Electronic Journal of Differential Equations, Vol. 2010(2010), No. 83, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF WEAK SOLUTIONS FOR MIXED PROBLEMS OF PARABOLIC SYSTEMS

PHAM TRIEU DUONG, DO VAN LOI

ABSTRACT. The purpose of this paper is to investigate the existence of generalized solutions for strongly parabolic systems in a cylindrical domain. The decay of the solution at infinity, depending on the right-hand of the equation, is also studied in this article.

1. Introduction

The study of boundary value problems for non-stationary systems of PDE's in arbitrary domains differs from the study of stationary systems. In previous works of the first author [4, 5], the existence of solutions for parabolic system has been obtained by a Galerkin's approximation scheme. Following the method in [1], which have been successfully applied for studying the Stokes equation, we prove in this article the existence and uniqueness of solutions for general parabolic systems. This approach allows us to study the decay of solutions for large time for bounded and for unbounded domains. However, from these proofs it is not clear how to find the solutions.

This paper is organized as follows. In the second section we introduce the necessary notation and functional spaces for our problem. The third section is devoted to formulate and prove the main theorem. The last section is intended to conduct the research on the asymptotic behavior of solution at infinity.

2. Notation

Let Ω be a domain in $\mathbb{R}^n$ and T be arbitrary, $0 < T \leq \infty$. We denote by Q_T the cylinder $\Omega \times (0, T)$. We consider the differential operator

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p(a_{pq}D^q),$$

where a_{pq} are bounded functions in $C^\infty(Q_T)$ together with $\frac{\partial a_{pq}}{\partial t}$. For $|p| = |q| = m$ we assume the condition $a_{pq} = a^\star_{qp}$, where the asterisk denotes the transposed complex conjugate of a matrix.

2000 *Mathematics Subject Classification.* 35K35.
Key words and phrases. Parabolic system; generalized solution; asymptotic behavior; Garding's inequality.
©2010 Texas State University - San Marcos.
Submitted May 20, 2010. Published June 18, 2010.
Supported by National Foundation for Science and Technology Development (Nafosted).

The operator L is strongly elliptic. Then there exists a constant $C > 0$ such that for all real vectors $\xi \in \mathbb{R}^n$ and all complex vectors $\eta \in \mathbb{C}^s$

$$\sum_{|p|,|q|=m} a_{pq}\xi^p\xi^q\eta\overline{\eta} \geq C|\xi|^{2m}|\eta|^2 \quad \forall (x,t) \in Q_T\,. \tag{2.1}$$

The spaces $H^m(\Omega), H^{m,k}(Q_T)$ are the usual Sobolev spaces, where m, k denote the order of derivatives with respect to x and t respectively. In addition $H_0^{m,k}(Q_T)$ is the completion with respect to $H^{m,k}(Q_T)$ norm of functions from $C^\infty(Q_T)$ which vanish near the lateral surface $S_T = \partial\Omega \times (0,T)$.

In this study, we consider the initial-boundary value problem (IBVP) in the cylinder Q_T for the system of PDE's:

$$Pu \equiv u_t + (-1)^m L(t,x,D_x)u = f(x,t) \quad \text{in } Q_T, \tag{2.2}$$

$$\frac{\partial^j u}{\partial \nu^j} = 0, \quad 0 \leq j \leq m-1 \quad \text{on } S_T, \tag{2.3}$$

$$u(x,0) = \varphi(x) \tag{2.4}$$

where ν is the unit outer normal vector to the lateral surface S_T. Here $f(x,t)$ and $\varphi(x)$ are given functions.

The coercivity of the bilinear form associated with the operator $\mathcal{L}$ is a direct consequence of Garding's inequality which is formulated as follows.

