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OSCILLATION CRITERIA FOR FORCED SECOND-ORDER MIXED TYPE QUASILINEAR DELAY DIFFERENTIAL EQUATIONS

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Abstract. This article presents new oscillation criteria for the second-order delay differential equation

$$(p(t)(x'(t))^{\alpha})' + q(t)x^{\alpha}(t-\tau) + \sum_{i=1}^{n} q_i(t)x^{\alpha_i}(t-\tau) = e(t)$$

where $\tau \geq 0$, $p(t) \in C^1[0,\infty)$, $q(t), q_i(t), e(t) \in C[0,\infty)$, $p(t) > 0$, $\alpha_1 > \dots > \alpha_m > \alpha > \alpha_{m+1} > \dots > \alpha_n > 0$ $(n > m \geq 1)$, $\alpha_1, \dots, \alpha_n$ and α are ratio of odd positive integers. Without assuming that $q(t), q_i(t)$ and $e(t)$ are nonnegative, the results in [6, 8] have been extended and a mistake in the proof of the results in [3] is corrected.

1. Introduction

In this paper, we are concerned with the oscillatory behavior of the quasilinear delay differential equation

$$(p(t)(x'(t))^{\alpha})' + q(t)x^{\alpha}(t-\tau) + \sum_{i=1}^{n} q_i(t)x^{\alpha_i}(t-\tau) = e(t) \tag{1.1}$$

where $\tau \geq 0$, $p(t), q(t), q_i(t) \in C[0,\infty)$, $p(t)$ is positive, nondecreasing and differentiable, $\alpha_1, \dots, \alpha_n, \alpha$ are ratio of odd positive integers, and $\alpha_1 > \dots > \alpha_m > \alpha > \alpha_{m+1} > \dots > \alpha_n > 0$.

A solution $x(t)$ of (1.1) is said to be oscillatory if it is defined on some ray $[T,\infty)$ with $T \geq 0$ and has unbounded set of zeros. Equation (1.1) is said to be oscillatory if all solutions extendable throughout $[0,\infty)$ are oscillatory.

For $\tau = 0$ and $\alpha = 1$, the oscillatory behavior of (1.1) has been studied in Sun and Wong [8] and Sun and Meng [6]. When $\alpha = 1$, Chen and Li [3] extended the results established by Sun and Meng [6] to (1.1). A close look into the proof of [3, Theorem 1] reveals that the authors used $x''(t) \leq 0$ for $t \in [a_1-\tau, b_1]$ instead of taking $(p(t)x'(t))' \leq 0$ for $t \in [a_1-\tau, b_1]$. We wish not only to correct the proof of the theorem but also extend the results given in [1, 2, 4, 8] for ordinary and delay differential equations.

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In Section 2, we present some new oscillation criteria for the (1.1) and in Section 3 we provide some examples to illustrate the results.

2. Oscillation Results

We first present a lemma which is a generalization of Lemma 1 of Sun and Wong [8].

Lemma 2.1. *Let $\{\alpha_i\}$, $i = 1, 2, \dots, n$ be the n-tuple satisfying $\alpha_1 > \dots > \alpha_m > \alpha > \alpha_{m+1} > \dots > \alpha_n > 0$. Then there is an n-tuple $(\eta_1, \eta_2, \dots, \eta_n)$ satisfying*

$$\sum_{i=1}^{n} \alpha_i \eta_i = \alpha \tag{2.1}$$

which also satisfies

$$\sum_{i=1}^{n} \eta_i < 1, \quad 0 < \eta_i < 1, \tag{2.2}$$

or

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1. \tag{2.3}$$

Lemma 2.2. *Suppose X and Y are nonnegative, then*

$$X^\gamma - \gamma\, Y^{\gamma-1} X + (\gamma - 1) Y^\gamma \geq 0, \quad \gamma > 1,$$

where the equality holds if and only if $X = Y$.

The proof of the above lemma can be found in [5].

