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INFINITY LAPLACE EQUATION WITH NON-TRIVIAL RIGHT-HAND SIDE

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Abstract. We analyze the set of continuous viscosity solutions of the infinity Laplace equation $-\Delta_\infty^N w(x) = f(x)$, with generally sign-changing right-hand side in a bounded domain. The existence of a least and a greatest continuous viscosity solutions, up to the boundary, is proved through a Perron's construction by means of a strict comparison principle. These extremal solutions are proved to be absolutely extremal solutions.

1. Introduction

In this article, we consider a nonlinear differential operator known as the normalized infinity Laplacian and symbolically defined as

$$-\Delta_\infty^N w(x) = -\frac{1}{|\nabla w(x)|^2}\sum_{i,j=1}^{n} \partial_{x_i} w(x)\partial_{x_j} w(x)\partial^2_{x_i x_j} w(x), \tag{1.1}$$

which is abbreviated as

$$-\Delta_\infty^N w(x) = -\frac{1}{|\nabla w(x)|^2}\langle D^2 w(x)\nabla w(x), \nabla w(x)\rangle. \tag{1.2}$$

Here $\langle\cdot,\cdot\rangle$ denotes the inner product in the Euclidean space $\Re^n$. The expression $\langle D^2 w(x)\nabla w(x), \nabla w(x)\rangle$ stands for the un-normalized infinity Laplacian of w at x, sometimes denoted by $\Delta_\infty w(x)$.

We assume $\Omega \in \Re^n$ is a bounded open set and consider the boundary-value problem

$$\begin{aligned} -\Delta_\infty^N w(x) &= f(x) \quad \text{in } \Omega \\ w(x) &= g(x) \quad \text{on } \partial\Omega. \end{aligned} \tag{1.3}$$

Here we assume that $f \in C(\Omega)$ and $g \in C(\partial\Omega)$. Next, we make clear the meaning of $\Delta_\infty^N w(x)$. For a twice differentiable function φ, the normalized infinity Laplacian

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is

$$\Delta_\infty^N \varphi(x_0) = \begin{cases} \frac{1}{|\nabla\varphi(x_0)|^2}\langle D^2\varphi(x_0)\nabla\varphi(x_0), \nabla\varphi(x_0)\rangle & \text{if } \nabla\varphi(x_0) \neq 0 \\ [\lambda_{\min}(D^2\varphi(x_0)), \lambda_{\max}(D^2\varphi(x_0))] & \text{if } \nabla\varphi(x_0) = 0, \end{cases} \tag{1.4}$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ denote respectively the least and greatest eigenvalues of a square matrix M. Another pair of symbols are $\Delta_\infty^+\varphi$ and $\Delta_\infty^-\varphi$ which are given equivalently by $\Delta_\infty^+\varphi(x) = \Delta_\infty^-\varphi(x) = \Delta_\infty^N\varphi$ if $\nabla\varphi(x) \neq 0$, and $\Delta_\infty^+\varphi(x) = \lambda_{\max}(D^2\varphi(x))$ and $\Delta_\infty^-\varphi(x) = \lambda_{\min}(D^2(\varphi(x)))$ if $\nabla\varphi(x) = 0$. In case $\nabla\varphi(x) = 0$, $\Delta_\infty^N\varphi(x) \geq f(x)$ means that $\lambda_{\max}(D^2\varphi(x)) \geq f(x)$ and $\Delta_\infty^N\varphi(x) \leq f(x)$ means that $\lambda_{\min}(D^2\varphi(x)) \leq f(x)$. Equivalently, $\Delta_\infty^N\varphi(x) \geq f(x)$ means that $\Delta_\infty^+\varphi(x) \geq f(x)$ and $\Delta_\infty^N\varphi(x) \leq f(x)$ means that $\Delta_\infty^-\varphi(x) \leq f(x)$. For a detailed explanation of this definition, we refer to [27].

An upper semi-continuous function ,$u \in USC(\Omega)$, is *a viscosity sub-solution* of the infinity Laplace equation

$$-\Delta_\infty^N u(x) = f(x) \tag{1.5}$$

if the condition $u \prec_{x_0} \varphi$ for $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ implies $-\Delta_\infty^N\varphi(x_0) \leq f(x_0)$. Here $USC(\Omega)$ and $LSC(\Omega)$ denote the sets of upper semi-continuous and lower semi-continuous functions in Ω respectively, and $u \prec_{x_0} \varphi$ means $u - \varphi$ attains a local maximum at x_0. Similarly, *a viscosity super-solution* of (1.5) is a function $u \in LSC(\Omega)$ which satisfies the condition that $\varphi \prec_{x_0} u$ for $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ implies $-\Delta_\infty^N\varphi(x_0) \geq f(x_0)$. *A viscosity solution* of (1.5) is both a viscosity sub-solution and super-solution.