Proposition 2.1. *Assume that the coefficients of L satisfy the condition* (2.1) *on Q_T. Moreover, we suppose that a_{pq} are continuous functions on the x variable, uniformly with respect to $t \in (0,T)$. Then there exist constants $\mu > 0, \lambda \geq 0$ such that for all $u(x,t) \in H_0^{m,k}(Q_T)$,*

$$\int_\Omega \sum_{|p|,|q|=0}^{m} (-1)^{|p|+m} a_{pq} D^q u \overline{D^p u} dx \geq \mu\|u\|^2_{H^m(\Omega)} - \lambda\|u\|^2_{L_2(\Omega)}\,.$$

We note that the constant λ can be equal 0, since by a substitution $v = e^{-\lambda t}u$ the initial problem can be transformed to a problem with constant $\lambda = 0$.

3. The existence result

We obtained previously in [4, 5] some results on the solvability of above problem. Although, in these works we studied this problem only in non-smooth cylinders, the assertion there is still valid for arbitrary domains; since there we considered only the generalized solutions possessing weak derivatives of orders less or equal m. In this section we present another proof of this result by introducing an operator associating to the initial- boundary problem and its properties.

Theorem 3.1. *The problem* (2.2)–(2.4) *is uniquely solvable for arbitrary functions $f(x,t) \in L_2(Q_T)$ and $\varphi(x,t) \in V$. The solution $u(x,t)$ has the derivative u_t and the value $\mathcal{L}u$ (in distributional sense) from the space $L_2(Q_T)$ and it satisfies system* (2.2)–(2.4) *in the classical sense in each subset $\Omega' \times (0,T)$, where Ω' is a compact subset of Ω.*

Proof. We will use the Lax-Milgram's theorem. First, we will use the coercivity and the self-adjoint property of $-\mathcal{L}$ to show the solvability of the elliptic problem.

The Lax-Milgram procedure. Let

$$a(t,u,v)=\int_\Omega \sum_{|p|,|q|=0}^{m}(-1)^{|p|+m}a_{pq}D^q u\overline{D^p v}dx$$

which is a bilinear form defined on the space $V=\mathring{H}^m(\Omega)$. Since a is coercive, by virtue of Lax-Milgram's theorem, for each function $f(x)\in L_2(\Omega)$ there exists a solution $u\in V$ of the variational problem

$$a(t,u,v)=(f,v),\forall v\in V$$

It is easy to show that $-\mathcal{L}$ is a closed, symmetric operator with the range dense in space $H=L_2(\Omega)$; therefore, $-\mathcal{L}$ is a self-adjoint operator.

Operator A: Let us introduce the space

$$H^t=\{u(x,t)|f\in L_2(\Omega) \text{ for almost } t\in(0,T); u=0 \text{ near } \partial\Omega\}.$$

We define an operator A which associates the function $u(x,t)$ with the right-hand f and the initial value φ by the formula:

$$Au=\big(Pu,u(x,0)\big).$$

This operator is defined in the space $D(A)$ consisting of functions $u(x,t)$ which are represented by the formula

$$u=\varphi_0(x)+\int_0^T\varphi_1(x,t)\,dx\,dt,$$

where $\varphi_0(x)\in V$ and $\varphi_1(x,t)\in V$ for a.e. $t\in(0,T)$.

We can easily verify that $D(A)$ is a dense subset in $\mathring{H}^m(Q_T)$. The image of A is considered to be the elements in a certain Hilbert space W, which consists of the pairs $(f(x,t);\varphi(x)$ with $f(x,t)\in H^t$ and $\varphi(x)\in V$. In W the scalar product is defined as

$$\{(f_1;\varphi_1),(f_2;\varphi_2)\}=\int_0^T(f_1,f_2)+[\varphi_1,\varphi_2]dt$$

where $(,)$ and $[,]$ are the inner products in the spaces H and V respectively.

Operator A acts from V^t to W. Now we will show that A can be extended to its closure $\overline{A}$, with range $R(\overline{A})$ exhausting entire the space W. It is equivalent to conclude that the problem (2.2)–(2.4) has a solution u which belongs to $D(\overline{A})$ for arbitrary functions $f\in H^t,\varphi\in V$.

The existence of $\overline{A}$. Let $\{u_n(x,t)\}\subset D(A)$ is such sequence of functions that u_n tend to 0 in H^t and $A(u_n)$ tend to (f,φ) in W for $n\to\infty$. We have to verify that $(f,\varphi)=(0,0)$.