Following Philos [1], we say a continuous function $H(t, s)$ belongs to a function class $D_{a,b}$, denoted by $H \in D_{a,b}$, if $H(b, b) = H(a, a) = 0$, $H(b, s) > 0$ and $H(s, a) > 0$ for $b > s > a$, and $H(t, s)$ has continuous partial derivatives with $\frac{\partial H(t,s)}{\partial t}$ and $\frac{\partial H(t,s)}{\partial s}$ in $[a, b] \times [a, b]$. Set

$$\frac{\partial H(t, s)}{\partial t} = (\alpha + 1) h_1(t, s) \sqrt{H(t, s)}, \quad \frac{\partial H(t, s)}{\partial s} = -(\alpha + 1) h_2(t, s) \sqrt{H(t, s)}. \tag{2.4}$$

Theorem 2.3. *If for any $T \geq 0$, there exist a_1, b_1, c_1, a_2, b_2 and c_2 such that $T \leq a_1 < c_1 < b_1$, $T \leq a_2 < c_2 < b_2$ and*

$$\begin{gathered} q_i(t) \geq 0, \quad q(t) \geq 0, \quad t \in [a_1 - \tau, b_1] \cup [a_2 - \tau, b_2],\ i = 1, 2, \dots, n, \\ e(t) \leq 0, \quad t \in [a_1 - \tau, b_1], \\ e(t) \geq 0, \quad t \in [a_2 - \tau, b_2], \end{gathered} \tag{2.5}$$

and there exist $H_j \in D_{a_j, b_j}$, $j = 1, 2$, such that

$$\begin{aligned} &\frac{1}{H_j(c_j, a_j)} \int_{a_j}^{c_j} H_j(s, a_j) \Big[Q_j(s) - \frac{p(s)}{\alpha^\alpha} \Big(\frac{h_{j_1}(s, a_j)}{\sqrt{H_j(s, a_j)}} \Big)^{\alpha+1} \Big] ds \\ &+ \frac{1}{H_j(b_j, c_j)} \int_{c_j}^{b_j} H_j(b_j, s) \Big[Q_j(s) - \frac{p(s)}{\alpha^\alpha} \Big(\frac{h_{j_2}(b_j, s)}{\sqrt{H_j(b_j, s)}} \Big)^{\alpha+1} \Big] ds > 0 \end{aligned} \tag{2.6}$$

where h_{j_1} and h_{j_2} are defined as in (2.4),

$$Q_j(t) = \beta_j(t) \Big[q(t) + k_0 |e(t)|^{\eta_0} \prod_{i=1}^{n} q_i^{\eta_i}(t) \Big], \quad k_0 = \prod_{i=0}^{n} \eta_i^{-\eta_i}, \quad \eta_0 = 1 - \sum_{i=1}^{n} \eta_i, \tag{2.7}$$

and $\eta_1, \eta_2, \dots, \eta_n$ *are positive constants satisfying (a) and (b) in Lemma 2.1 and* $\beta_j(t) = \left(\frac{(t-a_j)}{(t-a_j+\tau)}\right)^\alpha$ *then* (1.1) *is oscillatory.*

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \geq t_0 - 2\tau > 0$ where t_0 depends on the solution $x(t)$. When $x(t)$ is eventually negative, the proof follows the same argument by using the interval $[a_2, b_2]$ instead of $[a_1, b_1]$. Choose $a_1, b_1 \geq t_0$ such that $q_i(t) \geq 0, q(t) \geq 0$ and $e(t) \leq 0$ for $t \in [a_1 - \tau, b_1]$ and $i = 1, 2, \dots, n$. From (1.1), we have $(p(t)(x'(t))^\alpha)' \leq 0$ for $t \in [a_1 - \tau, b_1]$. Therefore for $a_1 - \tau < s < t \leq b_1$, we have

$$x(t) - x(a_1 - \tau) = \frac{p^{\frac{1}{\alpha}}(s)x'(s)}{p^{\frac{1}{\alpha}}(s)}(t - a_1 + \tau)$$

or

$$x(t) \geq \frac{p^{\frac{1}{\alpha}}(t)x'(t)}{p^{\frac{1}{\alpha}}(s)}(t - a_1 + \tau)$$

where $t \in (a_1 - \tau, b_1]$. Noting that $x(a_1 - \tau) > 0$ and $p(t)$ is nondecreasing, we have

$$\frac{1}{(t - a_1 + \tau)} \geq \frac{x'(t)}{x(t)}, \quad t \in (a_1 - \tau, b_1]. \tag{2.8}$$