The study of the infinity Laplace equation $\Delta_\infty u(x) = 0$ was initiated in the 1960s by Aronsson in [2, 3, 4], where he deduced the infinity Laplace equation $\Delta_\infty u(x) = 0$ as the Euler-Aronsson equation for smooth absolute minimizers. Partly due to the lack of a proper notion of solutions of the highly degenerate nonlinear infinity Laplace equation, the study had been dormant for quite a while until the introduction of viscosity solutions by Evans, Crandall, Ishii,Lions, et al (see [15] and the references therein). The existence of a solution of the equation was proved by Bhattacharya, DiBenedetto and Manfredi in [7]. Jensen presented the first proof of the uniqueness of a viscosity solution of the Dirichlet problem for the homogeneous infinity Laplace equation in a bounded domain in 1993 in [19], which revived the study of the infinity Laplacian. Since then, the Dirichlet problem for the infinity Laplace equation has received extensive attention. The works [20, 11, 12, 6, 10, 18, 24, 9, 5, 16, 14] give a partial list of the references in the literature. Among them, [6] contains a second proof of the uniqueness of a viscosity solutions of the Dirichlet problem for the homogeneous infinity Laplacian in a bounded domain. A third uniqueness proof is given in [14] which works for unbounded domains. Meanwhile, the study of the eigenvalue problem for the infinity Laplacian and the evolution problem for the infinity Laplacian were also taken up (see [23, 16, 22, 21]). The authors of the current paper investigated in the well-posedness of the inhomogeneous problems $\Delta_\infty u(x) = f(x)$ and $\Delta_\infty^N u(x) = f(x)$, with $f(x) > 0$, in [26, 27], where the existence and uniqueness of a viscosity solution of the Dirichlet problem are proved. Peres-Schramm-Sheffield-Wilson provided interpretation of the normalized infinity Laplacian from the point of view of the differential game theory in [28]. Quoted from [28], the continuum value of a differential game called the "tug-of-war" verifies the inhomogeneous infinity Laplace equation $-\Delta_\infty^N u(x) = 2f(x)$, where f is the running payoff function which satisfies $\inf f > 0$ in the domain. A counter-example was also provided in [28] to show the uniqueness of a viscosity solution of the Dirichlet problem for the inhomogeneous equation fails if f could change sign. It is unclear what one can say about the multiple viscosity solutions of the Dirichlet problem (1.3) for a general "payoff" function f, though. The theme of this paper is to answer at least partially this question. In fact, we prove that there always exist continuous viscosity solutions of the Dirichlet problem (1.3) for the normalized infinity Laplacian and for any continuous right-hand-side f (Theorem 3.1). Moreover, the greatest and least viscosity solutions are constructed (Theorem 3.1) through the Perron's method combined with a strict comparison theorem (Theorem 2.4).

This article is organized as follows. The second section is devoted to the derivation of the local Lipschitz continuity of a viscosity sub-solution (Lemma 2.2) and a strict comparison principle (Theorem 2.4). The third section contains the construction of the least and the greatest solutions, i.e. the main theorem (Theorem 3.1). The last section contains closely related problems yet to be solved.

In this article, especially when the inhomogeneous term f is not continuous in its arguments, the strict differential inequality $-\Delta_\infty^N w(x) < f(x, \nabla w(x))$ in Ω in the viscosity sense is understood in *the locally uniform sense* that for any $x_0 \in \Omega$, there exist a neighborhood N of x_0 in Ω and a $\delta > 0$ such that $-\Delta_\infty^N w(x) \le f(x, \nabla w(x)) - \delta$ in the viscosity sense in N. The differential inequality $-\Delta_\infty^N w(x) > f(x, \nabla w(x))$ is similarly understood.

We recall [27, Lemma 1.10], the proof of which may also be found therein.

Lemma 1.1. *Assume Ω is an open subset of $\Re^n$ and $f \in C(\Omega)$. Λ is an index set.*

(a) Suppose $u(x) = \sup_{\lambda\in\Lambda} u_\lambda(x) < \infty$, $x \in \Omega$, where $-\Delta_\infty^N u_\lambda \le f$ in Ω in the viscosity sense for every $\lambda \in \Lambda$. If $u \in C(\Omega)$, then $-\Delta_\infty^N u \le f$ in Ω in the viscosity sense.

(b) Similarly, if $u(x) = \inf_{\lambda\in\Lambda} u_\lambda(x) > -\infty$, $x \in \Omega$, where $-\Delta_\infty^N u_\lambda \ge f$ in Ω in the viscosity sense for every $\lambda \in \Lambda$. Then $u \in C(\Omega)$ implies that $-\Delta_\infty^N u \ge f$ in Ω in the viscosity sense.

A similar result holds for the infinity Laplace equation $-\Delta_\infty u = f$, the proof of which is simpler as the singularity caused by $\nabla u = 0$ does not present in this case.

2. A Comparison Theorem

For a nonzero vector x, $\hat{x} = x/|x|$ denotes its normalized vector. The notation $C_b(\Omega)$ denotes the set of bounded continuous functions defined in Ω. For two sets V and U, $V \subset\subset U$ means that V is compactly contained in U.

We start out to prove a lifting lemma stated as follows.

Lemma 2.1. *If $u \in USC(\Omega)$ is a viscosity sub-solution of $\Delta_\infty u = k_1$ in Ω, and $v \in C^2(\Sigma)$ verifies $\Delta_\infty v \ge k_2$ in Σ, for constants k_1 and k_2, then the function $w : (x, y) \mapsto u(x) + v(y)$ is a viscosity sub-solution of $\Delta_\infty w(x, y) = k_1 + k_2$ in $\Omega \times \Sigma$.*

Proof. It suffices to prove $\Delta_\infty \varphi(x_0, y_0) \ge k_1 + k_2$ for any $\varphi \in C^2(\Omega)$ and $(x_0, y_0) \in \Omega$ such that $u(x)+v(y) \prec_{(x_0,y_0)} \varphi(x, y)$. Without the loss of generality, we may assume $(x_0, y_0) = (0, 0)$, $u(0) = 0$, $v(0) = 0$, $\varphi(0, 0) = 0$, and φ is a quadratic polynomial.

