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MULTIPLICITY OF SOLUTIONS FOR GRADIENT SYSTEMS

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Abstract. We establish the existence of nontrivial solutions for an elliptic system which is resonant both at the origin and at infinity. The resonance is given by an eigenvalue problem with indefinite weight, and the nonlinear term is permitted to be unbounded. Also, we consider the case where the resonance at infinity and at the origin can occur with different weights. Our main tool is the computation of critical groups.

1. Introduction

In this article, we present results on the existence and multiplicity of solutions for the system

$$\begin{gathered} -\Delta u = f(x,u,v) \quad \text{in } \Omega \\ -\Delta v = g(x,u,v) \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$ is bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $f, g \in C^1(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$. We assume that there is a function $F \in C^2(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ such that $\nabla F = (f, g)$. In this paper, ∇F denotes the gradient in the variables u and v. In this case, (1.1) has a variational structure. More precisely, we have a system of gradient type studied by many authors; see [1, 7, 11] and references therein.

The main goal of this paper is to find nontrivial solutions for (1.1) under resonance conditions at infinity and at the origin using Morse theory. More specifically, we assume resonance conditions at infinity and the origin using an eigenvalue problem with weights. Resonant problems have been the subject of a vast amount of research since the appearance of the pioneering paper by Landesman and Lazer [14]. For gradient system with weights see [1, 3, 5, 11], and for problems with a single equation where there is resonance at infinity and the origin see [2, 18, 26, 27].

From a variational stand point, finding weak solutions of (1.1) in $H = H_0^1(\Omega) \times H_0^1(\Omega)$ is equivalent to finding critical points of the C^2 functional

$$J(z) = \frac{1}{2}\int_\Omega |\nabla u|^2 + |\nabla v|^2 dx - \int_\Omega F(x,u,v)dx, z = (u,v) \quad \text{in } H. \tag{1.2}$$

Throughout this paper we assume that

$$\nabla F(x,0,0) = 0, \quad x \in \Omega.$$

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Then (1.1) admits the trivial solution $(u, v) = 0$. In this case, the key point is to ensure the existence of nontrivial solutions for (1.1). The existence of nontrivial solutions for (1.1) depends on the behavior of F near the origin and at infinity.

There is an extensive bibliography on the study of variational elliptic systems, both of the gradient type and the Hamiltonian, see [1, 3, 6, 11, 22, 29, 30] and references therein. In two recent articles [29, 30] the problem (1.1) has been studied under resonant conditions at infinity and at the origin. We complement the results in [30] by considering resonant conditions using an eigenvalue problem for some continuous functions. Furthermore, these functions are not necessary positive, see Section 2.

We also recall that elliptic problems for a single semilinear equation at resonance have been studied in the recent years. We refer the reader to [15, 16, 24, 25, 26] where several problems were studied under different conditions on the nonlinear term. More specifically, those works used the well known angle conditions at zero and infinity. introduced by Bartsch-Li [2]. In this paper we will find an extension for these angle conditions for our gradient systems (1.1).

We note that (1.1) represents a steady state case of reaction-diffusion systems of interest in biology, chemistry, physics and ecology. Mathematically, reaction-diffusion systems take the form of nonlinear parabolic partial differential equations which have been intensively studied during recent years; see [23, 21] where many references can be found.

On the other hand, resonant problems have a great interest due to the additional difficulty coming from the fact that the associated functional may not satisfy the classical Palais-Smale condition. In order to obtain nontrivial solutions of (1.1), overcoming this difficulty, we will impose some conditions in the behavior of F at infinity and at the origin.

Let us denote by $\mathcal{M}_2(\Omega)$ the set of all continuous, cooperative and symmetric functions $A \in C(\overline{\Omega}, M_{2\times 2}(\mathbb{R}))$. More precisely, if $A \in \mathcal{M}_2(\Omega)$ then it has the form

$$A(x) = \begin{pmatrix} a(x) & b(x) \\ b(x) & c(x) \end{pmatrix},$$

where the functions $a, b, c \in C(\overline{\Omega}, \mathbb{R})$ satisfy the hypotheses:

- (M1) A is cooperative; that is, $b(x) \geq 0$ for all $x \in \overline{\Omega}$.
- (M2) $\max_{x\in\Omega} \max\{a, c\} > 0$.

Here, $M_{2\times 2}(\mathbb{R})$ denotes the set of all real matrices of order 2. In this case, given $A \in \mathcal{M}_2(\Omega)$, we consider the weighted eigenvalue problem

$$\begin{gathered} -\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.3}$$

Using conditions (M1) and (M2) above, we apply the spectral theory for compact operators [9] and some results in [3]. We obtain a sequence of distinct eigenvalues

$$0 < \lambda_1(A) < \lambda_2(A) < \lambda_3(A) < \dots$$

such that $\lambda_k(A) \to +\infty$ as $k \to \infty$; see [3, 11] for more details. To state the behavior of F at infinity and at the origin we introduce the following hypotheses:

(MI) There exist $A_\infty \in \mathcal{M}_2(\Omega)$ and a function G_∞ such that

$$G_\infty(x,z) = F(x,z) - \frac{1}{2}\langle A_\infty(x)z, z\rangle \quad \forall\,(x,z) \in \overline{\Omega} \times \mathbb{R}^2; \tag{1.4}$$

(M0) There is $A_0 \in \mathcal{M}_2(\Omega)$ and a function G_0 such that

$$G_0(x,z) = F(x,z) - \frac{1}{2}\langle A_0(x)z, z\rangle, \quad \forall\,(x,z) \in \overline{\Omega} \times \mathbb{R}^2, \tag{1.5}$$

where ∇G_∞ and ∇G_0 satisfy the following growth conditions:

(BI) There exist $\alpha \in (0,1)$ such that

$$|\nabla G_\infty(x,z)| \le C(1+|z|^\alpha) \quad \text{for a.e. } x \in \Omega,\, \forall\, z \in \mathbb{R}^2;$$

(B0) There exist $\beta \in (1, 2^*-1)$ and $\delta > 0$ such that

$$|\nabla G_0(x,z)| \le C|z|^\beta \quad \text{for a.e.} x \in \Omega, \forall\, |z| < \delta.$$

Under these hypotheses, system (1.1) is called asymptotically quadratic both at infinity and at the origin. Moreover, when $\lambda_k(A_\infty) = \lambda_m(A_0) = 1$ with $k, m \ge 2$, problem (1.1) becomes resonant at infinity and at the origin. In addition, the resonance phenomena occurs at higher eigenvalues.