Integrating (2.8) from $t - \tau$ to $t > a_1$, we obtain

$$\frac{x(t-\tau)}{x(t)} \geq \frac{t - a_1}{t - a_1 + \tau}, \quad t \in (a_1, b_1]. \tag{2.9}$$

Define $w(t) = -p(t)\frac{(x'(t))^\alpha}{x^\alpha(t)}$. From (1.1) and (2.9) we find that $w(t)$ satisfies the inequality

$$\begin{aligned} w'(t) \geq q(t)\beta_1(t) + \sum_{i=1}^n q_i(t)\beta_1(t)x^{\alpha_i - \alpha}(t - \tau) \\ - e(t)\beta_1(t)x^{-\alpha}(t - \tau) + \alpha\frac{|w(t)|^{1+\frac{1}{\alpha}}}{p^{1/\alpha}(t)}, \quad t \in [a_1, b_1]. \end{aligned} \tag{2.10}$$

Recall the arithmetic-geometric mean inequality

$$\sum_{i=0}^n \eta_i u_i \geq \prod_{i=0}^n u_i^{\eta_i}, \quad u_i \geq 0, \tag{2.11}$$

where $\eta_0 = 1 - \sum_{i=1}^n \eta_i$ and $\eta_i > 0, i = 1, 2, \dots, n$, are chosen according to given $\alpha_1, \dots, \alpha_n$ as in Lemma 2.1 satisfying (a) and (b). Now return to (2.10) and identify $u_0 = \eta_0^{-1}|e(t)|x^{-\alpha}(t - \tau)$ and $u_i = \eta_i^{-1}q_i(t)x^{\alpha_i - \alpha}(t - \tau)$ in (2.11) to obtain

$$\begin{aligned} w'(t) &\geq \beta_1(t)q(t) + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(t)} + \beta_1(t)\eta_0^{-\eta_0}|e(t)|^{\eta_0}\prod_{i=1}^n \eta_i^{-\eta_i}q_i^{\eta_i}(t) \\ &= Q_1(t) + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(t)}, \quad t \in [a_1, b_1], \end{aligned} \tag{2.12}$$

where $Q_1(t)$ is defined by (2.7). Multiply (2.12) by $H_1(b_1,t) \in D_{a_1,b_1}$ and integrating by parts, we find

$$\begin{aligned} &-H_1(b_1,c_1)w(c_1)\\ &\geq \int_{c_1}^{b_1} Q_1(s)H_1(b_1,s)ds\\ &\quad+\int_{c_1}^{b_1}\Big[-|w(s)|(\alpha+1)h_{12}(b_1,s)\sqrt{H_1(b_1,s)}+\frac{\alpha|w(s)|^{1+\frac{1}{\alpha}}}{p^{\frac{1}{\alpha}}(s)}H_1(b_1,s)\Big]ds. \end{aligned}$$

Using Lemma 2.2 to the right side of the last inequality, we have

$$-H_1(b_1,c_1)w(c_1) \geq \int_{c_1}^{b_1}\Big[Q_1(s)H_1(b_1,s)-\frac{p(s)}{\alpha^\alpha}H_1(b_1,s)\Big(\frac{h_{12}(b_1,s)}{\sqrt{H_1(b_1,s)}}\Big)^{\alpha+1}\Big]ds.$$

It follows that

$$-w(c_1) \geq \frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}\Big[Q_1(s)H_1(b_1,s)-\frac{p(s)}{\alpha^\alpha}H_1(b_1,s)\Big(\frac{h_{12}(b_1,s)}{\sqrt{H_1(b_1,s)}}\Big)^{\alpha+1}\Big]ds. \tag{2.13}$$