Denote

$$\nabla\varphi(0,0) = \begin{pmatrix} \varphi_x(0,0) \\ \varphi_y(0,0) \end{pmatrix}$$

and

$$D^2\varphi(0,0) = \begin{pmatrix} \varphi_{xx}(0,0) & \varphi_{xy}(0,0) \\ \varphi_{yx}(0,0) & \varphi_{yy}(0,0) \end{pmatrix}$$

Then

$$\begin{aligned} u(x) + v(y) \le \varphi_x(0,0)\cdot x + \varphi_y(0,0)\cdot y + \frac{1}{2}\langle \varphi_{xx}(0,0)x, x\rangle \\ + \langle \varphi_{xy}(0,0)x, y\rangle + \frac{1}{2}\langle \varphi_{yy}(0,0)y, y\rangle. \end{aligned} \tag{2.1}$$

We write

$$v(y) = \nabla v(0)\cdot y + \frac{1}{2}\langle D^2v(0)y, y\rangle + \circ(|y|^2).$$

Replacing this in (2.1), we obtain

$$\begin{aligned} &u(x) + \nabla v(0)\cdot y + \frac{1}{2}\langle D^2v(0)y, y\rangle + \circ(|y|^2) \\ &\le \varphi_x(0,0)\cdot x + \varphi_y(0,0)\cdot y + \frac{1}{2}\langle \varphi_{xx}(0,0)x, x\rangle \\ &\quad + \langle \varphi_{xy}(0,0)x, y\rangle + \frac{1}{2}\langle \varphi_{yy}(0,0)y, y\rangle \end{aligned}$$

or equivalently

$$\begin{aligned} u(x) \le \varphi_x(0,0)\cdot x + (\varphi_y(0,0) - \nabla v(0))\cdot y + \frac{1}{2}\langle \varphi_{xx}(0,0)x, x\rangle \\ + \langle \varphi_{xy}(0,0)x, y\rangle + \frac{1}{2}\langle (\varphi_{yy}(0,0) - D^2v(0))y, y\rangle + \circ(|y|^2) \end{aligned}$$

for any small x and y.

It is clear that $\varphi_y(0,0) = \nabla v(0)$ and $\varphi_{yy}(0,0) - D^2v(0) \ge 0$. Denote $B = \varphi_{yy}(0,0) - D^2v(0)$. Then

$$\begin{aligned} u(x) \le \varphi_x(0,0)\cdot x + \frac{1}{2}\langle \varphi_{xx}(0,0)x, x\rangle \\ + \langle \varphi_{xy}(0,0)x, y\rangle + \frac{1}{2}\langle By, y\rangle + \circ(|y|^2). \end{aligned} \tag{2.2}$$

First, we assume the matrix B is invertible. So $B = A^2$ for some symmetric invertible matrix A. Then the right-hand-side of (2.2) is equal to

$$\begin{aligned} &\varphi_x(0,0)\cdot x + \frac{1}{2}\langle \varphi_{xx}(0,0)x, x\rangle + \langle A^{-1}\varphi_{xy}(0,0)x, Ay\rangle \\ &+ \frac{1}{2}\langle Ay, Ay\rangle + \circ(|y|^2) \\ &= \varphi_x(0,0)\cdot x + \frac{1}{2}\langle (\varphi_{xx}(0,0) - \varphi_{xy}(0,0)B^{-1}\varphi_{xy}(0,0))x, x\rangle \\ &\quad + \frac{1}{2}|A^{-1}\varphi_{xy}(0,0)x + Ay|^2 + \circ(|y|^2), \end{aligned} \tag{2.3}$$

where x and y are any small vectors. Take $y = -B^{-1}\varphi_{xy}(0,0)x$ for each small x. Then

$$u(x) \le \varphi_x(0,0)\cdot x + \frac{1}{2}\langle (\varphi_{xx}(0,0) - \varphi_{xy}(0,0)B^{-1}\varphi_{xy}(0,0))x, x\rangle + \circ(|x|^2)$$

for all small vector x. Therefore, on account of the fact that $\Delta_\infty u \geq k_1$ in Ω, we obtain

$$\langle(\varphi_{xx}(0,0) - \varphi_{xy}(0,0)B^{-1}\varphi_{xy}(0,0))\varphi_x(0,0), \varphi_x(0,0)\rangle \geq k_1. \tag{2.4}$$

As a result, the following equalities and inequalities hold at $(0,0)$:

$$\begin{aligned}
&\langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle\varphi_{yy}\varphi_y, \varphi_y\rangle \\
&= \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle B\varphi_y, \varphi_y\rangle + \langle D^2 v\nabla v, \nabla v\rangle \\
&\geq \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle B\varphi_y, \varphi_y\rangle + k_2 \\
&= \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle A^{-1}\varphi_{xy}\varphi_x, A\varphi_y\rangle + \langle A\varphi_y, A\varphi_y\rangle + k_2 \\
&= \langle(\varphi_{xx} - \varphi_{xy}B^{-1}\varphi_{xy})\varphi_x, \varphi_x\rangle + |A^{-1}\varphi_{xy}\varphi_x + A\varphi_y|^2 + k_2 \\
&\geq k_1 + k_2,
\end{aligned}$$

according to (2.4). In general, when B is not invertible, we define $B^\varepsilon = B + \varepsilon I$ for every small $\varepsilon > 0$. Then B^ε is invertible and the inequalities (2.2) and (2.4) still hold with B replaced by B^ε. Let $B^\varepsilon = A^2$ for a positive definite matrix A. In the end, we have, at $(0,0)$,