To avoid the resonance, we make the following assumptions on the behavior of ∇G_∞ and ∇G_0 near infinity and near the origin, respectively:

(CI) There exist $F_1, F_2 \in C(\overline{\Omega}, \mathbb{R})$ such that

$$F_1(x) \le \liminf_{|z|\to\infty} \frac{\nabla G_\infty(x,z)\cdot z}{|z|^{1+\alpha}} \le \limsup_{|z|\to\infty} \frac{\nabla G_\infty(x,z)\cdot z}{|z|^{1+\alpha}} \le F_2(x) \tag{1.6}$$

with $\int F_j \neq 0$ for j=1,2.

(C0) There exist $f_1, f_2 \in C(\overline{\Omega}, \mathbb{R})$ such that

$$f_1(x) \le \liminf_{|z|\to 0} \frac{\nabla G_0(x,z)\cdot z}{|z|^{1+\beta}} \le \limsup_{|z|\to 0} \frac{\nabla G_0(x,z)\cdot z}{|z|^{1+\beta}} \le f_2(x) \tag{1.7}$$

with $\int f_j \neq 0$ for j=1,2.

In what follows we assume $\lambda_k(A_\infty) = \lambda_m(A_0) = 1$ where $k, m \ge 1$. In this way, we shall prove the following results.

Theorem 1.1. *Assume* (MI), (M0), (BI), (B0), (CI), (C0). *In addition, suppose that either one of the following two conditions holds:*

(a) $F_2(x) \le 0$, $f_1(x) \ge 0$ *in* Ω *and* $m \neq k-1$.
(b) $F_1(x) \ge 0$, $f_2(x) \le 0$ *in* Ω *and* $k \neq m-1$.

Then (1.1) *has at least one nontrivial solution* $z_\star \neq 0$.

Theorem 1.2. *Assume* (MI), (M0), (BI), (B0), (CI), (C0). *In addition, suppose that either one of the following two conditions holds:*

(a) $F_1(x) \ge 0$, $f_1(x) \ge 0$ *in* Ω *and* $k \neq m$.
(b) $F_2(x) \le 0$, $f_2(x) \le 0$ *in* Ω *and* $k \neq m$.

Then (1.1) *has at least one nontrivial solution* $z_\star \neq 0$.

Remark 1.3. In Theorems 1.1 and 1.2 we have resonance both at infinity and at the origin given by the weights $A_\infty, A_0 \in \mathcal{M}_2(\Omega)$ respectively. Moreover, these functions can be different; i. e., we allow the resonance with two distinct weights. In addition when $A_\infty = A_0$, Theorem 1.1 is similar to first result in [30] with constant functions in $\mathcal{M}_2(\Omega)$. But, the conditions in Theorem 1.2-a and Theorem 1.2-b are new.

Again, under the hypotheses of Theorem 1.1 or Theorem 1.2, problem (1.1) has one nontrivial solution. Now, an interesting question is: Are there more nontrivial solutions?

For the next result, we will add further hypotheses on F and we find another nontrivial solution. Firstly, we have the following definition.

Definition 1.4. Let $A, B \in \mathcal{M}_2(\Omega)$. We define $A \le B$ if $\langle A(x)z, z\rangle \le \langle B(x)z, z\rangle$, for all $z \in \mathbb{R}^2$, and all $x \in \Omega$. Moreover, we define $A \preceq B$ if $A \le B$ and $B - A$ is positive definite on $\widetilde{\Omega} \subseteq \Omega$ where $|\widetilde{\Omega}| > 0$. Here $|\cdot|$ denotes the Lebesgue measure.

Remark 1.5. Let $F \in C^2$ and $A, B \in \mathcal{M}_2(\Omega)$. Then the inequalities $A \le F'' \le B$ mean $\langle A(x)z, z\rangle \le \langle F''(x)z, z\rangle \le \langle B(x)z, z\rangle$ for all $(x, z) \in \Omega \times \mathbb{R}^2$. Here F'' denotes the Hessian matrix of F in the variables u and v.

The second and third result of this paper can be stated as follows.

Theorem 1.6. *Assume* (MI), (M0), (BI), (B0), (CI), (C0). *In addition, suppose that either one of the following two cases holds:*

- (a) *$F_2(x) \le 0$ and $f_1(x) \ge 0$ in Ω, with $F'' \ge \beta \succeq \lambda_{k-1}A_\infty$ and $m > k - 1$,*
- (b) *$F_1(x) \ge 0$ and $f_2(x) \le 0$ in Ω, with $F'' \le \beta \preceq \lambda_{k+1}A_\infty$ and $k > m - 1$,*

where β is a function in $\mathcal{M}_2(\Omega)$. Then (1.1) *has at least two nontrivial solutions.*

Theorem 1.7. *Assume* (MI), (M0), (BI), (B0), (CI), (C0). *In addition, suppose that either one of the following two cases holds:*

- (a) *$F_1(x) \ge 0$ and $f_1(x) \ge 0$ in Ω, with $F'' \le \beta \prec \lambda_{k+1}A_\infty$ and $m < k$,*
- (b) *$F_2(x) \le 0$ and $f_2(x) \le 0$ in Ω, with $F'' \ge \beta \succeq \lambda_{k-1}A_\infty$ and $m > k$, where β is a function in $\mathcal{M}_2(\Omega)$.*

Then problem (1.1) *has at least two nontrivial solutions.*

Remark 1.8. Theorems 1.6 and 1.7 improve the second result in [30]. Again, we allow the resonance at infinity and the origin with distinct functions $A_\infty, A_0 \in \mathcal{M}_2(\Omega)$.

Ours main results are compared to those of [30] when $A_\infty, A_0 \in \mathcal{M}_2(\Omega)$ are the same and constant. However, in Theorem 1.1-a and Theorem 1.1-b we have a new result where the resonance in the same matrix with distinct or same eigenvalues was allowed.