On the other hand, multiplying both sides of (2.12) by $H_1(t,a_1) \in D_{a_1,b_1}$, integrating by parts, and similar to the above analysis we can easily obtain

$$w(c_1) \geq \frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}\Big[Q_1(s)H_1(s,a_1)-\frac{p(s)}{\alpha^\alpha}H_1(s,a_1)\Big(\frac{h_{11}(s,a_1)}{\sqrt{H_1(s,a_1)}}\Big)^{\alpha+1}\Big]ds. \tag{2.14}$$

From (2.13) and (2.14) we have

$$\begin{aligned} &\frac{1}{H_1(c_1,a_1)}\int_{a_1}^{c_1}\Big[Q_1(s)H_1(s,a_1)-\frac{p(s)}{\alpha^\alpha}H_1(s,a_1)\Big(\frac{h_{11}(s,a_1)}{\sqrt{H_1(s,a_1)}}\Big)^{\alpha+1}\Big]ds\\ &+\frac{1}{H_1(b_1,c_1)}\int_{c_1}^{b_1}\Big[Q_1(s)H_1(b_1,s)-\frac{p(s)}{\alpha^\alpha}H_1(b_1,s)\Big(\frac{h_{12}(b_1,s)}{\sqrt{H_1(b_1,s)}}\Big)^{\alpha+1}\Big]ds \leq 0 \end{aligned}$$

which contradicts (2.6) for $j=1$. The proof is now complete. □

The following theorem gives an interval oscillation criteria for the unforced (1.1) with $e(t) \equiv 0$.

Theorem 2.4. *If for any $T>0$ there exist a,b and c such that $T \leq a < c < b$ and $q(t) \geq 0, q_i(t) \geq 0$ for $t \in [a-\tau,b]$ and $i=1,2,\dots,n$, and there exists $H \in D_{a,b}$ such that*

$$\begin{aligned} &\frac{1}{H(c,a)}\int_a^c H(s,a)\Big[\overline{Q}(s)-\frac{p(s)}{\alpha^\alpha}\Big(\frac{h_1(s,a)}{\sqrt{H(s,a)}}\Big)^{\alpha+1}\Big]ds\\ &+\frac{1}{H(b,c)}\int_c^b H(b,s)\Big[\overline{Q}(s)-\frac{p(s)}{\alpha^\alpha}\Big(\frac{h_2(b,s)}{\sqrt{H(b,s)}}\Big)^{\alpha+1}\Big]ds > 0 \end{aligned}$$

where h_1 and h_2 are defined by (2.4),

$$\overline{Q}(t)=\beta(t)\Big[q(t)+k_1\prod_{i=1}^n q_i^{\eta_i}(t)\Big], \quad k_1=\prod_{i=1}^n \eta_i^{-\eta_i},$$

and $\eta_1,\eta_2,\dots,\eta_n$ are positive constants satisfying (a) and (c) of Lemma 2.1, $\beta(t)=\big(\frac{(t-a)}{(t-a+\tau)}\big)^\alpha$, then (1.1) with $e(t)\equiv 0$ is oscillatory.

The proof of the above theorem is in fact a particular version of the proof of Theorem 2.3. We need only to note that $e(t) \equiv 0$ and $\eta_0 = 0$ and apply conditions (a) and (c) of Lemma 2.1.

Remark 2.5. When $\tau = 0$, Theorems 2.3 and 2.4 reduce to the main results in [9]. Moreover if $\tau = 0$ and $\alpha = 1$, then Theorems 2.3 and 2.4 reduce to [6, Theorems 1 and 2].

Before stating the next result we introduce another function class. Say $u(t) \in E_{a,b}$ if $u \in C^1[a,b]$, $u^{\alpha+1}(t) > 0$, and $u(a) = u(b) = 0$.