$$\begin{aligned}
&\langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle\varphi_{yy}\varphi_y, \varphi_y\rangle + \varepsilon|\varphi_y|^2 \\
&= \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle B^\varepsilon\varphi_y, \varphi_y\rangle + \langle D^2 v\nabla v, \nabla v\rangle \\
&\geq \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2\langle\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle B^\varepsilon\varphi_y, \varphi_y\rangle + k_2 \\
&= \langle\varphi_{xx}\varphi_x, \varphi_x\rangle + 2 < A^{-1}\varphi_{xy}\varphi_x, A\varphi_y\rangle + \langle A\varphi_y, A\varphi_y\rangle + k_2 \\
&= \langle(\varphi_{xx} - \varphi_{xy}(B^\varepsilon)^{-1}\varphi_{xy})\varphi_x, \varphi_x\rangle + |A^{-1}\varphi_{xy}\varphi_x + A\varphi_y|^2 + k_2 \\
&\geq k_1 + k_2,
\end{aligned}$$

for every small $\varepsilon > 0$. Then $\langle\varphi_{xx}\varphi_x, \varphi_x\rangle + \langle 2\varphi_{xy}\varphi_x, \varphi_y\rangle + \langle\varphi_{yy}\varphi_y, \varphi_y\rangle \geq k_1 + k_2$. The proof is complete. □

Lemma 2.2. *Suppose $f \in C(\Omega)$, and $w \in USC(\Omega)$ is locally bounded.*

(a) *If $-\Delta_\infty w(x) \leq f(x)$ in Ω, then $w : \Omega \to \Re$ is locally Lipschitz continuous.*

(b) *If $-\Delta_\infty^N w(x) \leq f(x)$ in Ω, then $w : \Omega \to \Re$ is locally Lipschitz continuous.*

Furthermore, the Lipschitz constant of w over $\Omega' \subset\subset \Omega$ may be taken as $C(1 + \|w\|_{L^\infty(\tilde\Omega)})$, where $\Omega' \subset\subset \tilde\Omega \subset\subset \Omega$ and C depends on $\|f\|_{L^\infty(\Omega')}$.

Proof. Assume $f(x) \leq K$, $x \in \Omega' \subset\subset \Omega$. Without loss of generality, we also assume $\|w\|_{L^\infty(\Omega)} < \infty$.

(a) Define $u(x,y) = w(x) + Cy^{4/3}$, for $x \in \Omega$, $1 \leq y \leq 2$. We notice that $Cy^{4/3}$ is a C^2 solution of the equation $\Delta_\infty w(y) = \frac{64}{81}C^3$ for $y \neq 0$. Then the preceding lift lemma (2.1) implies that u is an infinity sub-harmonic function in $\Omega \times (1,2)$, if C is sufficiently large. A well-known fact about semi-continuous infinity sub-harmonic functions (see, for example, [5, Lemma 2.9] for continuous functions, and [12], or [25] for semi-continuous functions) states that u is Lipschitz continuous on $\Omega' \times [1.1, 1.9]$ with some Lipschitz constant $L = L(\|w\|_{L^\infty(\Omega)})$. As a result, w is Lipschitz continuous on Ω' with Lipschitz constant L.

(b) Clearly, that $-\Delta_\infty^N w(x) \leq f(x)$ in Ω in the viscosity sense implies that $-\Delta_\infty w(x) \leq |\nabla w(x)|^2 f(x)$ in the viscosity sense in Ω (but not the converse). Without the loss of generality, we assume that $w > 0$ in Ω. Take $\lambda > 0$ small so that $\lambda\|w\|_{L^\infty(\Omega)} < \frac{1}{2}$. Define $u = G(w) = w + \frac{\lambda}{2}w^2$ in Ω. For simplicity, we assume

w is C^2. All steps of the following computation can be made rigorous by means of viscosity solutions. We leave the details to the reader. Then $G'(w) = 1 + \lambda w$ and $G''(w) = \lambda$. In particular, $2 > G'(w) > \frac{1}{2}$. Moreover, $\nabla u = G'(w)\nabla w$, $D^2u = G'(w)D^2w + G''(w)\nabla w \otimes \nabla w$, and

$$\begin{aligned}
-\Delta_\infty u &= -(G'(w))^3\Delta_\infty w - (G'(w))^2G''(w)|\nabla w|^4 \\
&\le (G'(w))^3\{f(x) - \frac{G''(w)}{G'(w)}|\nabla w|^2\}|\nabla w|^2 \\
&= \frac{(1+\lambda w)^4}{\lambda}\{f(x) - \frac{\lambda}{1+\lambda w}|\nabla w|^2\}\frac{\lambda|\nabla w|^2}{1+\lambda w} \\
&\le \frac{(1+\lambda w)^4}{4\lambda}(f(x))^2, \quad \text{(due to the Cauchy-Schwarz inequality)} \\
&< \frac{4K^2}{\lambda}
\end{aligned}$$

By (a), one deduces that u is locally Lipschitz continuous in Ω. So $w = \frac{2u}{1+\sqrt{1+2\lambda u}}$ is also locally Lipschitz continuous in Ω. □

The following comparison theorem is a generalization of a strict comparison principle stated in [27, Theorem 3.1].