On the other hand, in Theorem 1.6 and Theorem 1.7, we have new multiplicity results without restriction in the nullity at the origin. More specifically, these theorems given us multiplicity of solutions for (1.1) controlling the second derivative of F. Hence, our approach permits to extend of [30] for elliptic systems using the eigenvalue problem (1.3) for any functions in $\mathcal{M}_2(\Omega)$.

We point out that the main idea for finding the second nontrivial solution in Theorems 1.6 and 1.7 was first used in Li-Willem [17] on elliptic problems for a single equation.

In the proof of ours results, we study (1.1) using some results related to the critical groups at an isolated critical point, see [2, 4, 18, 20]. So, we compute the critical groups at infinity and the origin.

This paper is organized as follows: In Section 2, we recall the abstract framework of Problem (1.1) and highlight the properties for the eigenvalue problem (1.3). In Section 3 we determine the critical groups at infinity and the origin. Section 4 is devoted to the proofs of Theorems 1.6 and 1.7.

2. Abstract framework and eigenvalue problem for (1.1)

Firstly, we denote by $H = H_0^1(\Omega) \times H_0^1(\Omega)$ the Hilbert space endowed with the norm

$$\|z\|^2 = \int_\Omega |\nabla u|^2 + |\nabla v|^2 dx, \quad z = (u, v) \in H.$$

We denote by $\langle \cdot, \cdot \rangle$ the scalar product in H.

Now, we recall the properties of the eigenvalue problem

$$\begin{gathered} -\Delta \begin{pmatrix} u \\ v \end{pmatrix} = \lambda A(x) \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{in } \Omega \\ u = v = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.1}$$

Let $A \in \mathcal{M}_2(\Omega)$, then there is a compact self-adjoint linear operator $T_A : H \to H$ such that

$$\langle T_A z, w \rangle = \int_\Omega \langle A(x) z, w \rangle dx, \quad \forall z, w \in H.$$

This operator has the propriety that λ is eigenvalue of (2.1) if and only if $T_A z = \frac{1}{\lambda} z$, for some $z \in H$. Thus, for each $A \in \mathcal{M}_2(\Omega)$ there exist a sequence of eigenvalues for system (2.1) and a Hilbertian basis for H formed by eigenfunctions of (2.1). Moreover, denoting by $\lambda_k(A)$ the eigenvalues of (2.1) and $\Phi_k(A)$ the associated eigenfunctions, then $0 < \lambda_1(A) < \lambda_2(A) \leq \dots \lambda_k(A) \to \infty$ as $k \to \infty$, and we have

$$\frac{1}{\lambda_k(A)} = \sup\{\langle T_A z, z \rangle, \|z\| = 1, \quad z \in V_{k-1}^\perp\},$$

where $V_{k-1}^\perp = \operatorname{span}\{\Phi_1(A), \dots, \Phi_{k-1}(A)\}$. Thus, we get $H = V_k \oplus V_k^\perp$ for $k \geq 1$, and the following variational inequalities hold

$$\|z\|^2 \leq \lambda_k(A) \langle T_A z, z \rangle, \quad \forall z \in V_k, k \geq 2, \tag{2.2}$$

$$\|z\|^2 \geq \lambda_{k+1}(A) \langle T_A z, z \rangle, \quad \forall z \in V_k^\perp, k \geq 1. \tag{2.3}$$

The variational inequalities will be used in the next section, for more properties to the problem (2.1) see [3, 5, 9, 11].

3. The computations of critical groups

In this section we present some lemmas for the computations of critical groups at infinity and at the origin. As stated in the Introduction, we will look for the critical points of the C^2 functional $J : H \to \mathbb{R}$ given by equation (1.2).

We divide this section into two parts. The first part is devoted to find the critical groups at the origin. To do that, we will use a result proved in [30]. Namely, we will consider the following lemma.

Lemma 3.1 ([30]). *Suppose* (M0), (B0), (C0) *hold. Let* $H = V_0 \oplus W_0$ *where* $V_0 = \ker(I - T_{A_0}), W_0 = V_0^\perp$. *Let* $(z_n)_{n \in \mathbf{N}} \in H, z_n = z_n^0 + w_n$, $z_n^0 \in V_0, w_n \in W_0$ *such that* $\|z_n\| \to 0$, $\frac{w_n}{\|z_n\|} \to 0$ *as* $n \to \infty$. *Then we have the following two alternatives:*

(a) *If* $f_1(x) \geq 0$ *a.e.* $x \in \Omega$ *then*

$$\liminf_{n\to\infty} \int_\Omega \frac{\nabla G_0(x,z_n)\cdot z_n}{\|z_n\|^{1+\beta}} > 0$$

(b) *If* $f_2(x) \leq 0$ *a.e.* $x \in \Omega$ *then*

$$\limsup_{n\to\infty} \int_\Omega \frac{\nabla G_0(x,z_n)\cdot z_n}{\|z_n\|^{1+\beta}} < 0.$$

Using the previous result we have the following characterization for critical groups at the origin.

Lemma 3.2. *Suppose* (M0), (B0), (C0) *hold. Then we have*

(a) *If* $f_1(x) \geq 0$, *then* $C_q(J,0) = \delta_{q,\mu_0+\nu_0}\mathcal{G}$, $q \in \mathbb{N}$.

(b) *If* $f_2(x) \leq 0$, *then* $C_q(J,0) = \delta_{q,\mu_0}\mathcal{G}$, $q \in \mathbb{N}$.

Here, $\mathcal{G}$ *is an Abelian group and* μ_0 *and* ν_0 *denote the index of Morse and the nullity at the origin, respectively.*

Proof. Case (a). We will divide the proof of this case into two steps.