Theorem 2.6. *If for any $T \ge 0$, there exist a_1, b_1 and a_2, b_2 such that $T \le a_1 < b_1$, $T \le a_2 < b_2$ and (2.5) holds, and there exists $H_j \in E_{a_j,b_j}$ and a positive nondecreasing function $\phi \in C^1([0,\infty),\mathbb{R})$ such that*

$$\int_{a_j}^{b_j} \phi(t)\Big[Q_j(t)H_j^{\alpha+1}(t) - p(t)\Big(|H_j'(t)| + \frac{H_j(t)\phi'(t)}{(\alpha+1)\phi(t)}\Big)^{\alpha+1}\Big]dt > 0 \tag{2.15}$$

for $j = 1, 2$, where

$$\begin{gathered} Q_j(t) = \beta_j(t)\Big[q(t) + k_0|e(t)|^{\eta_0}\prod_{i=1}^{n} q_i^{\eta_i}(t)\Big],\ k_0 = \prod_{i=0}^{n}\eta_i^{-\eta_i}, \\ \beta_j(t) = \Big(\frac{(t-a_j)}{(t-a_j+\tau)}\Big)^{\alpha} \end{gathered} \tag{2.16}$$

then (1.1) is oscillatory.

Proof. Suppose that $x(t)$ is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \ge t_0 - 2\tau > 0$ where t_0 depends on the solution $x(t)$. When $x(t)$ is eventually negative, the proof follows the same argument by using the interval $[a_2, b_2]$ instead of $[a_1, b_1]$. Choose $q(t) \ge 0, q_i(t) \ge 0$ and $e(t) \le 0$ for $t \in [a_1 - \tau, b_1]$ and $i = 1, 2, \dots, n$. As in the proof of Theorem 2.3

$$\Big(\frac{x(t-\tau)}{x(t)}\Big)^{\alpha} \ge \beta_1(t), \quad t \in (a_1, b_1]. \tag{2.17}$$

Define $w(t) = -\phi(t)\frac{p(t)(x'(t))^{\alpha}}{x^{\alpha}(t)}$. From (1.1) and (2.17) we have

$$\begin{aligned} w'(t) &\ge \phi(t)q(t)\beta_1(t) + \sum_{i=1}^{n}\phi(t)q_i(t)\beta_1(t)x^{\alpha_i-\alpha}(t-\tau) + \frac{w(t)\phi'(t)}{\phi(t)} \\ &\quad - e(t)\beta_1(t)x^{-\alpha}(t-\tau) + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}}. \end{aligned}$$

Using Lemma 2.1, we have

$$w'(t) \ge \phi(t)Q_1(t) + \frac{w(t)\phi'(t)}{\phi(t)} + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}}. \tag{2.18}$$

Multiply (2.18) by $H^{\alpha+1}(t)$ and integrating from a_1 to b_1 using the fact that $H(a_1) = H(b_1) = 0$, we obtain

$$\begin{aligned} 0 \ge \int_{a_1}^{b_1} H^{\alpha+1}(t)\phi(t)Q_1(t)dt + \int_{a_1}^{b_1}\Big\{&\frac{\alpha H^{\alpha+1}(t)|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}} \\ &- \Big[(\alpha+1)H(t)^{\alpha}|H'(t)| + \frac{H^{\alpha+1}(t)\phi'(t)}{\phi(t)}\Big]|w(t)|\Big\}dt. \end{aligned} \tag{2.19}$$

Using Lemma 2.2 in (2.19), we have

$$0 \geq \int_{a_1}^{b_1} \phi(t)\Big[Q_1(t)H^{\alpha+1}(t) - p(t)\Big(|H'(t)| + \frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\Big)^{\alpha+1}\Big]dt$$

which contradicts (2.15) with $j = 1$. This completes the proof. $\square$

Corollary 2.7. *Suppose that $\phi(t) \equiv 1$ in Theorem 2.6, and (2.15) is replaced by*

$$\int_{a_j}^{b_j} [Q_j(t)H^{\alpha+1}(t) - p(t)|H'(t)|^{\alpha+1}]dt > 0$$

for $j = 1, 2$. Then (1.1) is oscillatory.