Theorem 2.3. *Assume $f \in C(\Omega \times \Re^n)$, and the modulus of continuity of the function $x \mapsto f(x,p)$ is independent of $p \in \Re^n$. Suppose $u_j \in C(\Omega)$, $j = 1, 2$, verify in the viscosity sense either*

$$\Delta_\infty^N u_1(x) < f(x, \nabla u_1) \quad \textit{and} \quad \Delta_\infty^N u_2(x) \ge f(x, \nabla u_2)$$

or

$$\Delta_\infty^N u_1(x) \le f(x, \nabla u_1) \quad \textit{and} \quad \Delta_\infty^N u_2(x) > f(x, \nabla u_2)$$

in Ω.

If $\limsup_{x\in\Omega\to z}(u_2(x) - u_1(x)) \le 0$ for any $z \in \partial\Omega$, then $u_2(x) \le u_1(x)$ in Ω.

Proof. One may follow the proof of [27, Theorem 3.1] which can be simplified substantially with the application of a fourth order penalty function $w_\varepsilon(x,y) = u_2(x) - u_1(y) - \frac{1}{4\varepsilon}|x-y|^4$, $(x,y) \in \Omega \times \Omega$, used in [13] and [22]. We leave the details to the reader. □

The preceding comparison theorem and the lemma (2.2) imply the following theorem immediately.

Theorem 2.4. *Assume $f \in C(\Omega)$. Suppose $u_1 \in LSC(\Omega)$, $u_2 \in USC(\Omega)$, and they verify either*

$$\Delta_\infty^N u_1(x) < f(x) \quad \textit{and} \quad \Delta_\infty^N u_2(x) \ge f(x)$$

or

$$\Delta_\infty^N u_1(x) \le f(x) \quad \textit{and} \quad \Delta_\infty^N u_2(x) > f(x)$$

in the viscosity sense in Ω.

If $\limsup_{x\in\Omega\to z}(u_2(x) - u_1(x)) \le 0$ for any $z \in \partial\Omega$, then $u_2(x) \le u_1(x)$ in Ω.

3. Continuous Solutions Of The Dirichlet Problem

There are different approaches to the existence of a viscosity solution of the boundary value problem (1.3). The approach used here is the Perron's method combined with a delicate albeit elementary analysis which depends essentially on the strict comparison theorem (2.4).

For $f \in C(\Omega)$ and $g \in C(\partial\Omega)$, we define the set of strict super-solutions

$$\mathcal{A}^+_{f,g} = \{v \in C(\bar{\Omega}) : -\Delta^N_\infty v(x) > f(x) \text{ in } \Omega, \text{ and } v \geq g \text{ on } \partial\Omega\} \tag{3.1}$$

and the set of strict sub-solutions

$$\mathcal{A}^-_{f,g} = \{v \in C(\bar{\Omega}) : -\Delta^N_\infty v(x) < f(x) \text{ in } \Omega, \text{ and } v \leq g \text{ on } \partial\Omega\}. \tag{3.2}$$

Whenever there is little confusion, we will write $\mathcal{A}^+$ and $\mathcal{A}^-$ for $\mathcal{A}^+_{f,g}$ and $\mathcal{A}^-_{f,g}$, respectively. Obviously, $\mathcal{A}^+$ and $\mathcal{A}^-$ are both nonempty.

Define $w^+(x) = \inf_{v \in \mathcal{A}^+} v(x)$ and $w^-(x) = \sup_{v \in \mathcal{A}^-} v(x)$, $x \in \bar{\Omega}$. By definition, w^+ is upper semi-continuous and w^- is lower semi-continuous on $\bar{\Omega}$. Obviously, w^+ and w^- are both bounded on $\bar{\Omega}$, and $w^-(x) \leq g(x) \leq w^+(x)$ on $\partial\Omega$ according to the preceding comparison theorem (2.4).

Take a super-solution ϕ in $\mathcal{A}^+_{f,g}$. For example, $\phi(x) = -C|x-z|^2 + D$ for suitable C and D. Define

$$\mathcal{A}^+_{f,g,\phi} = \{\min(v, \phi) : v \in \mathcal{A}^+_{f,g}\}. \tag{3.3}$$

Clearly, $\mathcal{A}^+_{f,g,\phi} \subset \mathcal{A}^+_{f,g}$ and $w^+(x) = \inf_{v \in \mathcal{A}^+_{f,g,\phi}} v(x)$. For every v in $\mathcal{A}^+_{f,g,\phi}$, Lemma (2.2) says v is locally Lipschitz continuous with Lipschitz constant $\leq C(1 + \|\phi\|_{L^\infty(\Omega)})$, i. e. $\mathcal{A}^+_{f,g,\phi}$ is locally Lipschitz equi-continuous.

On the other hand, one may pick a sequence $\{v_k\}$ in $\mathcal{A}^+_{f,g,\phi}$ such that v_k converges to w^+ on a countable dense subset E of $\bar{\Omega}$. Define $\tilde{v}_k = \min\{v_1, v_2, \dots, v_k\}$, $k = 1, 2, \dots$. Then $\tilde{v}_k \in \mathcal{A}^+_{f,g,\phi}$ and $\tilde{v}_k$ converges to w^+ on E. Replacing v_k by $\tilde{v}_k$, one may assume that $v_k \geq v_{k+1}$ for all k. Consequently, a subsequence of $\{v_k\}$, which will still be denoted by $\{v_k\}$, converges to some v locally uniformly on Ω. Then $v \in C(\Omega)$ and $v_k \geq v \geq w^+$ on Ω. Clearly, $v = w^+$ on $E \cap \Omega$. As w^+ is upper semi-continuous on $\bar{\Omega}$, for any $x \in \Omega$,

$$w^+(x) \geq \limsup_{z \in E \to x} w^+(z) = \limsup_{z \in E \to x} v(z) = v(x). \tag{3.4}$$

So $w^+ = v$ on Ω and whence $\{v_k\}$ converges to w^+ locally uniformly on Ω. Therefore $w^+ \in C(\Omega)$ and $-\Delta^N_\infty w^+(x) \geq f(x)$ on Ω on account of Lemma (1.1). Similarly, $w^- \in C(\Omega)$ and $-\Delta^N_\infty w^-(x) \leq f(x)$ on Ω.