Step 1. We claim that there are $\rho > 0$ and $\epsilon \in (0,1)$ such that

$$\langle J'(z), z^0 + z^-\rangle \leq 0, \quad \forall\, z \in C(\rho,\epsilon), \tag{3.1}$$

where

$$C(\rho,\epsilon) = \big\{z = z^0 + z^+ + z^- \in H = V_0 \oplus W; \|z\| \leq \rho \text{ and } \|z^+ + z^-\| \leq \epsilon\|z\|\big\}$$

with $W = W_0^+ \oplus W_0^-$. More precisely, we chose the sets

$$V_0 = \ker(I - T_{A_0}), \quad W_0^+ = \oplus_{j=m+1}^{\infty} \ker(I\lambda_j^{-1}(A_0) - T_{A_0}),$$
$$W_0^- = \oplus_{j=1}^{m-1} \ker(I\lambda_j^{-1}(A_0) - T_{A_0}).$$

In this way, if statement (3.1) is false, we have for each $\rho = \epsilon = \frac{1}{n}$ a point $z_n \in H; z_n = z_n^0 + z_n^+ + z_n^-$ satisfying the inequalities

$$\|z_n\| \leq \frac{1}{n}, \quad \|z_n^+ + z_n^-\| \leq \frac{1}{n}\|z_n\|, \quad \langle J'(z_n), z_n^0 + z_n^-\rangle > 0.$$

Therefore, $\|z_n\| \to 0$, $\frac{z_n^\pm}{\|z_n\|} \to 0$ as $n \to \infty$.

On the other hand, there is a linear operator $T_{A_0} : H \to H$ which is self-adjoint and compact. Thus, using the variational inequality (2.2) for T_{A_0}, we obtain

$$\begin{aligned}
0 &< \langle J'(z_n), z_n^0 + z_n^-\rangle \\
&= \langle (I - T_{A_0})z_n, z_n^0 + z_n^-\rangle - \int_\Omega \nabla G_0(x,z_n)(z_n^0 + z_n^-)dx \\
&= \langle (I - T_{A_0})z_n^-, z_n^-\rangle + \langle (I - T_{A_0})z_n^0, z_n^0\rangle - \int_\Omega \nabla G_0(x,z_n)(z_n^0 + z_n^-)dx \\
&= \langle (I - T_{A_0})z_n^-, z_n^-\rangle - \int_\Omega \nabla G_0(x,z_n)(z_n^0 + z_n^-)dx \\
&\leq -\int_\Omega \nabla G_0(x,z_n)(z_n^0 + z_n^-)dx.
\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \int_\Omega \frac{\nabla G_0(x,z_n)(z_n^0 + z_n^-)}{\|z_n\|^{1+\beta}} dx \leq 0. \tag{3.2}$$

Now, using the Hölder's inequality and Sobolev Embedding Theorem we have

$$\Big|\int_\Omega \frac{\nabla G_0(x,z_n)z_n^+}{\|z_n\|^{1+\beta}}dx\Big| \le C\int_\Omega \frac{|z_n|^\beta|z_n^+|}{\|z_n\|^{1+\beta}}dx = C\frac{\|z_n\|^\beta\|z_n^+\|}{\|z_n\|^{1+\beta}} = C\frac{\|z_n^+\|}{\|z_n\|} \to 0,$$

as $n\to\infty$. Therefore,

$$\limsup_{n\to\infty}\int_\Omega \frac{\nabla G_0(x,z_n)\cdot z_n}{\|z_n\|^{1+\beta}}dx = \limsup_{n\to\infty}\int_\Omega \frac{\nabla G_0(x,z_n)(z_n^0+z_n^-)}{\|z_n\|^{1+\beta}}dx \le 0. \tag{3.3}$$

By Lemma 3.1, we have

$$\liminf_{n\to\infty}\int_\Omega \frac{\nabla G_0(x,z_n)\cdot z_n}{\|z_n\|^{1+\beta}}dx > 0.$$

which contradicts the preceding estimate (3.3). Therefore, there are $\delta>0, \epsilon\in(0,1)$ satisfying (3.1). So we finish the proof of this claim.

Step 2. Let $t\in[0,1]$. We consider the following homotopy $J_t : H\to\mathbb{R}$ given by

$$J_t(z) = J(z) - \frac{1}{2}t\|z_0\|^2, \quad z\in H$$

where $z = z^0+z^++z^-\in H = V_0\oplus W_0^-\oplus W_0^+$. Clearly, J_1 possesses $z=0$ as a nondegenerate critical point with the Morse index $\mu_0+\nu_0$.

We claim that there exists a $\rho>0$ small enough such that

$$J_t'(z)\ne 0, \quad \forall\, z\in B_\rho\backslash\{0\},\ t\in[0,1],$$

where B_ρ is the open ball in H centered at the origin with radius ρ. Using this fact, by the characterization of critical groups of a nondegenerate critical point see [4], the homotopy J_t is admissible. So, we obtain

$$C_q(J,0) = C_q(J_0,0) = C_q(J_1,0) = \delta_{q,\mu_0+\nu_0}\mathcal{G}, \quad \forall\, q\in\mathbb{N}.$$

Now we prove the claim just above. By Step 1 for each $z\in C(\rho,\epsilon)\backslash\{0\}$ we obtain $z^0\ne 0$,$z^0+z^-\ne 0$ and

$$\langle J_t'(z), z^0+z^-\rangle = \langle J'(z), z^0+z^-\rangle - t\langle z^0,z^0\rangle \le -t\|z^0\|^2 < 0.$$

On the other hand, if $z\in B_\rho\backslash C(\rho,\epsilon)$ with ρ small, we have the following inequalities

$$\begin{aligned}
&\langle J_t'(z), z^+-z^-\rangle\\
&= \langle (I-T_{A_0})z, z^+\rangle - \langle (I-T_{A_0})z, z^-\rangle - \int_\Omega \nabla G_0(x,z)(z^+-z^-)dx\\
&\ge \|z^+\|^2 - \langle T_{A_0}z^+, z^+\rangle - \big(\|z^-\|^2 - \langle T_{A_0}z^-, z^-\rangle\big)\\
&\quad - \int_\Omega \nabla G_0(x,z)(z^+-z^-)dx\\
&\ge \Big(1-\frac{1}{\lambda_{m+1}(A_0)}\Big)\|z^+\|^2 - \Big(1-\frac{1}{\lambda_{m-1}(A_0)}\Big)\|z^-\|^2 - \int_\Omega \nabla G_0(x,z)(z^+-z^-)dx\\
&\ge \delta\|z^++z^-\|^2 - \int_\Omega \nabla G_0(x,z)(z^+-z^-)dx\\
&\ge \|z^++z^-\|^2\Big[\delta - \frac{1}{\|z\|^2}\int_\Omega \nabla G_0(x,z)(z^+-z^-)dx\Big]\\
&\ge \|z^++z^-\|^2\Big[\delta - \frac{1}{\|z\|^2}\int_\Omega C|z|^\beta|z^+-z^-|dx\Big]
\end{aligned}$$

$$\geq \|z^+ + z^-\|^2 \big[\delta - \frac{C}{\|z\|^2}\|z\|^\beta \|z^+ - z^-\|\big] > 0$$

for all $\|z\| \leq \rho$ and uniformly on $t \in [0,1]$. Where we used the hypothesis (B0) and the variational inequalities (2.2) and (2.3). Clearly, we take $\delta > 0$ such that

$$\delta \leq \min\big\{1 - \frac{1}{\lambda_{m+1}(A_0)}, -1 + \frac{1}{\lambda_{m-1}(A_0)}\big\}.$$

