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MAXIMUM PRINCIPLE AND EXISTENCE RESULTS FOR ELLIPTIC SYSTEMS ON $\mathbb{R}^N$

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Abstract. In this work we give necessary and sufficient conditions for having a maximum principle for cooperative elliptic systems involving p-Laplacian operator on the whole $\mathbb{R}^N$. This principle is then used to yield solvability for the cooperative elliptic systems by an approximation method.

1. Introduction

This work is mainly concerned with the elliptic system

$$\begin{gathered}
-\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)|v|^{\beta}v + f \quad \text{in } \mathbb{R}^N, \\
-\Delta_q v = cn_1(x)|u|^{\alpha}u + dn(x)|v|^{q-2}v + g \quad \text{in } \mathbb{R}^N, \\
u(x) \to 0, v(x) \to 0 \quad \text{as } |x| \to +\infty.
\end{gathered} \tag{1.1}$$

Here $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < +\infty$, is the so-called p-Laplacian operator; a, b, c, d, α and β are reals parameters; f, g, m, n, m_1 and n_1 are weights whose properties will be specified later.

We are concerned with the existence of positive solutions and with the following form of maximum principle: If $f, g \geq 0$ in $\mathbb{R}^N$ then $u, v \geq 0$ in $\mathbb{R}^N$ for any solution (u, v) of (1.1). It is well known that maximum principle plays an important role in the theory on nonlinear equations. For instance, it is used to access existence results and qualitative properties of solutions for linear and nonlinear differential equations, (see for instance [14] and [18] for a survey).

Many works have been devoted to the study of linear and nonlinear elliptic systems either on a bounded domain or an unbounded domain of $\mathbb{R}^N$ (in particular the whole $\mathbb{R}^n$) (cf.[3, 5, 6, 7, 8, 9, 19]). In [12, 13] for the linear case (i.e $p = q = 2$), it was presented necessary and sufficient conditions for having maximum principle and existence of positive solutions. These results have been later extended in [9] to

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the nonlinear system

$$-\Delta_p u_i = \sum_{j=1}^{n} a_{ij}|u_j|^{p-2}u_j + f_i \quad \text{in } \Omega$$
$$u_i = 0 \quad \text{on } \partial\Omega,\ i = 1, 2, \dots n$$

where Ω is a bounded domain of $\mathbb{R}^N$.

For specific interest for our purposes is the work in [19] where a study of problems such as (1.1) was carried out in the case of $\mathbb{R}^N$ in the presence of some weight functions. In our work we consider problem (1.1) with coefficients $b, c > 0$, and the weight functions $m(x)$, $n(x)$, $m_1(x)$, $n_1(x)$ positive. Here m belongs to $L^{N/p}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$ and n belongs to $L^{N/q}(\mathbb{R}^N) \cap L^\infty_{\rm loc}(\mathbb{R}^N)$. Then we state necessary and sufficient conditions for a maximum principle to hold. Moreover our technique can be developed to get a related result for the following class of cooperative systems

$$\begin{gathered} -\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)|u|^\alpha|v|^\beta v + f \quad \text{in } \mathbb{R}^N, \\ -\Delta_q v = cn_1(x)|v|^\beta|u|^\alpha u + dn(x)|v|^{q-2}v + g \quad \text{in } \mathbb{R}^N, \\ u(x) \to 0, \quad v(x) \to 0 \quad \text{as } |x| \to +\infty \end{gathered} \tag{1.2}$$

where the coefficients a, b, c, d, and the weights $m(x), n(x), m_1(x), n_1(x)$ are as above. When $a = b = c = d = 1$, problem (1.2) is relaxed to the particular case of system considered in [19] where the necessary condition for the maximum principle to hold given by the authors is depend on x. The arguments developed in this paper enable us to obtain a non dependance on x necessary condition.

The remainder of the paper is organized as follows: In Section 3, the maximum principle for (1.1) is given and is shown to be proven full enough to yield existence results of solutions for (1.1) in Section 4. In section 5, we briefly give a version of our result for the cooperative systems (1.2). In the preliminary Section 2, we collect some known results relative to the principal positive eigenvalue and to various Sobolev imbeddings.

2. Preliminaries

Throughout this work, we will assume that $1 < p, q < N$ and

- (H1) $m, n > 0; m \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^{N/p}(\mathbb{R}^N)$ and $n \in L^\infty_{\rm loc}(\mathbb{R}^N) \cap L^{N/q}(\mathbb{R}^N)$
- (H2) $0 < m_1(x) \le [m(x)]^{\frac{1}{p}}[n(x)]^{\frac{\beta+1}{q}}$ and $0 < n_1(x) \le [n(x)]^{\frac{1}{q}}[m(x)]^{\frac{\alpha+1}{p}}$ a.e. in $\mathbb{R}^N$
- (H3) $f \ge 0$ and $f \in L^{(p^*)'}(\mathbb{R}^N)$; $g \ge 0$ and $g \in L^{(q^*)'}(\mathbb{R}^N)$
- (H4) $b, c \ge 0$; $\alpha, \beta \ge 0$; $\frac{\alpha+1}{p} + \frac{1}{q} = 1$ and $\frac{\beta+1}{q} + \frac{1}{p} = 1$

Here $p^* = \frac{Np}{N-p}$, $q^* = \frac{Nq}{N-q}$ denote the critical Sobolev exponent of p and q respectively; p' is the Hölder conjugate of p. It is clear that $\frac{1}{p'} = \frac{\beta+1}{q}$ and $\frac{1}{q'} = \frac{\alpha+1}{p}$.

We denote by $D^{1,s}(\mathbb{R}^N)$ (with $1 < s < N$) the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{D^{1,s}(\mathbb{R}^N)} = \Big(\int_{\mathbb{R}^N} |\nabla u|^s\Big)^{1/s}.$$

It can be shown that (cf [16])

$$D^{1,s}(\mathbb{R}^N) = \{u \in L^{s^*}(\mathbb{R}^N) : \nabla u \in (L^s(\mathbb{R}^N))^N\}$$

and for any positive weight $g \in L^{\infty}_{\rm loc}(\mathbb{R}^N) \cap L^{N/s}(\mathbb{R}^N)$ the following embeddings hold (cf.[10, 11, 15])

$$D^{1,s}(\mathbb{R}^N) \hookrightarrow L^{s^*(\mathbb{R}^N)} \quad \text{and} \quad D^{1,s}(\mathbb{R}^N) \hookrightarrow\hookrightarrow L^s(g,\mathbb{R}^N)$$

where $L^s(g,\mathbb{R}^N)$ is the L^s space on $\mathbb{R}^N$ with the weight g (cf. [11]).