Next, we show $w^+ = w^- = g$ on $\partial\Omega$. For any $z \in \partial\Omega$ and any $\varepsilon > 0$, there exists $r > 0$ such that $g(x) \leq g(z) + \varepsilon$ for all $x \in \partial\Omega$ with $|x - z| \leq r$. Take $C > \|f\|_{L^\infty(\Omega)}$ and $D \geq C diam(\Omega) + 2\|g\|_{L^\infty(\partial\Omega)}$. Define $v \in C(\bar{\Omega})$ by

$$v(x) = g(z) + \varepsilon - C|x - z|^2 + D|x - z|. \tag{3.5}$$

Then $-\Delta^N_\infty v(x) = 2C > f(x)$ for $x \in \Omega$. For $x \in \partial\Omega$ with $|x - z| \leq r$, $v(x) \geq g(z) + \varepsilon + |x - z|\{D - Cr\} \geq g(z) + \varepsilon \geq g(x)$, while for $x \in \partial\Omega$ with $|x - z| > r$, $v(x) \geq g(z) + \varepsilon + r\{D - C\operatorname{diam}(\Omega)\} \geq g(z) + \varepsilon + 2\|g\|_{L^\infty(\partial\Omega)} \geq g(x)$. So $v \in \mathcal{A}^+_{f,g}$. As a result, $w^+(z) \leq v(z) = g(z) + \varepsilon$, for all $\varepsilon > 0$. So $w^+ = g$ on $\partial\Omega$. Similarly $w^- = g$ on $\partial\Omega$.

Since w^+ is upper semi-continuous and w^- is lower semi-continuous on $\bar{\Omega}$, for any $z \in \partial\Omega$, the following inequalities hold

$$g(z) = w^+(z) \geq \limsup_{x\in\Omega\to z} w^+(x) \geq \liminf_{x\in\Omega\to z} w^+(x) \geq \liminf_{x\in\Omega\to z} w^-(x) \geq w^-(z) = g(z). \tag{3.6}$$

Consequently, all the above inequalities are indeed equalities. So

$$\lim_{x\in\Omega\to z} w^+(x) = w^+(z) = g(z). \tag{3.7}$$

Similarly, one obtains

$$\lim_{x\in\Omega\to z} w^-(x) = w^-(z). \tag{3.8}$$

Therefore w^+ and w^- are in $C(\bar{\Omega})$.

We now show that $-\Delta_\infty^N w^+(x) = f(x)$ and $-\Delta_\infty^N w^-(x) = f(x)$ in Ω. We need only prove that $-\Delta_\infty^N w^+(x) \leq f(x)$ in Ω. Suppose the contrary that there exist a C^2 function φ and a point $x_0 \in \Omega$ such that $w^+ \prec_{x_0} \varphi$ and $\Delta_\infty^+\varphi(x_0) < f(x_0)$.

For any small $\varepsilon > 0$, we define

$$\varphi_\varepsilon(x) = \varphi(x_0) + \nabla\varphi(x_0)\cdot(x-x_0) + \frac{1}{2}\langle D^2\varphi(x_0)(x-x_0), x-x_0\rangle + \varepsilon|x-x_0|^2 \tag{3.9}$$

so that x_0 is a strict local maximum point of $w^+ - \varphi_\varepsilon$. We claim that $-\Delta_\infty^+\varphi_\varepsilon(x) < f(x)$ for all x sufficiently close to x_0 if ε is small enough.

In fact, if $\nabla\varphi(x_0) \neq 0$, then $\nabla\varphi(x) \neq 0$ in a neighborhood of x_0, and in this neighborhood,

$$\Delta_\infty^+\varphi_\varepsilon(x) = \langle D^2\varphi_\varepsilon(x)\hat{\nabla}\varphi_\varepsilon(x), \hat{\nabla}\varphi_\varepsilon(x) >= \Delta_\infty^+\varphi(x_0) + O(\varepsilon). \tag{3.10}$$

The claim follows from the continuity of $\Delta_\infty^+\varphi$ and f.

If $\nabla\varphi(x_0) = 0$, then $\lambda_{\max}(D^2\varphi(x_0)) = \Delta_\infty^+\varphi(x_0) < f(x_0)$. As $\lambda_{\max}(D^2\varphi_\varepsilon(x)) \leq \lambda_{\max}(D^2\varphi(x)) + C\varepsilon$,

$$\Delta_\infty^+\varphi_\varepsilon(x) \leq \lambda_{\max}(D^2\varphi_\varepsilon(x)) < f(x) \tag{3.11}$$

holds for x sufficiently close to x_0.

We take $\delta > 0$ small enough so that the function $\hat{\varphi}(x) := \varphi_\varepsilon(x) - \delta$ satisfies $\hat{\varphi} < w^+$ in a neighborhood of x_0 which is contained in the set $\{x \in \Omega : \Delta_\infty^+\varphi_\varepsilon(x) < f(x)\}$, and $\hat{\varphi} \geq w^+$ outside this neighborhood of x_0.