Therefore, there exists a neighborhood $B_\rho \subset H = H_0^1(\Omega)^2$ of 0 such that

$$J_t'(z) \neq 0, \quad \forall z \in B_\rho \text{ with } t \in [0,1].$$

So the claim above follows and the proof of this lemma for the case (a) is now complete.

Case (b) In this case we consider the homotopy

$$J_t(z) = J(z) + \frac{1}{2}t\|z^0\|^2, \quad z \in H = V_0 \oplus W_0^+ \oplus W_0^-,\ z \in H,\ t \in [0,1].$$

Again, the homotopy J_t is admissible; i.e, there exists an open ball $B_\rho \subset H = H_0^1(\Omega)^2$ such that

$$J_t'(z) \neq 0, \quad \forall z \in B_\rho \text{ for each } t \in [0,1].$$

Actually, is sufficient to prove that there exist $\rho > 0$ and $\epsilon \in (0,1)$ small such that

$$\langle J_t'(z), z^0 - z^- \rangle \geq 0, \quad \forall\, z \in C(\rho,\epsilon) \text{ for each } t \in [0,1].$$

The proof of this inequality follows the same ideas discussed in case (a). So, we will omit it. $\square$

In the second part we will show that the functional J satisfies the Cerami condition at any level $c \in \mathbb{R}$. So, using a result given by [2], we compute the critical groups at infinity. In order to do that, we have the following lemmas.

Lemma 3.3 ([30]). *Assume* (MI), (BI), (CI). *Let $H = V_\infty \oplus W_\infty$, where $V_\infty = \ker(I - T_{A_\infty})$, $W_\infty = V_\infty^\perp$. Suppose also that there is a sequence $z_n = z_n^0 + w_n \in H$ with $z_n^0 \in V_\infty, w_n \in W_\infty$ where $\|z_n\| \to \infty$ and $\frac{w_n}{\|z_n\|} \to 0$ as $n \to \infty$. So we have the following alternatives:*

(a) *If $F_1(x) \geq 0$ a.e. $x \in \Omega$ then $\liminf_{n\to\infty} \int_\Omega \frac{\nabla G_\infty(x,z_n)\cdot z_n}{\|z_n\|^{1+\alpha}} > 0$,*
(b) *If $F_2(x) \leq 0$ a.e. $x \in \Omega$ then $\limsup_{n\to\infty} \int_\Omega \frac{\nabla G_\infty(x,z_n)\cdot z_n}{\|z_n\|^{1+\alpha}} < 0$.*

Lemma 3.4. *Assume* (MI), (BI), (CI). *Let $R > 0$, $\epsilon \in (0,1)$ and consider the set*

$$C(R,\epsilon) = \big\{z = z^0 + z^- + z^+ \in H = V_\infty \oplus W_\infty^- \oplus W_\infty^+, \|z\| \geq R, \|z^+ + z^-\| \leq \epsilon\|z\|\big\}.$$

Then we have the following alternatives:

(a) *$F_1(x) \geq 0$ implies that there exist $R > 0$, $\epsilon \in (0,1)$, and $\delta > 0$ such that*

$$\langle J'(z), z^0 \rangle \leq -\delta; \quad \forall\, z \in C(R,\epsilon).$$

(b) *$F_2(x) \leq 0$ implies that there exist $R > 0$, $\epsilon \in (0,1)$, and $\delta > 0$ such that*

$$\langle J'(z), z^0 \rangle \geq \delta; \quad \forall\, z \in C(R,\epsilon).$$

Proof. Case (a). Let us assume, by contradiction, that for $\epsilon = \delta = \frac{1}{n}$, there is a sequence $z_n = z_n^0 + z_n^- + z_n^+ \in H = V_\infty \oplus W_\infty^- \oplus W_\infty^+$ satisfying the inequalities

$$\|z_n\| \geq n, \quad \|z_n^- + z_n^+\| \leq \frac{1}{n}\|z_n\|, \quad \langle J(z_n), z_n^0\rangle > -\frac{1}{n}.$$

Here we put $V_\infty = \ker(I - T_{A_\infty})$, $W_\infty = W_\infty^- \oplus W_\infty^+$ where

$$W_\infty^+ = \oplus_{j=k+1}^\infty \ker(I\lambda_j^{-1}(A_\infty) - T_{A_\infty}), W_\infty^- = \oplus_{j=1}^{k-1} \ker(I\lambda_j^{-1}(A_\infty) - T_{A_\infty}).$$

Therefore, we have the following estimates

$$\begin{aligned}\frac{-1}{n} &< \langle J'(z_n), z_n^0\rangle = \langle (I - T_{A_\infty})z_n, z_n^0\rangle - \int_\Omega \nabla G_\infty(x, z_n) z_n^0 dx \\ &\leq -\int_\Omega \nabla G_\infty(x, z_n) z_n^0 dx.\end{aligned}$$

This implies

$$\limsup_{n\to\infty} \int_\Omega \frac{\nabla G_\infty(x, z_n) z_n^0}{\|z_n\|^{1+\alpha}} dx \leq 0. \tag{3.4}$$