By solution (u,v) of (1.1) (or related equations), we mean a weak solution; i.e., $(u,v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ with

$$\begin{gathered} \int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u.\nabla w = \int_{\mathbb{R}^N} [am(x)|u|^{p-2}uw + bm_1(x)|v|^{\beta}vw + fw] \\ \int_{\mathbb{R}^N} |\nabla v|^{q-2}\nabla v.\nabla z = \int_{\mathbb{R}^N} [cn_1(x)|u|^{\alpha}uz + dn(x)|v|^{q-2}vz + gz] \end{gathered} \tag{2.1}$$

for all $(w,z) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$. Note that by the above embeddings, every integral in (2.1) is well-defined. Regularity results from [20, 21] on general quasilinear equations imply that such a weak solution (u,v) belong to $C^1(\mathbb{R}^N) \times C^1(\mathbb{R}^N)$. It is also known that a weak solution of (1.1) decays to zero at infinity (cf. [4, 10]).

To conclude this introduction, let us briefly recall some properties of the spectrum of $-\Delta_p$ with weight to be used later (cf. [1, 11]). We denote by

$$\lambda_1(m,p) := \min\big\{ \int_{\mathbb{R}^N} |\nabla u|^p : u \in D^{1,p}(\mathbb{R}^N) \text{ and } \int_{\mathbb{R}^N} m|u|^p = 1 \big\} \tag{2.2}$$

the unique principal eigenvalue of

$$\begin{gathered} -\Delta_p u = \lambda m(x)|u|^{p-2}u \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to +\infty;\ u > 0 \text{ in } \mathbb{R}^N \end{gathered} \tag{2.3}$$

and by $\varphi_1(m) = \varphi_1(m,p) \in D^{1,p}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$ the associated positive eigenvalue such that $\int_{\mathbb{R}^N} m|\varphi_1(m)|^p = 1$. It is well known that $\lambda_1(m,p)$ is simple and isolated.

Here and henceforth, we will denote by $\Phi = \varphi_1(m,p)$ (respectively by $\Psi = \varphi_1(n,q)$) the positive eigenfunction associated to $\lambda_1(m,p)$ (respectively $\lambda_1(n,q)$) and normalized by

$$\int_{\mathbb{R}^N} m\Phi(x)^p = \int_{\mathbb{R}^N} n\Psi(x)^q = 1 \tag{2.4}$$

3. Maximum principle

We assume that $1 < p,\ q < N$ and that hypothesis (H1), (H2), (H3) and (H4) are satisfied. We begin by consider the problem

$$\begin{gathered} -\Delta_p u = \mu m(x)|u|^{p-2}u + h(x) \quad \text{in } \mathbb{R}^N \\ u(x) \to 0 \quad \text{as } |x| \to +\infty \end{gathered} \tag{3.1}$$

The following results were proved in [10, 11]

Proposition 3.1. *(1) Let $h \in L^{(p^*)'}(\mathbb{R}^N)$ and assume that* (H1) *is satisfied. If $\mu < \lambda_1(m,p)$ then* (3.1) *admits a solution in $D^{1,p}(\mathbb{R}^N)$.*

(2) Let $h \in L^{(p^*)'}(\mathbb{R}^N)$ with $h \geq 0$ a.e. in $\mathbb{R}^N$ and $h \not\equiv 0$.

- (a) *If $\mu \in [0, \lambda_1(m,p)[$, then any solution u of* (3.1) *is positive in $\mathbb{R}^N$.*
- (b) *If $\mu = \lambda_1(m,p)$ then* (3.1) *has no solution*
- (c) *If $\mu > \lambda_1(m,p)$ then* (3.1) *has no positive solution.*

Using [20, 21], one also has a regularity result.

Proposition 3.2. *For all $r > 0$, any solution (u, v) of (1.1)) belongs to $C^{1,\gamma}(B_r) \times C^{1,\gamma}(B_r)$, where $\gamma = \gamma(r) \in]0, 1[$ and B_r is the ball of radius r centered at the origin.*

Let

$$a_1(r) := \inf_{B_r} k_1(x), \quad a_2(r) := \sup_{B_r} k_2(x), \tag{3.2}$$

where

$$k_1(x) := \Big[\frac{n_1(x)}{n(x)}\Big]^{\frac{\beta+1}{q}} \Big[\frac{\Phi(x)^p}{\Psi(x)^q}\Big]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}},$$

$$k_2(x) := \Big[\frac{m(x)}{m_1(x)}\Big]^{\frac{\alpha+1}{p}} \Big[\frac{\Phi(x)^p}{\Psi(x)^q}\Big]^{\frac{\alpha+1}{p} \frac{\beta+1}{q}}.$$

We denote $a_{1\infty} = \lim_{r\to+\infty} a_1(r)$ and $a_{2\infty} = \lim_{r\to+\infty} a_2(r)$. Let

$$\Theta = \frac{a_{1\infty}}{a_{2\infty}}. \tag{3.3}$$

One can easily prove that

$$\Theta \le \frac{a_1(r)}{a_2(r)} \quad \text{for all } r > 0 \quad \text{and} \quad 0 \le \Theta \le 1 \tag{3.4}$$

We say that (1.1) satisfies the maximum principle (in short (MP)) if for $f, g \ge 0$ a.e in $\mathbb{R}^N$, any solution (u, v) of (1.1) is such that $u > 0, v > 0$ a.e. in $\mathbb{R}^N$.

We now turn to our first main result, i.e., the validity of the (MP) which is stated as follows

Theorem 3.3. *Assume that hypothesis* (H1)–(H) *are satisfied. Then the (MP) holds for* (1.1) *if*

- (C1) $\lambda_1(m, p) > a$
- (C2) $\lambda_1(n, q) > d$
- (C3) $[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} > b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}$

Conversely, if the (MP) holds, then (C1), (C2) and (C4) are satisfied, where

- (C4) $[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} > \Theta b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}$.