Theorem 2.8. *Assume that for any $T \geq 0$, there exist a, b such that $T \leq a < b$ and $q(t) \geq 0, q_i(t) \geq 0$ for $t \in [a, b]$ and $i = 1, 2, \dots, n$. Suppose there exists $H \in E_{a,b}$ and a positive nondecreasing function $\phi \in C'([0, \infty), \mathbb{R})$ such that*

$$\int_a^b \phi(t)\Big[\overline{Q}(t)H^{\alpha+1}(t) - p(t)\Big(|H'(t)| + \frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\Big)^{\alpha+1}\Big]dt > 0$$

where

$$\overline{Q}(t) = \beta(t)\Big[q(t) + k_1 \prod_{i=1}^{n} q_i^{\eta_i}(t)\Big], \ k_1 = \prod_{i=1}^{n} \eta_i^{-\eta_i},$$
$$\beta(t) = \Big(\frac{(t-a)}{(t-a+\tau)}\Big)^{\alpha}.$$

Then (1.1) with $e(t) \equiv 0$ is oscillatory.

The proof of the above theorem is in fact a particular version of the proof of Theorem 2.6. We need only to note that $e(t) \equiv 0$ and $\eta_0 = 0$ and apply conditions (a) and (c) of Lemma 2.1

Remark 2.9. When $\tau = 0, \alpha = 1,$ and $\phi(t) \equiv 1$, then Theorem 2.6 and 2.8 reduced to [8, Theorems 1 and 2].

If $n = 1$ and $e(t) \equiv 0$ then we see that Theorems 2.3–2.8 are not valid. Therefore in the following we state and prove some new oscillation criteria for the equation

$$(p(t)(x'(t))^{\alpha})' + q(t)x^{\alpha}(t-\tau) + q_1(t)x^{\alpha_1}(t-\tau) = 0, \quad t \geq 0. \tag{2.20}$$

Theorem 2.10. *Assume that for any $T \geq 0$ there exist a, b such that $T \leq a < b$ and $q(t) \geq 0, q_1(t) \geq 0$ for $t \in [a, b]$. Suppose there exists $H \in E_{a,b}$ and positive nondecreasing function $\phi \in C'([0, \infty), \mathbb{R})$ such that*

$$\int_a^b \phi(t)\Big[Q_3(t)H^{\alpha+1}(t) - p(t)\Big(|H'(t)| + \frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\Big)^{\alpha+1}\Big]dt > 0 \tag{2.21}$$

where

$$Q_3(t) = \beta(t)[q(t) - M_1 q_1(t)], \quad M_1 = (\alpha_1 - \alpha - 1)\Big(\frac{1}{\alpha_1 - \alpha}\Big)^{\frac{(\alpha_1-\alpha)}{(\alpha_1-\alpha-1)}},$$
$$\beta(t) = \Big(\frac{(t-a)}{(t-a+\tau)}\Big)^{\alpha}$$

and $\alpha_1 > \alpha + 1$, then (2.20) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.6, we obtain

$$\begin{aligned} w'(t) &\geq \frac{w(t)\phi'(t)}{\phi(t)} + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}} + \phi(t)q(t)\beta(t) + \phi(t)q_1(t)\beta(t)x^{\alpha_1-\alpha}(t-\tau), \\ &\geq \frac{w(t)\phi'(t)}{\phi(t)} + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}} + \phi(t)q(t)\beta(t) \\ &\quad + \phi(t)q_1(t)\beta(t)(x^{\alpha_1-\alpha}(t-\tau) - x(t-\tau)). \end{aligned} \tag{2.22}$$

Set $F(x) = x^{\alpha_1-\alpha} - x$. Using differential calculus, we find that $F(x) \geq -M_1$. From (2.22), we have

$$w'(t) \geq \phi(t)Q_3(t) + \frac{\phi'(t)}{\phi(t)}w(t) + \frac{\alpha|w(t)|^{\frac{\alpha+1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}}.$$

The rest of the proof is similar to that of Theorem 2.6. This completes the proof. □