On the other hand, using the Holder's inequality and Sobolev embedding, we obtain

$$\begin{aligned}\Big|\int_\Omega \frac{\nabla G_\infty(x, z_n)(z_n^+ + z_n^-)}{\|z_n\|^{1+\alpha}} dx\Big| &\leq \int_\Omega \Big|\frac{\nabla G_\infty(x, z_n)(z_n^+ + z_n^-)}{\|z_n\|^{1+\alpha}}\Big| dx \\ &\leq \int_\Omega C \frac{(1 + |z_n|^\alpha)|z_n^+ + z_n^-|}{\|z_n\|^{1+\alpha}} dx \\ &\leq C \frac{\|z_n\|^\alpha \|z_n^+ + z_n^-\|}{\|z_n\|^{1+\alpha}}| + C \frac{\|z_n^+ + z_n^-\|}{\|z_n\|^{1+\alpha}} \to 0\end{aligned}$$

as $n \to \infty$. Therefore,

$$\liminf_{n\to\infty} \int_\Omega \frac{\nabla G_\infty(x, z_n) z_n^0}{\|z_n\|^{1+\alpha}} dx = \liminf_{n\to\infty} \int_\Omega \frac{\nabla G_\infty(x, z_n) z_n}{\|z_n\|^{1+\alpha}} dx \leq 0.$$

This is a contradiction with the estimate (a) of Lemma 3.3. Thus, there are $R > 0$ large enough and $\epsilon \in (0, 1)$ such that $\langle J'(z), z_0\rangle \leq -\delta$ for all $z \in C(R, \epsilon)$ for some $\delta > 0$. So we completed the proof of case (a). The proof of case (b) is similar to the previous case, therefore we omit it. □

Now we prove the compactness condition required for the proof of Theorem 1.1. First, we recall that $J : H \to \mathbb{R}$ is said to satisfy Palais-Smale condition at the level $c \in \mathbb{R}$ ($(\mathrm{PS})_c$ in short), if any sequence $(z_n) \subseteq H$ such that

$$J(z_n) \to c \quad \text{and} \quad J'(z_n) \to 0$$

as $n \to \infty$, possesses a convergent subsequence in H.

Moreover, we say that $J : H \to \mathbb{R}$ satisfies the Cerami condition at the level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ in short), if any sequence $(z_n) \subseteq H$ such that

$$J(z_n) \to c \quad \text{and} \quad (1 + \|z_n\|)\|J'(z_n)\| \to 0$$

as $n \to \infty$, possesses a convergent subsequence in H.

Lemma 3.5. *Assume* (MI), (BI), (CI). *If $F_2(x) \leq 0$ or $F_1(x) \geq 0$ for a.e. $x \in \Omega$ then the functional $J : H \to \mathbb{R}$ satisfies the compactness condition $(Ce)_c$ for all $c \in \mathbb{R}$.*

Proof. First, we suppose $F_1(x) \geq 0$ a.e. $x \in \Omega$, we shall show that all Cerami sequences $(z_n)_{n\in\mathbb{N}} \in H$ are bounded. Assume, by contraction, that $(z_n)_{n\in\mathbb{N}}$ in H is unbounded. Therefore, up to a subsequence, we have $\|z_n\| \to \infty$. Letting $z_n = z_n^+ + z_n^- + z_n^0$, $z_n^+ \in V_\infty^+$, $z_n^- \in V_\infty^-$, and $z_n^0 \in V_\infty^0$. Let $T_{A_\infty} : H \to H$ be a linear operator given by eigenvalue problem (2.1), see Section 2. Then we have the following estimates

$$
\begin{aligned}
&\langle J'(z_n), z_n^+ - z_n^- \rangle \\
&= \langle (I - T_{A_\infty}) z_n, z_n^+ - z_n^- \rangle - \int_\Omega \nabla G_\infty(x, z_n)(z_n^+ - z_n^-) dx \\
&= \langle (I - T_{A_\infty}) z_n^+, z_n^+ \rangle - \langle (I - T_{A_\infty}) z_n^-, z_n^- \rangle - \int_\Omega \nabla G_\infty(x, z_n)(z_n^+ - z_n^-) dx \\
&\geq \delta \|z_n^+ - z_n^-\|^2 - C \int_\Omega (1 + |z_n|^\alpha)|z_n^+ - z_n^-| dx \\
&\geq \delta \|z_n^+ - z_n^-\|^2 - C\|z_n^+ - z_n^-\| - C\|z_n\|^\alpha \|z_n^+ - z_n^-\| \\
&\geq \delta \|z_n^+ - z_n^-\|^2 - C\|z_n^+ - z_n^-\| - C\epsilon^2 \|z_n^+ - z_n^-\|^2 - \frac{C}{\epsilon^2}\|z_n\|^{2\alpha} \\
&\geq \delta_\epsilon \|z_n^+ - z_n^-\|^2 - C\|z_n^+ - z_n^-\| - \frac{C}{\epsilon^2}\|z_n\|^{2\alpha}.
\end{aligned}
$$

where we used Hölder's inequality, Sobolev embedding and Young's inequality with $\epsilon > 0$. For small $\epsilon > 0$, we have $\delta_\epsilon > 0$ which shows that

$$
\begin{aligned}
\frac{\delta_\epsilon \|z_n^+ - z_n^-\|^2}{\|z_n\|^2} &\leq \langle J'(z_n), \frac{z_n^+ - z_n^-}{\|z_n\|^2} \rangle + C\frac{\|z_n^+ - z_n^-\|}{\|z_n\|^2} + C_\epsilon \frac{\|z_n\|^{2\alpha}}{\|z_n\|^2} \\
&\leq \frac{C\|J'(z_n)\|}{\|z_n\|} + C\frac{\|z_n^+ - z_n^-\|}{\|z_n\|^2} + C_\epsilon \frac{\|z_n\|^{2\alpha}}{\|z_n\|^2}.
\end{aligned} \tag{3.5}
$$

From the above inequality, it follows that

$$\frac{z_n^\pm}{\|z_n\|} \to 0 \quad \text{as } n \to \infty.$$

Moreover, by Lemma 3.4 and for n large, we conclude that $z_n \in C(R, \epsilon)$ and

$$\langle J'(z_n), z_n^0 \rangle \ \leq -\delta < 0.$$

However, by Cerami condition, we recall that

$$\|J'(z_n)\|(1 + \|z_n\|) \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction. Consequently, all $(Ce)_c$ sequences $(z_n)_{n\in\mathbb{N}}$ are bounded. Using standard arguments, we can conclude that $z_n \to z \in H$ up to a subsequence. Also, the proof of the case where $F_2(x) \leq 0$ is similar. Thus we will omit it. □

Now we will find all critical groups at infinity using a result given in [2].