Corollary 3.4. *If $p = q$ and $m \equiv n$ a.e. in $\mathbb{R}^N$, then the (MP) holds for* (1.1) *if only if (C1), (C2) and (C4) are satisfied*

Proof of Theorem 3.3. The condition is necessary. The proof of (C1) or (C2) is standard (cf. for instance [2, 3, 19]). We give here the sketch of this proof.

If $\lambda_1(m, p) \le a$, then the functions $f := [a - \lambda_1(m, p)]m\Phi^{p-1}$ and $g := cn_1\Phi^{\alpha+1}$ are nonnegative and $(-\Phi, 0)$ is a solution of (1.1), which contradicts the (MP).

Similarly, if $\lambda_1(n, q) \le d$, then the functions $f := bm_1\Psi^{\beta+1}$ and $g := [d - \lambda_1(n, q)]n\Psi^{q-1}$ are nonnegative and $(0, -\Psi)$ is a solution of (1.1), a contradiction.

The proof of (C4) can be adapted from [19] as follow. We assume that $\lambda_1(m, p) > a$ and $\lambda_1(n, q) > d$. If one of the coefficients Θ, b or c vanishes, then (C4) is satisfied. We will then assume that $\Theta \ne 0$, $b \ne 0$, $c \ne 0$ and that (C4) does not hold, i.e.

$$[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} \le \Theta b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}} \tag{3.5}$$

Set $A = \big(\frac{\lambda_1(m,p)-a}{b}\big)^{\frac{\alpha+1}{p}}$ and $B = \big(\frac{\lambda_1(n,q)-d}{c}\big)^{\frac{\beta+1}{q}}$. Then, by (3.5), one has $AB \le \Theta$, which clear implies that $Aa_{2\infty} \le \frac{1}{B}a_{1\infty}$. One deduces that there exists $\xi \in \mathbb{R}_+^*$ such that

$$Aa_{2\infty} \le \xi \le \frac{1}{B}a_{1\infty}.$$

Since the function $a_1(r)$ (respectively $a_2(r)$) is decreasing (respectively increasing) on $\mathbb{R}_+^*$, one has

$$Aa_2(r) \le Aa_{2\infty} \le \xi \le \frac{1}{B}a_{1\infty} \le \frac{1}{B}a_1(r), \quad \text{for all } r > 0.$$

But for any $x \in \mathbb{R}^N$, there exists $r > 0$ such that

$$Ak_2(x) \le Aa_2(r) \quad \text{and} \quad \frac{1}{B}a_1(r) \le \frac{1}{B}k_1(x).$$

Consequently we set

$$Ak_2(x) \le Aa_2(r) \le \xi \le \frac{1}{B}a_1(r) \le \frac{1}{B}k_1(x)$$

for all $x \in \mathbb{R}^N$, i.e.,

$$Ak_2(x) \le \xi \quad \forall x \in \mathbb{R}^N \tag{3.6}$$

$$\frac{B}{k_1(x)} \le \frac{1}{\xi} \quad \forall x \in \mathbb{R}^N\,. \tag{3.7}$$

Next let we set $\xi = (\frac{c_1^q}{c_2^p})^{\frac{\alpha+1}{p}\frac{\beta+1}{q}}$, where c_1 and c_2 are positive constants.

From (3.6) and (H4), one easily gets,

$$-[\lambda_1(m,p) - a]m(x)[c_2\Phi(x)]^{p-1} + bm_1(x)[c_1\Psi(x)]^{\beta+1} \ge 0 \quad \text{for all } x \in \mathbb{R}^N.$$

Similarly, using (3.7) and (H4), one has

$$-[\lambda_1(n,q) - d]n(x)[c_1\Psi(x)]^{q-1} + cn_1(x)[c_2\Phi(x)]^{\alpha+1} \ge 0 \quad \text{for all } x \in \mathbb{R}^N.$$

Hence

$$f := -[\lambda_1(m,p) - a]m(x)[c_2\Phi(x)]^{p-1} + bm_1(x)[c_1\Psi(x)]^{\beta+1} \ge 0 \quad \text{for all } x \in \mathbb{R}^N$$

and

$$g := -[\lambda_1(n,q) - d]n(x)[c_1\Psi(x)]^{q-1} + cn_1(x)[c_2\Phi(x)]^{\alpha+1} \ge 0 \quad \text{for all } x \in \mathbb{R}^N$$

are nonnegative functions and $(-c_2\Phi, -c_1\Psi)$ is a solution of (1.1). This is a contradiction with the (MP).

The condition is sufficient. A detailed proof of this part can be found in [3, 20]. We give a sketch here. Assume that the conditions (C1), (C2) and (C3) are satisfied. Let (u,v) be a solution of (1.1) for $f, g \ge 0$. Moreover, suppose that $u^- \not\equiv 0$ and $v^- \not\equiv 0$ and taking those functions as test function in (1.1), we find by Hölder inequality that

$$\begin{aligned}
&[(\lambda_1(m,p) - a)^{\frac{\alpha+1}{p}}(\lambda_1(n,q) - d)^{\frac{\beta+1}{q}} - b^{\frac{\alpha+1}{p}}c^{\frac{\beta+1}{q}}] \\
&\times \Big[\Big(\int_{\mathbb{R}^N} m|u^-|^p\Big)\Big(\int_{\mathbb{R}^N} n|v^-|^q\Big)\Big]^{\frac{\alpha+1}{p}\frac{\beta+1}{q}} \le 0,
\end{aligned}$$

which contradicts assumption (C4). By applying regularity results of [20, 21] and the maximum principle of [22], one has in fact $u > 0$ and $v > 0$ a.e in $\mathbb{R}^N$. $\square$

4. Existence of positive solutions

In this section, we prove the existence of positive solutions for (1.1) under conditions (C1), (C2) and (C3), by an approximation method used in [2, 3]. For $\epsilon \in]0,1[$, we define the following expression

$$\begin{aligned}
X_k &:= \frac{|u_k|^{p-2}u_k}{1+|\epsilon^{1/p}u_k|^{p-1}}, \quad X := \frac{|u|^{p-2}u}{1+|\epsilon^{1/p}u|^{p-1}},\\
Y_k &:= \frac{|u_k|^{\alpha}u_k}{1+|\epsilon^{1/p}u_k|^{\alpha+1}}, \quad Y := \frac{|u|^{\alpha}u}{1+|\epsilon^{1/p}u|^{\alpha+1}},\\
X'_k &:= \frac{|v_k|^{q-2}v_k}{1+|\epsilon^{1/q}v_k|^{q-1}}, \quad X' := \frac{|v|^{q-2}v}{1+|\epsilon^{1/q}v|^{q-1}},\\
Y'_k &:= \frac{|v_k|^{\beta}v_k}{1+|\epsilon^{1/q}v_k|^{\beta+1}}, \quad Y' := \frac{|v|^{\beta}v}{1+|\epsilon^{1/q}v|^{\beta+1}}.
\end{aligned}$$

On has the following result which will be useful later.