We know from the previous part of the proof that there exists a sequence $\{v_k\}$ in $\mathcal{A}_{f,g}^+$ that converges to w^+ locally uniformly in Ω. Therefore there is an element v of $\mathcal{A}_{f,g}^+$ such that $\hat{\varphi} < v$ in a neighborhood N of x_0 which is a subset of the set $\{x \in \Omega : \Delta_\infty^+\varphi_\varepsilon(x) < f(x)\}$, and $\hat{\varphi} \geq v$ outside N and in some $\Omega' \subset\subset \Omega$, if δ is taken smaller as needed. We may without loss of generality modify the values of $\hat{\varphi}$ near $\partial\Omega$ so that $\hat{\varphi} \geq v$ in $\Omega\backslash N$.

Take $\hat{v} = \min\{\hat{\varphi}, v\}$. Then $\hat{v} = \hat{\varphi}$ in the neighborhood N of x_0 and $\hat{v} = v$ elsewhere. So $\hat{v} \in \mathcal{A}_{f,g}^+$. But $\hat{v} = \hat{\varphi} < w^+$ in a neighborhood of x_0, which is a contradiction to the definition of w^+. So $-\Delta_\infty^N w^+(x) \leq f(x)$ in Ω. Similarly, $-\Delta_\infty^N w^-(x) \geq f(x)$ in Ω.

Furthermore, the comparison theorem (2.4) implies that for any solution $w \in C(\bar{\Omega})$ of the Dirichlet problem

$$\begin{gathered} -\Delta_\infty^N w(x) = f(x) \quad \text{in } \Omega \\ w(x) = g(x) \quad \text{on } \partial\Omega \end{gathered}$$

$w^- \le w \le w^+$ holds on $\bar{\Omega}$, as it holds in $\bar{\Omega}$ that $v_2 \le w \le v_1$ for any $v_1 \in \mathcal{A}^+$ and any $v_2 \in \mathcal{A}^-$. We have proved the following existence theorem.

Theorem 3.1. *There exists at least one solution in $C(\bar{\Omega})$ of the boundary value problem* (1.3). *Every continuous solution of* (1.3) *is locally Lipschitz continuous in Ω. Among all the continuous solutions of the boundary value problem* (1.3)*, there are one least solution w^- and one greatest solution w^+ as constructed above.*

Furthermore, we can acquire a clearer picture of the set of continuous solutions of the Dirichlet problem (1.3) by inspecting the solutions in the following *absolute* way. First, the construction of w^+ and w^- and the above theorem imply the following theorem.

Theorem 3.2. *For any open set $V \subset\subset \Omega$, if $w \in C(\bar{V})$ satisfies*

$$\begin{gathered} -\Delta_\infty^N w(x) = f(x) \quad (x \in V) \\ w(x) = w^+(x) \quad (x \in \partial V), \end{gathered} \tag{3.12}$$

then $w \le w^+$ in $\bar{V}$.

Similarly, if $w \in C(\bar{V})$ satisfies

$$\begin{gathered} -\Delta_\infty^N w(x) = f(x) \quad (x \in V) \\ w(x) = w^-(x) \quad (x \in \partial V), \end{gathered} \tag{3.13}$$

then $w \ge w^-$ in $\bar{V}$.

Proof. According to the preceding Theorem 3.1, we may assume that w is the greatest solution in the region V with boundary data w^+ on ∂V. Then $w^+ \le w$ on $\bar{V}$. We need to prove the reverse inequality $w \le w^+$. Define $\tilde{w}$ on $\bar{\Omega}$ by

$$\tilde{w}(x) = \begin{cases} w(x), & x \in V \\ w^+(x), & x \in \bar{\Omega}\backslash V. \end{cases}$$

Then $-\Delta_\infty^N \tilde{w}(x) \le f(x)$ in Ω in the viscosity sense. In fact, if $\tilde{w} \prec_{x_0} \varphi$ for a point $x_0 \in \Omega$ and a C^2 function φ, and if $x_0 \notin \partial V$, then clearly $-\Delta_\infty^N \varphi(x_0) \le f(x_0)$. If $x_0 \in \partial V$, then $w^+ \prec_{x_0} \varphi$ as $w^+ \le w$ in V and $w^+ = w$ on ∂V. As a result, $-\Delta_\infty^N \varphi(x_0) \le f(x_0)$ holds.

For any $v \in \mathcal{A}_{f,g}^+$, Theorem 2.4 implies that $v \ge \tilde{w}$. Consequently, $w^+(x) \ge \tilde{w}(x)$, $x \in \Omega$, and in particular $w^+(x) \ge w(x)$, $x \in \bar{V}$.

The proof of the second part is similar. □

Define the set of viscosity solutions of the Dirichlet problem (1.3) by

$$\mathcal{A}_{f,g} = \{u \in C(\bar{\Omega}) : -\Delta_\infty^N u(x) = f(x) \text{ in } \Omega, \text{ and } u = g \text{ on } \partial\Omega.\} \tag{3.14}$$

According to the preceding theorem (3.2), w^+ and w^- are the extremal solutions in $\mathcal{A}_{f,h}$ in an absolute sense as mentioned above.

We conclude this section with a lemma which will be used in the next section. The proof of the following partial continuity of the infinity Laplacian lemma is straightforward if one observes that if $\nabla\varphi(x_0) = 0$ for a smooth function φ, then $\Delta_\infty^+ \varphi(x_0) = \lambda_{\max}(D^2\varphi(x_0))$.