Proposition 3.6 ([2]). *Let $J : H \to \mathbb{R}$ be a functional given by $J(z) = \frac{1}{2}\langle Az, z\rangle + G(z)$, where $A : H \to H$ is a bounded self-adjoint linear operator, such that 0 is an isolated point in the spectrum of A. Assume also that $J \in C^1(H, \mathbb{R})$ and G is of class C^2 in a neighborhood of infinity such that*

$$\frac{\|G'(z)\|}{\|z\|} \to 0 \quad \textit{as } \|z\| \to \infty.$$

In addition, suppose that the all critical values of J are bounded below and it satisfies $(PS)_c$ condition or $(Ce)_c$ condition for $c < 0$. Setting $V_\infty = \ker A$, $V_\infty^\perp = W_\infty^- \oplus W_\infty^+$ with W_∞^+ and W_∞^- invariant under A, $A\mid_{W_\infty^+}$ is positive definite, $A\big|_{W_\infty^-}$ negative definite. Let $\mu_\infty = \dim W_\infty^-$ and $\nu_\infty = \dim V_\infty$ the Morse index and the nullity of J at infinity, respectively. Then, we have the following alternatives:

(a) $(AC)_\infty^+$ *If there exist $R > 0$ and $\epsilon \in (0,1)$ such that $\langle J'(z), z^0\rangle \geq 0$ for $z = z^0 + z^- + z^+ \in V_\infty \oplus W_\infty^+ \oplus W_\infty^-$ with $\|z\| > R$ and $\|z^+ + z^-\| \leq \epsilon\|z\|$ then*

$$C_q(J,\infty) = \delta_{q,\mu_\infty}\mathcal{G}, \forall q \in \mathbb{N}.$$

(b) $(AC)_\infty^-$ *If there exist $R > 0$ and $\epsilon \in (0,1)$ such that $\langle J'(z), z^0\rangle \leq 0$ for $z = z^0 + z^- + z^+ \in V_\infty \oplus W_\infty^+ \oplus W_\infty^-$ with $\|z\| > R$ and $\|z^+ + z^-\| \leq \epsilon\|z\|$ then*

$$C_q(J,\infty) = \delta_{q,\mu_\infty+\nu_\infty}\mathcal{G}, \forall q \in \mathbb{N}.$$

The conditions $(AC)_\infty^+$ and $(AC)_\infty^-$ are well known as the angle conditions at infinity. We will use the previous result in order to compute the critical groups $C_q(J,\infty)$ under hypotheses (MI), (BI), and (CI).

Remark 3.7. In [2, Proposition 3.10], Bartsch-Li supposed the condition

$$G''(z) \to 0, \quad \text{as } \|z\| \to \infty. \tag{3.6}$$

However, we mention that there is a modified condition which was considered by Jiabao Su [25]. More precisely, in this work was used the assumption

$$\frac{\|G'(z)\|}{\|z\|} \to 0 \quad \text{as } \|z\| \to \infty \tag{3.7}$$

which is sufficient for the proof of [2, Proposition 3.10]. This modified condition was recently used in [27]. In this case, the proof could be done by using a result of Wang [28].

Evidently, the later assumption is slight weaker than the preceding hypothesis. Moreover, it is not a slightly modification, since the condition (3.6) can be not verified in many applications. But the condition (3.7) where the function G belongs to C^1 is easily verified and it is enough in applications.

Lemma 3.8. *Assume* (MI), (BI), (CI)). *Then we have the following alternatives:*

(a) *If $F_1(x) \geq 0$ a.e. in Ω then $C_q(J,\infty) = \delta_{q,\mu_\infty+\nu_\infty}\mathcal{G}$, for all $q \in \mathbb{N}$.*
(b) *If $F_2(x) \leq 0$ a.e. in Ω then $C_q(J,\infty) = \delta_{q,\mu_\infty}\mathcal{G}$, for all $q \in \mathbb{N}$.*

Here we define $\mu_\infty = \dim \oplus_{j=1}^{k-1} \ker(I\lambda_j^{-1} - T_{A_\infty})$ and $\nu_\infty = \dim\ker(I - T_{A_\infty})$ where $T_{A_\infty} : H \to H$ is a compact self-adjoint linear operator, see Section 2.

Proof. We recall that

$$J(z) = \frac{1}{2}\langle (I - T_{A_\infty})z, z\rangle + G(z), \quad G(z) = -\int_\Omega G_\infty(x,z)dx.$$

In this case (BI) implies $G'(z)/\|z\| \to 0$ as $\|z\| \to \infty$. Thus, for the proof of case (a), we have the $(AC)_\infty^-$ condition, which was provided by Lemma 3.4- a and Proposition 3.6-b.

Moreover, for the proof of case (b), we have the $(AC)_\infty^+$ condition, which was showed by Lemma 3.4-b and Proposition 3.6-a. This completes the proof. $\square$

In the next result we will use an interesting result proved by [8] for a single equation. This result has a similar version adapted for gradient systems. However, to the best our knowledge, this result is not well known for gradient systems. In the proof of this result we use the Strong Unique Continuation Property, in short (SUCP), for the eigenfunctions of problem (2.1). For the proof of this property we refer the reader to [10, 13, 19]. So we can prove the following result

Proposition 3.9. *Let $\beta, \alpha \in \mathcal{M}_2(\Omega)$. Then we have*

(a) *If $F'' \leq \beta(x) \preceq \lambda_{k+1} A_\infty(x)$, a.e. $x \in \Omega$ then there exist $\delta > 0$ such that*

$$\|z\|^2 - \int_\Omega \langle \beta(x) z, z\rangle dx \geq \delta \|z\|^2, \quad \forall z \in W_\infty^+ = \oplus_{j=k+1}^\infty \ker\left(I\lambda_j^{-1}(A_\infty) - T_{A_\infty}\right).$$

(b) *If $\lambda_{k-1} A_\infty(x) \prec \alpha(x) \leq F''$, a.e. $x \in \Omega$, then there exist $\delta > 0$ such that*

$$\|z\|^2 - \int_\Omega \langle \alpha(x) z, z\rangle dx \leq -\delta \|z\|^2, \quad \forall\, w \in W_\infty^- = \oplus_{j=1}^{k-1} \ker\left(I\lambda_j^{-1}(A_\infty) - T_{A_\infty}\right).$$

The proof of the above proposition is similar to the proof of [8, Proposition 2]. Thus, we omit it.