Lemma 4.1. *If (u_k, v_k) converges to (u, v) in $L^{p^*}(\mathbb{R}^N) \times L^{q^*}(\mathbb{R}^N)$ then*

(i) *$X_k \to X$ in $L^{\frac{p^*}{p-1}}(\mathbb{R}^N)$, $Y_k \to Y$ in $L^{\frac{p^*}{\alpha+1}}(\mathbb{R}^N)$ and in $L^{q'}(m, \mathbb{R}^N)$.*

(ii) *$X'_k \to X'$ in $L^{\frac{q^*}{q-1}}(\mathbb{R}^N)$, $Y'_k \to Y'$ in $L^{\frac{q^*}{\beta+1}}(\mathbb{R}^N)$ and in $L^{p'}(n, \mathbb{R}^N)$.*

Proof. We give the proof for (i) and indicate that the same arguments hold for (ii). If $u_k \to u$ in $L^{p^*}(\mathbb{R}^N)$, then there exists a subsequence denoted (u_k) such that $u_k \to u$ almost every where in $\mathbb{R}^N$ and $|u_k(x)| \le l_1(x)$ for some $l_1 \in L^{p^*}(\mathbb{R}^N)$. Hence

$$\begin{gathered}
X_k(x) \to X(x) \quad \text{a.e. in } \mathbb{R}^N,\\
|X_k(x)| \le |u_k(x)|^{p-1} \le |l_1(x)|^{p-1} \quad \text{in } L^{\frac{p^*}{p-1}},
\end{gathered}$$

which implies, by dominated convergence Theorem, that $X_k \to X$ in $L^{\frac{p^*}{p-1}}$.

Similarly, on deduces from the convergence of Y_k to Y in $L^{\frac{p^*}{\alpha+1}}$ that

$$\begin{gathered}
Y_k(x) \to Y(x) \quad \text{a.e in } \mathbb{R}^N,\\
|Y_k(x)| \le |u_k(x)|^{\alpha+1} \le |l_2(x)|^{\alpha+1} \quad \text{in } L^{\frac{p^*}{\alpha+1}},
\end{gathered}$$

and the conclusion follows. Moreover, using Hölder inequality, we have

$$\|Y_k - Y\|^{q'}_{L^{q'}(m,\mathbb{R}^N)} = \int_{\mathbb{R}^N} m|Y_k - Y|^{q'} \le \|m\|_{L^{N/p}(\mathbb{R}^N)} \|Y_k - Y\|^{q'}_{L^{\frac{p^*}{\alpha+1}}}.$$

□

We are now in position to give the main result of this section.

Theorem 4.2. *Assume that* (H1), (H2), (H3), (C1), (C2), (C3) *are satisfied. Then for all $f \in L^{(p^*)'}(\mathbb{R}^N)$ and $g \in L^{(q^*)'}(\mathbb{R}^N)$, the system* (1.1) *has at least one solution $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.*

The proof is partly adapted from [2, 3]. We choose $r > 0$ such that $a + r > 0$ and $d + r > 0$. The system (1.1) is then equivalent to

$$\begin{gathered}-\Delta_p u + rm|u|^{p-2}u = (a+r)m|u|^{p-2}u + bm_1|v|^{\beta}v + f \quad \text{in } \mathbb{R}^N \\ -\Delta_q v + rn|v|^{q-2}v = cn_1|u|^{\alpha}u + (d+r)n|v|^{q-2}v + g \quad \text{in } \mathbb{R}^N \\ u(x) \to 0, \quad v(x) \to 0 \quad \text{as } |x| \to +\infty\end{gathered} \tag{4.1}$$

For $\epsilon \in]0,1[$, let us introduce the system

$$\begin{gathered}-\Delta_p u_\epsilon + rm|u_\epsilon|^{p-2}u_\epsilon = mh(u_\epsilon) + m_1h_1(v_\epsilon) + f \quad \text{in } \mathbb{R}^N \\ -\Delta_q v_\epsilon + rn|v_\epsilon|^{q-2}v_\epsilon = n_1k_1(u_\epsilon) + nk(v_\epsilon) + g \quad \text{in } \mathbb{R}^N \\ u_\epsilon(x) \to 0, \quad v_\epsilon(x) \to 0 \quad \text{as } |x| \to +\infty\end{gathered} \tag{4.2}$$

where

$$\begin{gathered}h(u) := (a+r)\frac{|u|^{p-2}u}{1+|\epsilon^{1/p}u|^{p-1}}, \quad h_1(v) := b\frac{|v|^{\beta}v}{1+|\epsilon^{1/q}v|^{\beta+1}}, \\ k_1(u) := c\frac{|u|^{\alpha}u}{1+|\epsilon^{1/p}u|^{\alpha+1}}, \quad k(v) := (d+r)\frac{|v|^{q-2}v}{1+|\epsilon^{1/q}v|^{q-1}}.\end{gathered}$$

Lemma 4.3. *Under hypothesis of Theorem 4.2, system* (4.2) *admits at least a couple of solution* (u, v) *in* $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.

Proof. We give the proof in several steps.