Theorem 2.11. *Assume that for any $T \geq 0$ there exist a, b such that $T \leq a < b$ and $q(t) \geq 0$, $q_1(t) \geq 0$ for $t \in [a, b]$. Suppose there exists $H \in E_{a,b}$ and a positive nondecreasing function $\phi \in C'([0,\infty),\mathbb{R})$ such that*

$$\int_a^b \phi(t)\Big[Q_4(t)H^{\alpha+1}(t) - p(t)\Big(|H'(t)| + \frac{H(t)\phi'(t)}{(\alpha+1)\phi(t)}\Big)^{\alpha+1}\Big]dt > 0 \tag{2.23}$$

where

$$Q_4(t) = \beta(t)[q(t) - M_2q_1(t)], M_2 = \frac{(\alpha-\alpha_1-\beta)}{(\alpha-\alpha_1)}\left(\frac{\beta}{\alpha-\alpha_1}\right)^{\frac{\beta}{(\alpha-\alpha_1-\beta)}},$$

and $\alpha > \alpha_1 + \beta$, then (2.20) is oscillatory.

Proof. Proceeding as in the proof of Theorem 2.6, we obtain

$$\begin{aligned} w'(t) \geq\ & \frac{w(t)\phi'(t)}{\phi(t)} + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}} + \phi(t)q(t)\beta(t) \\ & + \phi(t)q_1(t)\beta(t)[x^{\alpha_1-\alpha}(t-\tau) - x^{-\beta}(t-\tau)]. \end{aligned} \tag{2.24}$$

Set $F(x) = x^{\alpha_1-\alpha} - x^{-\beta}$. Using differential calculus, we find $F(x) \geq -M_2$. From (2.24), we have

$$w'(t) \geq \phi(t)Q_4(t) + \frac{w(t)\phi'(t)}{\phi(t)} + \frac{\alpha|w(t)|^{1+\frac{1}{\alpha}}}{(p(t)\phi(t))^{1/\alpha}}.$$

The rest of the proof is similar to that of Theorem 2.6. This completes the proof. □

Remark 2.12. The results obtained here can also be extended to the following general equation

$$\begin{aligned} &(p(t)|(x'(t))|^{\alpha-1}x'(t))' + q(t)|x(t-\tau_0)|^{\alpha-1}x(t-\tau_0) \\ &+ \sum_{i=1}^n q_i(t)|x(t-\tau_i)|^{\alpha_i-1}x(t-\tau_i) = e(t) \end{aligned}$$

where $\tau_i \geq 0$, $i = 0, 1, \dots n$ and we left it to interesting readers.

3. Examples

In this section, we present some examples to illustrate the main results.

Example 3.1. Consider the delay differential equation

$$\begin{aligned}&(t(x'(t))^3)' + l_1 \cos t(x(t-\pi/8))^3 \\ &+ l_2(\sin t)^{20/11}(x(t-\pi/8))^5 + l_3 \cos^4 t(x(t-\pi/8)) \\ &= -m \cos^5 2t,\end{aligned} \tag{3.1}$$

where $t \geq 0$, l_1, l_2, l_3, m are positive constants. Here $p(t) = t$, $\alpha = 3$, $q(t) = l_1 \cos t$, $q_1(t) = l_2(\sin t)^{20/11}$, $q_2(t) = l_3(\cos t)^{1/4}$, $\alpha_1 = 5$, $\alpha_2 = 1$, $\tau = \frac{\pi}{8}$ and $e(t) = -m \cos 2t$. For any $T \geq 0$, we can choose $a_1 = 2n\pi + \frac{\pi}{8}, b_1 = 2n\pi + \frac{\pi}{4}$, $a_2 = 2n\pi + \frac{3\pi}{8}, b_2 = 2n\pi + \frac{\pi}{2}$ for sufficiently large n, where n is a positive integer. It is easy to find that

$$\begin{aligned}Q_j(t) &= k_0\Big[\frac{(t-a_j)}{(t-a_j+\pi/8)}\Big]^3(l_1 \cos t + (\cos^5 2t)^{1/5}(\sin^{20/11} t)^{11/20}(\cos^4 t)^{1/4}) \\ &= k_0\Big[\frac{(t-a_j)}{(t-a_j+\pi/8)}\Big]^3(l_1 \cos t + (\cos 2t)\sin t \cos t)\end{aligned}$$