4. Proof of our main results

4.1. **Proof of Theorem 1.1.** Firstly, suppose the case (a), i.e, when $F_2(x) \leq 0$ a.e. $x \in \Omega$ and $f_1(x) \geq 0$ a.e. $x \in \Omega$ holds. Then by Lemmas 3.2 and 3.8 we conclude that

$$C_q(J, \infty) = \delta_{q,\mu_\infty} \mathcal{G}, \quad C_q(J, 0) = \delta_{k,\mu_0+\nu_0} \mathcal{G}, \quad \forall\, q \in \mathbb{N}.$$

Thus, we get $C_{\mu_\infty}(J, \infty) \neq C_{\mu_\infty}(J, 0)$ for $m \neq k-1$. This information ensures the existence of a critical point $z_\star \in H$ such that $C_{\mu_\infty}(J, z_\star) \neq 0$. Therefore, $z_\star$ is a nontrivial solution for the system (1.1).

For the proof of case (b); i.e., $F_1(x) \geq 0$ a.e. $x \in \Omega$ and $f_2(x) \leq 0$ a.e. in Ω we use Lemmas 3.2 and 3.8 which imply

$$C_q(J, \infty) = \delta_{q,\mu_\infty+\nu_\infty} \mathcal{G}, \quad C_q(J, 0) = \delta_{q,\mu_0} \mathcal{G}, \quad \forall\, q \in \mathbb{N}.$$

In this case we obtain $C_{\mu_\infty+\nu_\infty}(J, \infty) \neq C_{\mu_\infty+\nu_\infty}(J, 0)$ where $k \neq m-1$. Therefore, we have at least one critical point $z_\star \in H$ such that $C_{\mu_\infty}(J, z_\star) \neq 0$. Thus $z_\star$ is a nontrivial solution for problem the (1.1) and the proof of this theorem is now complete.

4.2. **Proof of Theorem 1.2.** In this case, the proof of the cases (a) and (b) are analogues to the previous cases. Thus we omit it.

4.3. **Proof of Theorem 1.6.** Firstly, we will prove the case (b). In this case Theorem 1.1 ensures that

$$C_q(J, \infty) = \delta_{q,\mu_\infty+\nu_\infty} \mathcal{G}, \quad C_q(J, 0) = \delta_{q,\mu_0} \mathcal{G}, \quad \forall\, q \in \mathbb{N},$$
$$C_{\mu_\infty+\nu_\infty}(J, z_\star) \neq 0,$$

where $z_\star$ is a nontrivial solution. In addition, by Growoll-Meyer's Lemma [12], we get $\mu_\infty + \nu_\infty \in [m(z_\star), m(z_\star) + n(z_\star)]$ where $m(z_\star)$ is the index Morse for J at $z_\star$ and $n(z_\star)$ is the nullity in the same point.

On the other hand, we have

$$J''(z_\star)(w, w) = \|w\|^2 - \int_\Omega F''(x, z_\star)(w, w) dx$$

$$
\begin{aligned}
&\geq \|w\|^2 - \int_\Omega \langle \beta(x)w, w\rangle dx \\
&\geq \delta\|w\|^2 > 0, \quad \forall\, w \in W_\infty^+\backslash\{0\},
\end{aligned}
$$

where we used Proposition 3.9. Therefore, we conclude that

$$
\begin{aligned}
m(z_\star) + n(z_\star) &\leq \dim \oplus_{j=1}^k \ker\big(I\lambda_j^{-1}(A_\infty) - T_{A_\infty}\big) \\
&= \dim(W_\infty^- \oplus V) \\
&= \mu_\infty + \nu_\infty.
\end{aligned}
$$

The previous inequality provides the identity

$$m(z_\star) + n(z_\star) = \mu_\infty + \mu_\infty.$$

In that case, by [20], we obtain $C_q(J, z_\star) = \delta_{q,\mu_\infty+\nu_\infty}\mathcal{G}$, for all $q \in \mathbb{N}$.

Finally, if $0, z_\star$ are the unique critical points for J the Morse's identity implies that

$$(-1)^{\mu_0} + (-1)^{\mu_\infty+\nu_\infty} = (-1)^{\mu_\infty+\nu_\infty}$$

which is a contradiction. Therefore, the problem (1.1) has at least two nontrivial solutions, which completes the proof of case (b).

In the proof of case (a) we have the following critical groups

$$
\begin{gathered}
C_q(J, \infty) = \delta_{q,\mu_\infty}\mathcal{G}, \quad C_q(J, 0) = \delta_{q,\mu_0+\nu_0}\mathcal{G}, \quad \forall\, q \in \mathbb{N}, \\
C_{\mu_\infty}(J, z_\star) \neq 0.
\end{gathered}
$$

Again, we have $\mu_\infty \in [m(z_\star), m(z_\star) + n(z_\star)]$. In addition, we obtain

$$
\begin{aligned}
J''(z_\star)(w, w) &= \|w\|^2 - \int_\Omega F''(x, z_\star)(w, w)dx \\
&\leq \|w\|^2 - \int_\Omega \langle \beta(x)w, w\rangle dx \\
&\leq -\delta\|w\|^2 < 0, \quad \forall\, w \in W_\infty^-\backslash\{0\},
\end{aligned}
$$

where we used Proposition 3.9. This inequality yields

$$\mu_\infty \geq m(z_\star) \geq \dim W_\infty^- = \mu_\infty.$$

Hence $\mu_\infty = m(z^*)$. As a consequence, by [20], we conclude that

$$C_q(J, z_\star) = \delta_{q,\mu_\infty+\nu_\infty}\mathcal{G}, \quad \forall q \in \mathbb{N}.$$

In conclusion, if $0, z_\star$ are the unique critical points of J the Morse's identity implies

$$(-1)^{\mu_0+\nu_0} + (-1)^{\mu_\infty} = (-1)^{\mu_\infty}.$$

Clearly, we have a contradiction and problem (1.1) admits at least two nontrivial solutions in the case (a). So we completed the proof.

4.4. **Proof of Theorem 1.7.** The proof of the cases (a) and (b) are similar to the proof of Theorem 1.6; therefore, we omit them.

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