Step 1. Construction of sub-super solution for (4.2): Since the functions h, h_1, k and k_1 are bounded, there exists a constant $M > 0$ such that

$$|h(u)| \le M, \quad |h_1(v)| \le M, \quad |k_1(u)| \le M, \quad |k(v)| \le M$$

for all $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$. Let $\xi^0 \in D^{1,p}(\mathbb{R}^N)$ (respectively $\eta^0 \in D^{1,q}(\mathbb{R}^N)$) be a solution of

$$-\Delta_p u + rm|u|^{p-2}u = (m + m_1)M + f$$

(respectively $-\Delta_q v + rm|v|^{q-2}v = (n + n_1)M + g$), and let $\xi_0 \in D^{1,p}(\mathbb{R}^N)$ (respectively $\eta_0 \in D^{1,q}(\mathbb{R}^N)$) be solution of

$$-\Delta_p u + rm|u|^{p-2}u = -(m + m_1)M + f$$

(respectively $-\Delta_q v + rm|v|^{q-2}v = -(n + n_1)M + g$). Then (ξ^0, η^0) (respectively (ξ_0, η_0)) is a super solution (respectively sub solution) of system (4.2) since

$$\begin{aligned}&-\Delta_p\xi^0 + rm|\xi^0|^{p-2}\xi^0 - mh(\xi^0) - m_1h_1(\eta) - f \\ &\ge -\Delta_p\xi^0 + rm|\xi^0|^{p-2}\xi^0 - (m+m_1)M - f = 0 \quad \forall \eta \in [\eta_0, \eta^0], \\ &-\Delta_q\eta^0 + rn|\eta^0|^{q-2}\eta^0 - n_1k_1(\xi) - nk(\eta^0) - g \\ &\ge -\Delta_q\eta^0 + rn|\eta^0|^{q-2}\eta^0 - (n+n_1)M - g = 0 \quad \forall \eta \in [\xi_0, \xi^0], \\ &-\Delta_p\xi_0 + rm|\xi_0|^{p-2}\xi_0 - mh(\xi_0) - m_1h_1(\eta) - f \\ &\le -\Delta_p\xi_0 + rm|\xi_0|^{p-2}\xi_0 - (m+m_1)M - f = 0 \quad \forall \eta \in [\eta_0, \eta^0], \\ &-\Delta_q\eta_0 + rn|\eta_0|^{q-2}\eta_0 - n_1k_1(\xi) - nk(\eta_0) - g \\ &\le -\Delta_q\eta_0 + rn|\eta_0|^{q-2}\eta_0 - (n+n_1)M - g = 0 \quad \forall \eta \in [\xi_0, \xi^0]\,.\end{aligned}$$

Step 2. Definition of operator T. Denote by $K = [\xi_0, \xi^0] \times [\eta_0, \eta^0]$ and define the operator $T : (u, v) \to (w, z)$ such that

$$\begin{gathered} -\Delta_p w + rm|w|^{p-2}w = mh(u) + m_1 h_1(v) + f \quad \text{in } \mathbb{R}^N \\ -\Delta_q z + rm|z|^{q-2}z = n_1 k_1(u) + nk(v) + g \quad \text{in } \mathbb{R}^N \\ w(x) \to 0, \quad z(x) \to 0 \quad \text{as } |x| \to +\infty \end{gathered} \tag{4.3}$$

Step 3. Let us prove that $T(K) \subset K$. If $(u, v) \in K$ then we have

$$\begin{aligned} &-(\Delta_p w - \Delta_p \xi^0) + rm(|w|^{p-2}w - |\xi^0|^{p-2}\xi^0) \\ &= m[h(u) - M] + m_1[h_1(v) - M]) \end{aligned} \tag{4.4}$$

Taking $(w - \xi^0)^+$ as test function in (4.4), we have

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla w|^{p-2}\nabla w - |\nabla \xi^0|^{p-2}\nabla \xi^0)\nabla(w - \xi^0)^+ \\ &+ r\int_{\mathbb{R}^N} m(|w|^{p-2}w - |\xi^0|^{p-2}\xi^0)(w - \xi^0)^+ \\ &= \int_{\mathbb{R}^N} [m(h(u) - M) + m_1(h_1(v) - M)](w - \xi^0)^+ \leq 0. \end{aligned}$$

Hence by the monotonicity of the function $x \mapsto \|x\|^{p-2}x$ and by the monotonicity of the p-Laplacian, we deduce that $(w - \xi^0)^+ = 0$ and then $w \leq \xi^0$. Similarly we get $\xi_0 \leq w$ by taking $(w - \xi_0)^-$ as test function in (4.4). So we have $\xi_0 \leq w \leq \xi^0$ and $\eta_0 \leq z \leq \eta^0$ and the step is complete.

Step 4. T is completely continuous:

• We will first prove that T is continuous. Indeed let $(u_k, v_k) \to (u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$, we will prove that $(w_k, z_k) = T(u_k, v_k)$ converges to $(w, z) = T(u, v)$.

$$\begin{aligned} &(-\Delta_p w_k + rm|w_k|^{p-2}w_k) - (-\Delta_p w + rm|w|^{p-2}w) \\ &= m[h(u_k) - h(u)] + m_1[h_1(v_k) - h_1(v)] \\ &= (a + r)m(X_k - X) + bm_1(Y'_k - Y'), \end{aligned} \tag{4.5}$$

where X_k, X, Y'_k and Y' are previously define in Lemma 4.1. Then taking $(w_k - w)$ as test function in (4.5), we get

$$\begin{aligned} &\int_{\mathbb{R}^N} (|\nabla w_k|^{p-2}\nabla w_k - |\nabla w|^{p-2}w)\nabla(w_k - w) \\ &\leq \int_{\mathbb{R}^N} (|\nabla w_k|^{p-2}\nabla w_k - |\nabla w|^{p-2}w)\nabla(w_k - w) \\ &\quad + r\int_{\mathbb{R}^N} m(|w_k|^{p-2}w_k - |w|^{p-2}w)(w_k - w) \\ &= (a + r)\int_{\mathbb{R}^N} m(X_k - X)(w_k - w) + b\int_{\mathbb{R}^N} m_1(Y'_k - Y')(w_k - w). \end{aligned}$$