where $k_0 = (5m)^{1/5}(\frac{20l_2}{11})^{11/20}(4l_3)^{1/4}$. Let $H_1(t) = H_2(t) = \sin 8t$ and $\phi(t) = 1$. Based on Theorem 2.6, we have (3.1) is oscillatory if

$$\int_{a_j}^{b_j} \Big[k_0\Big(\frac{t-a_j}{t-a_j+\pi/8}\Big)^3\Big(l_1 \cos t + \frac{\sin 4t}{4}\Big)\sin^4 8t - 8t\cos^4 8t\Big]dt > 0, \quad j = 1, 2.$$

Example 3.2. Consider the delay differential equation

$$x''(t) + k_1 t^{-\lambda/3}(\sin t)x(t-\pi/2) + t^{-\delta}x^3(t-\pi/2) = 0,\ t \geq 1, \tag{3.2}$$

where $k_1, \lambda, \delta > 0$ are constants and $\alpha = 1, \alpha_1 = 3, \tau = \frac{\pi}{2}$ in Theorem 2.10. Since $\alpha < \alpha_1$ and $e(t) \equiv 0$, Theorem 2.4 and Theorem 2.8 are not applicable to this case. However, we can obtain oscillation of (3.2) with $H(t) = \sin 2t$ and $\phi(t) = 1$. For any $t_0 \geq 1$, we can choose $a = 2k\pi + \pi/2, b = 2k\pi + \pi$ for sufficiently large k, where k is a positive integer. It is easy to find that

$$Q_3(t) = \Big(\frac{t-a}{t-a+\pi/2}\Big)\Big[k_1 t^{-\lambda/3}\sin t - \frac{t^{-\delta}}{4}\Big],$$

$$\int_a^b \Big[\frac{t-a}{t-a+\pi/2}\Big(k_1 t^{-\lambda/3}\sin t - \frac{t^{-\delta}}{4}\Big)\sin^2 2t - 4\cos^2 2t\Big]dt > 0.$$

So by Theorem 2.10, Equation (3.2) is oscillatory if

$$\int_{2k\pi+\pi/2}^{2k\pi+\pi}\Big(\frac{t-a}{t-a+\pi/2}\Big)\Big(k_1 t^{-\lambda/3}\sin t - \frac{t^{-\delta}}{4}\Big)\sin^2 2t\,dt > \pi.$$

Example 3.3. Consider the delay differential equation

$$((x'(t))^3)' + k_1 t^{-\lambda}(\sin t)\,x^3(t-\pi/4) + k_2 t^{-\lambda}x(t-\pi/4) = 0, \tag{3.3}$$

where $t \geq 1, k_1, k_2$ and λ are positive constants and $\alpha = 3, \alpha_1 = 1$ in Theorem 2.11. Since other theorems cannot be applicable to this case but we can obtain oscillation of (3.3) with $\beta = 1, H(t) = \sin 4t$ and $\phi(t) = 1$. For any $t_0 \geq 1$, let

$a = 2n\pi + \pi/4, b = 2n\pi + \pi/2$ for n sufficiently large and n is a positive integer. It is easy to see that

$$\begin{aligned}&\int_a^b Q_4(t)H^4(t) - (H'(t))^4 \\ &= \int_a^b \Big[\Big(\frac{t-a}{t-a+\pi/4}\Big)^3\Big(k_1 t^{-\lambda}\sin t - \frac{1}{4}k_2 t^{-\lambda}\Big)\sin^4 4t - 256\cos^4 4t\Big]dt.\end{aligned}$$

So by Theorem 2.11, Equation (3.3) is oscillatory if

$$\int_{2n\pi+\pi/4}^{2n\pi+\pi/2} \Big(\frac{t-a}{t-a+\pi/4}\Big)^3\Big(k_1 t^{-\lambda}\sin t - \frac{1}{4}k_2 t^{-\lambda}\Big)\sin^4 4t\,dt > \frac{3\pi}{32}.$$

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