{ is a non positive function.} \tag{2.10}$$

Simple calculations show that anyone of the two conditions (2.9) or (2.10) leads to

$$\int_G \{-uf(x,u,\nabla u)\} dx \geq 0$$

and (2.7) applies.

The condition (A2) applies for example to the perturbed Schrodinger equation (see [3])

$$\Delta u + \langle \overrightarrow{b}(x), \nabla u\rangle + c(x)u = 0\,.$$

2.1. **Oscillation criteria.**

Definition. A function u is said to be oscillatory in $\mathbb{R}^n$ if for all $R > 0$, u has a simple zero in $\Omega_R := \{x \in \mathbb{R}^n : |x| > R\}$. A solution of (1.1) will be said to be oscillatory if its extension over $\mathbb{R}^n$ is oscillatory. Equation (1.1) is said to be oscillatory if it has oscillatory solutions. For the equation

$$Lu := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + c(x)u = 0 \quad \text{in } \mathbb{R}^n \tag{2.11}$$

and for $r > 0$ and $I_n := \{(i,j) : i,j \in \{1,2,\dots n\}\}$, define

$$A(r) := \max_{\{I_n:|x|=r\}}\{a_{ij}(x)\}, \quad C(r) := \min_{|x|=r} c(x),$$
$$p(r) := r^{n-1}A(r), \quad q(r) := r^{n-1}C(r)$$

and the associated equation

$$\big(p(r)y'\big)' + q(r)y = 0 \quad \text{in } \mathbb{R}_+ . \tag{2.12}$$

For $r_0 > 0$, define

$$P(t) := \int_{r_0}^{t} \frac{dr}{p(r)} \quad \text{if } \lim_{t\to\infty} p(t) = \infty$$

and

$$\Pi(t) := \int_{r_0}^{t} \frac{dr}{p(r)} \quad \text{if } \lim_{t\to\infty} p(t) < \infty.$$

From [2, Lemma 3.1 and Theorem 3.1], we have the following result (see also [7]).

Lemma 2.3. *Let* $r_0 > 0$,

(i) $\int_{r_0}^{\infty} q(r)dr = \infty$ *or*

$$\int_{r_0}^{\infty} q(r)dr < \infty \quad \textit{and} \quad \liminf_{r\nearrow\infty} \Big\{P(r)\int_{r}^{\infty} q(s)ds\Big\} > \frac{1}{4}$$

(ii) Π *is bounded and* $\int_{r_0}^{\infty} \Pi(r)^2 q(r)dr = \infty$, *or*

$$\int_{r_0}^{\infty} \Pi(r)^2 q(r)dr < \infty \quad \textit{and} \quad \liminf_{r\nearrow\infty} \Big\{\frac{1}{\Pi(r)}\int_{r}^{\infty} \Pi(s)^2 q(s)ds\Big\} > \frac{1}{4}$$

If either (i) or (ii) holds, then (2.12) *is oscillatory, and so is* (2.11).

The above lemma also holds when $A(r)$ and $C(r)$ are replaced, respectively, by

$$\overline{A}(r) := \frac{1}{\omega_n r^{n-1}}\int_{|x|=r} \max_{I_n}\{a_{ij}(x)\}ds \quad \text{and} \quad \overline{\mathrm{C}}(\mathrm{r}) := \frac{1}{\omega_{\mathrm{n}}\mathrm{r}^{\mathrm{n}-1}}\int_{|\mathrm{x}|=\mathrm{r}} \mathrm{c}(\mathrm{x})\mathrm{ds},$$

where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. ([7])

3. Main result

From Lemma 2.3 and the preceding results, we have the de following theorem.

Theorem 3.1. *Consider, in a bounded and regular domain* $G \subset \mathbb{R}^n$, *the equation*

$$\ell u := \sum_{ij=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial}{\partial x_j}\Big)u + f(x,u,\nabla u) + c(x)u = 0 \quad \textit{in } G, \tag{3.1}$$

where (H1), (H2) *hold in the whole space* $\mathbb{R}^n$. *If in addition*

(a) *either* (H3) *holds in* $\mathbb{R}^n$ *and the functions* a_{ij} *and* c *satisfy (i) or (ii) of Lemma 2.3, or*

(b) (2.8)–(2.10) *hold*

then (3.1) *is oscillatory in* $\mathbb{R}^n$*.*

Proof. From Lemma 2.3, conditions (i) and (ii) imply that (2.11) is oscillatory. From Lemma 2.1 and Remark 2.2, if (2.11) is oscillatory, so is (3.1). □

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