Using Hölder inequality, we obtain

$$\int_{\mathbb{R}^N} m(X_k - X)(w_k - w) \leq \|m\|_{L^{N/p}(\mathbb{R}^N)} \|X_k - X\|_{L^{p^*/(p-1)}(\mathbb{R}^N)} . \|w_k - w\|_{L^{p^*}(\mathbb{R}^N)}$$

and

$$\begin{aligned}\int_{\mathbb{R}^N} m_1(Y_k' - Y')(w_k - w) &\le \int_{\mathbb{R}^N} [m^{1/p}(w_k - w)][n^{(\beta+1)/q}(Y_k' - Y')] \\ &\le \|w_k - w\|_{L^p(m,\mathbb{R}^N)}.\|Y_k' - Y'\|_{L^{p'}(n,\mathbb{R}^N)},\end{aligned}$$

since $\frac{\beta+1}{q} = \frac{1}{p'}$. Consequently

$$\begin{aligned}&0 \le \int_{\mathbb{R}^N} (|\nabla w_k|^{p-2}\nabla w_k - |\nabla w|^{p-2} w)\nabla(w_k - w) \\ &\le (a + r)\|m\|_{L^{N/p}(\mathbb{R}^N)}\|X_k - X\|_{L^{p^*/(p-1)}(\mathbb{R}^N)}\|w_k - w\|_{L^{p^*}(\mathbb{R}^N)} \\ &\quad + b\|Y_k' - Y'\|_{L^{p'}(n,\mathbb{R}^N)}.\|w_k - w\|_{L^p(m,\mathbb{R}^N)}\end{aligned}$$

Using then the inequality

$$\|x - y\|^p \le c[(\|x\|^{p-2}x - \|y\|^{p-2}y)(x - y)]^{s/2}[\|x\|^p + \|y\|^p]^{1-s/2}, \tag{4.6}$$

where $x, y \in \mathbb{R}^N$, $c = c(p) > 0$ and $s = 2$ if $p \ge 2$, $s = p$ if $1 < p < 2$ (cf. e.g. [17]), one easily obtains that $w_k \to w$ in $D^{1,p}(\mathbb{R}^N)$. Similarly, we have $z_k \to z$ in $D^{1,q}(\mathbb{R}^N)$.

• We now prove that operator T is compact. Let (u_k, v_k) be a bounded sequence in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ and set $(w_k, z_k) = T(u_k, v_k)$. We have

$$-\Delta_p w_k + rm|w_k|^{p-2}w_k = mh(u_k) + m_1h_1(v_k) + f \tag{4.7}$$

Taking w_k as test function in (4.7), we get

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w_k|^p + r\int_{\mathbb{R}^N} m|w_k|^p \\ &= \int_{\mathbb{R}^N} mh(u_k)w_k + \int_{\mathbb{R}^N} m_1h_1(v_k)w_k + \int_{\mathbb{R}^N} fw_k \\ &\le (a + r)\int_{\mathbb{R}^N} m|u_k|^{p-1}|w_k| + b\int_{\mathbb{R}^N} m_1|v_k|^{\beta+1}|w_k| + \int_{\mathbb{R}^N} |f||w_k| \\ &\le \big[(a + r)\|u_k\|^{p-1}_{L^p(m,\mathbb{R}^N)} + b\|v_k\|^{\beta+1}_{L^q(n,\mathbb{R}^N)}\big].\|w_k\|_{L^p(m,\mathbb{R}^N)} \\ &\quad + \|f\|_{L^{(p^*)'}(\mathbb{R}^N)}\|w_k\|_{L^{p^*}(\mathbb{R}^N)}.\end{aligned}$$

Hence w_k is bounded in $D^{1,p}(\mathbb{R}^N)$ and consequently, up to a subsequence w_k converges to w weakly in $D^{1,p}(\mathbb{R}^N)$ and strongly in $L^p(m, \mathbb{R}^N)$. Now taking $(w_k - w_q)$ as test function in (4.7), we have

$$\begin{aligned}&\int_{\mathbb{R}^N} |\nabla w_k|^{p-2}\nabla w_k\nabla(w_k - w_q) + r\int_{\mathbb{R}^N} m|w_k|^{p-2}w_k(w_k - w_q) \\ &= \int_{\mathbb{R}^N} [mh(u_k) + m_1h_1(v_k)](w_k - w_q) + \int_{\mathbb{R}^N} f(w_k - w_q)\end{aligned}$$

and consequently

$$\begin{aligned}
&\int_{\mathbb{R}^N}(|\nabla w_k|^{p-2}\nabla w_k-|\nabla w_q|^{p-2}\nabla w_q)\nabla(w_k-w_q)\\
&\le \int_{\mathbb{R}^N}(|\nabla w_k|^{p-2}\nabla w_k-|\nabla w_q|^{p-2}\nabla w_q)\nabla(w_k-w_q)\\
&\quad + r\int_{\mathbb{R}^N} m(|w_k|^{p-2}w_k-|w_q|^{p-2}w_q)(w_k-w_q)\\
&= \int_{\mathbb{R}^N} m[h(u_k)-h(u_q)](w_k-w_q)+\int_{\mathbb{R}^N} m_1[h_1(v_k)-h_1(v_q)](w_k-w_q)\\
&\le \big[\|u_k\|^{p-1}_{L^p(m,\mathbb{R}^N)}+\|u_q\|^{p-1}_{L^p(m,\mathbb{R}^N)}+\|v_k\|^{\beta+1}_{L^q(n,\mathbb{R}^N)}\\
&\quad +\|v_q\|^{\beta+1}_{L^q(n,\mathbb{R}^N)}\big]\|w_k-w_q\|_{L^p(m,\mathbb{R}^N)}.
\end{aligned}$$

We then deduce that

$$\int_{\mathbb{R}^N}(|\nabla w_k|^{p-2}\nabla w_k-|\nabla w_q|^{p-2}\nabla w_q)\nabla(w_k-w_q)\to 0.$$

From (4.6), we conclude that w_k converges to w in $D^{1,p}(\mathbb{R}^N)$. Similarly, we prove that z_k converges to z in $D^{1,q}(\mathbb{R}^N)$.

Since the set K is convex, bounded and closed in $D^{1,q}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$, applying Schauder's fixed point theorem, then there exists a fixed point for T which gives the existence of solution of system (4.2). $\square$

Proof of Theorem 4.2. The proof will be given by three steps.