Lemma 3.3. *Suppose φ is a C^2 function, and $x_k \to x_0$.*

(i) *If $\nabla\varphi(x_0) \ne 0$, then $\Delta_\infty^N \varphi(x_k) \to \Delta_\infty^N \varphi(x_0)$.*
(ii) *If $\nabla\varphi(x_0) = 0$, then $\Delta_\infty^+ \varphi(x_0) \ge \limsup_k \Delta_\infty^+ \varphi(x_k)$.*

Remark: In Lemma 3.1(ii), the inequality holds obviously. In many cases, the inequality is indeed an equality. However, in general, the equality is not true. For example, in 2D, take $\varphi(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2$. Then $\Delta_\infty^+\varphi(0,0) = 1$ but $\Delta_\infty^+\varphi(x,y) = \frac{x^2-y^2}{x^2+y^2}$ does not necessarily converge to 1 as $(x,y) \to (0,0)$.

4. Unanswered Questions

Following the proof of the existence of the maximum and minimum solutions of the Dirichlet problem (1.3) with non-trivial right-hand-side in this work, some closely related problems need to be answered.

Naturally, one would ask when the uniqueness of a viscosity solution of the Dirichlet problem (1.3) holds even if f changes sign. More precisely, what is the necessary and sufficiency condition on f (and possibly on g as well) and on the domain Ω that ensures the Dirichlet problem (1.3) has a unique continuous solution? Are there always more than one viscosity solutions of the Dirichlet problem if f changes sign? A recent work by Armstrong and Smart, [1], answered part of the questions. Interested reader may read their work for up-to-date development.

One may also ask at most how many distinct solutions can the Dirichlet problem (1.3) have for any non-trivial right-hand-side? Under what condition are there infinitely many solutions? In case there exist multiple solutions, what is the structure of the set of the continuous solutions of the Dirichlet problem (1.3)? Do the extremal solutions w^+ and w^- determine all the solutions of the Dirichlet problem in some way? Or parallelly, "What is a criterion for a continuous function to be an element of $\mathcal{A}_{f,g}$?"

We will be more precise in our notations below and hope that the following discussion will justify our use of multiple subscripts. Let $\mathcal{A}_{f,g}(\Omega)$ denote the set of the viscosity solutions of the Dirichlet problem (1.3) in a bounded open set Ω. $w^+_{f,g,\Omega}$ and $w^-_{f,g,\Omega}$ denote the maximum and minimum solutions in $\mathcal{A}_{f,g}(\Omega)$. The following theorem is a criterion which is not quite up to the authors' satisfaction in that it depends on the maximum and minimum solutions for every open subset and does not give enough information about the solution u solely in terms of $w^+_{f,g,\Omega}$ and $w^-_{f,g,\Omega}$.

Theorem 4.1. *Suppose $u \in C(\Omega)$. Then $-\Delta_\infty^N u(x) = f(x)$ in Ω if and only if for every open set $V \subset\subset \Omega$,*

$$w^-_{f,g,V}(x) \le u(x) \le w^+_{f,g,V}(x), \quad \text{for } \; x \in V,$$

where $g = u|_{\partial V}$.

Proof. The necessity follows from the Theorem (3.1).

To show the sufficiency, we only prove $-\Delta_\infty^N u \le f$ in Ω, as the proof of $-\Delta_\infty^N u \ge f$ is similar. Suppose $u \prec_{x_0} \varphi$ for some $x_0 \in \Omega$ and some C^2 function φ. For any small $r > 0$, let $V = B_r(x_0)$ and w_r^+ be the maximum solution of the Dirichlet problem in V. As $w_r^+ \ge u$ in V and $w_r^+ = u$ on ∂V, it is clear that $w_r^+ \prec_{x_r} \varphi$ for some point $x_r \in V$. So $-\Delta_\infty^N\varphi(x_r) \le f(x_r)$. Sending r to 0, one obtains $-\Delta_\infty^N\varphi(x_0) \le f(x_0)$ on account of the continuity of f and the smoothness of φ, noticing the fact that $-\lambda_{\max}(D^2\varphi(x_0)) \le \liminf_{r\downarrow 0} -\Delta_\infty^N\varphi(x_r)$ if $\nabla\varphi(x_0) = 0$. $\square$

Clearly, $u \in C(\bar\Omega)$ is an element of $\mathcal{A}_{f,g}(\Omega)$ if and only if u verifies the condition stated in the preceding theorem and $u = g$ on $\partial\Omega$. On the other hand, it is unknown if the comparison property $\sup_V(u - w^+_{f,g,\Omega}) \le \max_{\partial V}(u - w^+_{f,g,\Omega})$ and $\inf_V(u - w^-_{f,g,\Omega}) \ge \min_{\partial V}(u - w^-_{f,g,\Omega})$ for every open subset $V \subset \Omega$ alone implies $u \in \mathcal{A}_{f,g}(\Omega)$.

In addition, can we anticipate a differential game theory interpretation of the Dirichlet problem (1.3) with the nontrivial right-hand-side f as we do with the case $\sup_\Omega f(x) < 0$ ([28], [8] and [17])? This question has been partially answered by Armstrong and Smart in [1]. Furthermore, one may still ask the questions such as "Are there any connections between the maximal and minimal solutions and the value functions of the players II and I in the generalized 'tug-of-war' game?"

In the end, one may also consider the inverse problem of the Dirichlet problem (1.3), "For what continuous functions u, are there continuous functions f such that $-\Delta^N_\infty u = f$?" The uniqueness of f was initially considered in [28] and has recently been proved by Y. Yu ([29]).

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