Step 1. We show that (u_ϵ,v_ϵ) is bounded in $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$. Indeed denoting by $t_\epsilon=\max(\|u_\epsilon\|^p_{D^{1,p}(\mathbb{R}^N)},\|v_\epsilon\|^q_{D^{1,q}(\mathbb{R}^N)})$, $z_\epsilon=t_\epsilon^{-1/p}u_\epsilon$ and $w_\epsilon=t_\epsilon^{-1/q}v_\epsilon$. Since (u_ϵ,v_ϵ) is solution of (4.2), we have

$$\begin{aligned}
-\Delta_p z_\epsilon+rm|z_\epsilon|^{p-2}z_\epsilon&=\frac{(a+r)m|z_\epsilon|^{p-2}z_\epsilon}{1+t_\epsilon^{1/p'}|\epsilon^{1/p}z_\epsilon|^{p-1}}+\frac{bm_1|w_\epsilon|^{\beta}w_\epsilon}{1+t_\epsilon^{1/p'}|\epsilon^{1/q}w_\epsilon|^{\beta+1}}+t_\epsilon^{-1/p'}f\,,\\
-\Delta_q w_\epsilon+rm|w_\epsilon|^{q-2}w_\epsilon&=\frac{cn_1|z_\epsilon|^{\alpha}z_\epsilon}{1+t_\epsilon^{1/q'}|\epsilon^{1/p}z_\epsilon|^{\alpha+1}}+\frac{(d+r)n|w_\epsilon|^{q-2}w_\epsilon}{1+t_\epsilon^{1/q'}|\epsilon^{1/q}w_\epsilon|^{q-1}}+t_\epsilon^{-1/q'}g.
\end{aligned}$$

Hence taking z_ϵ as test function in the first equation, we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\nabla z_\epsilon|^p+r\int_{\mathbb{R}^N} m|z_\epsilon|^p\\
&\le (a+r)\int_{\mathbb{R}^N} m|z_\epsilon|^p+b\int_{\mathbb{R}^N} m_1|w_\epsilon|^{\beta+1}|z_\epsilon|+t_\epsilon^{-1/p'}\int_{\mathbb{R}^N}|f||z_\epsilon|,
\end{aligned}$$

which implies, by Hölder inequality and (H2),

$$\begin{aligned}
\int_{\mathbb{R}^N}|\nabla z_\epsilon|^p&\le a\int_{\mathbb{R}^N} m|z_\epsilon|^p+b\Big(\int_{\mathbb{R}^N} m|z_\epsilon|^p\Big)^{1/p}\Big(\int_{\mathbb{R}^N} n|w_\epsilon|^q\Big)^{(\beta+1)/q}\\
&\quad +t_\epsilon^{-1/p'}\|f\|_{L^{(p^*)'}(\mathbb{R}^n)}\Big(\int_{\mathbb{R}^N}|z_\epsilon|^{p^*}\Big)^{1/p^*}
\end{aligned}$$

By the imbedding of $D^{1,p}(\mathbb{R}^N)$ in $L^{p^*}(\mathbb{R}^N)$, we deduce that

$$\begin{aligned}\|z_\epsilon\|^p_{D^{1,p}} \le \frac{a}{\lambda_1(m,p)}\|z_\epsilon\|^p_{D^{1,p}} + b\frac{\|z_\epsilon\|_{D^{1,p}}}{[\lambda_1(m,p)]^{1/p}}\frac{\|w_\epsilon\|^{\beta+1}_{D^{1,q}}}{[\lambda_1(n,q)]^{(\beta+1)/q}} \\ + c_1 t_\epsilon^{-1/p'}\|f\|_{L^{(p^*)'}}\|z_\epsilon\|_{D^{1,p}},\end{aligned}$$

where $c_1 = c_1(p,N)$ is the constant of the imbedding of $D^{1,p}(\mathbb{R}^N)$ into $L^{p^*}(\mathbb{R}^N)$, and consequently

$$\begin{aligned}\Big(\frac{\|z_\epsilon\|_{D^{1,p}}}{[\lambda_1(m,p)]^{1/p}}\Big)^{p-1} &\le b\Big(\frac{\|w_\epsilon\|_{D^{1,q}}}{[\lambda_1(n,q)]^{1/q}}\Big)^{\beta+1} \\ &\quad + c_1 t_\epsilon^{-1/p'}[\lambda_1(m,p)]^{1/p}\|f\|_{L^{(p^*)'}}.\end{aligned} \tag{4.8}$$

Similarly, we obtain

$$\begin{aligned}\Big(\frac{\|w_\epsilon\|_{D^{1,q}}}{[\lambda_1(n,q)]^{1/q}}\Big)^{q-1} &\le c\Big(\frac{\|z_\epsilon\|_{D^{1,p}}}{[\lambda_1(m,p)]^{1/p}}\Big)^{\alpha+1} \\ &\quad + c_2 t_\epsilon^{-1/q'}[\lambda_1(n,q)]^{1/q}\|g\|_{L^{(q^*)'}}\end{aligned} \tag{4.9}$$

Now assume that u_ϵ (or v_ϵ)is unbounded in $D^{1,p}(\mathbb{R}^N)$ (in $D^{1,q}(\mathbb{R}^N)$). Then $t_\epsilon \to +\infty$ and it follows from (4.8) and (4.9), that

$$\begin{aligned}&[\lambda_1(m,p)-a]^{\frac{\alpha+1}{p}}[\lambda_1(n,q)-d]^{\frac{\beta+1}{q}}\Big(\frac{\|z_\epsilon\|_{D^{1,p}}}{[\lambda_1(m,p)]^{1/p}}\Big)^{\frac{(\alpha+1)(\beta+1)}{q}} \\ &\times\Big(\frac{\|w_\epsilon\|_{D^{1,q}}}{[\lambda_1(n,q)]^{1/q}}\Big)^{\frac{(\alpha+1)(\beta+1)}{p}} \\ &\le b^{\frac{\alpha+1}{p}}c^{\frac{\beta+1}{q}}\Big(\frac{\|z_\epsilon\|_{D^{1,p}}}{[\lambda_1(m,p)]^{1/p}}\Big)^{\frac{(\alpha+1)(\beta+1)}{q}}\Big(\frac{\|w_\epsilon\|_{D^{1,q}}}{[\lambda_1(n,q)]^{1/q}}\Big)^{\frac{(\alpha+1)(\beta+1)}{p}},\end{aligned}$$

which implies

$$\begin{aligned}&[(\lambda_1(m,p)-a)^{\frac{\alpha+1}{p}}(\lambda_1(n,q)-d)^{\frac{\beta+1}{q}} - b^{\frac{\alpha+1}{p}}c^{\frac{\beta+1}{q}}] \\ &\times\Big(\frac{\|z_\epsilon\|_{D^{1,p}}}{\lambda_1(m,p)^{1/p}}\frac{\|w_\epsilon\|_{D^{1,q}}}{\lambda_1(n,q)^{1/q}}\Big)^{\frac{(\alpha+1)(\beta+1)}{p}} \le 0.\end{aligned}$$

But this is a contradiction since conditions (C1), (C2) and (C3) hold.

Step 2. Using the same arguments as in [19], we easily prove that $(\epsilon^{\frac{1}{p}}u_\epsilon, \epsilon^{\frac{1}{q}}v_\epsilon) \to (0,0)$ in $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$.

Step 3. Now we prove that (u_ϵ, v_ϵ) converges strongly in $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ as $\epsilon\to 0$. Indeed from Step 1 and Step 2, we have (u_ϵ,v_ϵ) is bounded in $D^{1,p}(\mathbb{R}^N)\times D^{1,q}(\mathbb{R}^N)$ and $\epsilon^{\frac{1}{p}}u_\epsilon \to 0$ a.e in $\mathbb{R}^n$. So up to a subsequence $(u_\epsilon,v_\epsilon)\to(u_0,v_0)$ in $L^p(m,\mathbb{R}^N)\times L^q(n,\mathbb{R}^N)$ and consequently

$$\begin{gathered}\Big|\frac{|u_\epsilon|^{p-2}u_\epsilon}{1+|\epsilon^{\frac{1}{p}}u_\epsilon|^{p-1}}\Big| \le |u_\epsilon|^{p-1} \le l_1^{p-1} \quad \text{in } L^{p'}(m,\mathbb{R}^N), \\ \frac{|u_\epsilon(x)|^{p-2}u_\epsilon(x)}{1+|\epsilon^{\frac{1}{p}}u_\epsilon(x)|^{p-1}} \to |u_0(x)|^{p-2}u_0(x) \quad \text{a.e. in } \mathbb{R}^N.\end{gathered}$$

From the dominated convergence theorem, we have $h(u_\epsilon)\to h(u_0)$ in $L^{p'}(m,\mathbb{R}^N)$ as $\epsilon\to 0$. Similarly, we get $h_1(v_\epsilon)\to h_1(v_0)$ in $L^{q'}(n,\mathbb{R}^N)$, $k_1(u_\epsilon)\to k_1(u_0)$ in $L^{p'}(m,\mathbb{R}^N)$ and $k(v_\epsilon)\to k(v_0)$ in $L^{q'}(n,\mathbb{R}^N)$. We finally use (4.6) to deduce that

$(u_\epsilon, v_\epsilon) \to (u_0, v_0)$ in $D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$ as $\epsilon \to 0$. Therefore, passing to the limit in (4.2), we obtain

$$\begin{gathered}
-\Delta_p u_0 = am|u_0|^{p-2}u_0 + bm_1|v_0|^\beta v_0 + f \quad \text{in } \mathbb{R}^N, \\
-\Delta_q v_0 = cn_1|u_0|^\alpha u_0 + dn|v_0|^{q-2}v_0 + g \quad \text{in } \mathbb{R}^N
\end{gathered}$$

which implies that (u_0, v_0) is solution of (1.1). □

We remark that when $\alpha = \beta = 0$ and $p = q = 2$, we obtain the results presented in [12, 13].

5. Related results

The tools used to establish the above results can be easily adapted for the problem

$$\begin{gathered}
-\Delta_p u = am(x)|u|^{p-2}u + bm_1(x)|u|^\alpha|v|^\beta v + f \quad \text{in } \mathbb{R}^n \\
-\Delta_q v = cn_1(x)|u|^\alpha u|v|^\beta + dn(x)|v|^{q-2}v + g \quad \text{in } \mathbb{R}^n \\
u(x) \to 0, \quad v(x) \to 0 \quad \text{as } |x| \to +\infty
\end{gathered} \tag{5.1}$$

where we assume that the conditions (H1),(H2'),(H3) and (H4') hold, with

(H2') $0 < m_1(x) \le m(x)^{\frac{\alpha+1}{p}} n(x)^{\frac{\beta+1}{q}}$ and $0 < n_1(x) \le m(x)^{\frac{\alpha+1}{p}} n(x)^{\frac{\beta+1}{q}}$ a.e. in $\mathbb{R}^N$

(H4') $b, c \ge 0$; $\alpha, \beta \ge 0$; $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$.

Under these assumptions, one has the following results.

Theorem 5.1. *Assume that hypothesis* (H1), (H2'), (H3), (H4') *are satisfied. Then the (MP) holds for* (5.1) *if*

(C1') $\lambda_1(m, p) > a$;

(C2') $\lambda_1(n, q) > d$;

(C3') $[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} > b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}$;

Conversely, if the (MP) holds, then (C1'), (C2') and (C4') are satisfied, where

(C4') $[\lambda_1(m, p) - a]^{\frac{\alpha+1}{p}} [\lambda_1(n, q) - d]^{\frac{\beta+1}{q}} > \Theta b^{\frac{\alpha+1}{p}} c^{\frac{\beta+1}{q}}$.

Theorem 5.2. *Assume that* (H1), (H2'), (H3), (C1'), (C2'), (C3') *hold. Furthermore assume that $m \in L^{(p^*)'}(\mathbb{R}^N)$ and $m \in L^{(q^*)'}(\mathbb{R}^N)$. Then for all $f \in L^{(p^*)'}(\mathbb{R}^N)$ and $g \in L^{(q^*)'}(\mathbb{R}^N)$, the system* (5.1) *has at least one solution $(u, v) \in D^{1,p}(\mathbb{R}^N) \times D^{1,q}(\mathbb{R}^N)$.*

The proofs of theorems 5.1 and 5.2 can be adapted from those of theorems 3.3 and 4.2 respectively.

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