To do that, let us multiply the equation $Pu_n=f_n$ by an arbitrary smooth function $\Phi(x,t)$ from $D(A)$ which equal 0 for $t=T$, and then integrate the result over Q_T. After that, we will pass all the derivatives from u to Φ, by integration by parts. In the end we arrive at

$$\begin{aligned}\int_{Q_T}f_n\Phi\,dx\,dt&=\int_{Q_T}(u_{nt}-\mathcal{L}u)\Phi\,dx\,dt\\&=\int_{Q_T}u_n(-\Phi_t-\mathcal{L}\Phi)\,dx\,dt-\int_\Omega\varphi_n(x)\Phi(x,0)dx\end{aligned}$$

Passing $n \to \infty$, by our assumption, we obtain

$$\int_{Q_T} f\Phi \, dx \, dt = -\int_\Omega \varphi \Phi(x,0) dx.$$

However, noting that the smooth functions $\Phi(x,t)$ from $D(A)$ which vanish for $t = T$ and $t = 0$ form a dense subset in H^t, we can conclude that $f(x,t) \equiv 0$. Since the values $\Phi(x,0)$ are dense in V, we have $\varphi \equiv 0$. From these facts we can guarantee the existence of the closure $\overline{A}$.

The domain of $\overline{A}$. We consider for $u \in D(A)$ the expression $\int_0^t (Pu, Pu) dt$. We can transform this by the integration by parts

$$\begin{aligned}
&\int_0^t \int_\Omega (Pu, Pu) \, dx \, dt \\
&= \int_{Q_t} [u_t.\overline{u_t} + \mathcal{L}u.\overline{\mathcal{L}u} + 2Re u_t \mathcal{L}u)] \, dx \, dt \\
&= \int_{Q_t} [u_t^2 + (\mathcal{L}u)^2] \, dx \, dt - 2\int_{Q_t} \sum_{|p|,|q|=0}^{m} (-1)^{|p|+m-1} a_{pq} D^q u \overline{D^p u_t} \, dx \, dt \,.
\end{aligned} \tag{3.1}$$

The second term in the right-hand side can be rewritten as

$$\begin{aligned}
&-2\,\mathrm{Re} \int_{Q_{t_0}} \sum_{|p|,|q|=0}^{m} (-1)^{|p|+m-1} a_{pq} D^p u \overline{D^q u_t} \, dx \, dt \\
&= -\int_{Q_{t_0}} \Big[\frac{\partial}{\partial t}\Big(\sum_{|p|,|q|=0}^{m} (-1)^{|p|+m-1} a_{pq} D^q u \overline{D^p u}\Big) \\
&\quad - \sum_{|p|,|q|=0}^{m} (-1)^{|p|+m-1} \frac{\partial a_{pq}}{\partial t} D^q u \overline{D^p u}\Big].
\end{aligned} \tag{3.2}$$

Therefore, from Garding's inequality and the boundedness of $\frac{\partial a_{pq}}{\partial t}$ we can show that

$$c\int_0^{t_0} (|u_t|^2 + |\mathcal{L}u|^2) dt + \mu \|u(.,t_0)\|_V^2 \le \int_0^{t_0} |Pu|^2 dt + C\|(u.,0)\|_V^2 + C\int_0^{t_0} \|u\|_V^2 dt. \tag{3.3}$$

with the constant C depends on the bound of a_{pq} and $\frac{\partial a_{pq}}{\partial t}$.

From Gronwall-Bellman's inequality, as was carried out in [4], from (3.3), it is obvious that by assuming the convergence of u_n to u in H^t and Au_n to w in W, we attain immediately:

$$\begin{gathered}
\frac{\partial u_n}{\partial t} \to \frac{\partial u}{\partial t} \quad \text{in } H \\
\mathcal{L}u_n \to \mathcal{L}u \quad \text{in } H \\
D^p u_n \to D^p u \quad \text{uniformly w.r.t to } t \text{ in } L_2(\Omega), \forall p : 0 \le |p| \le m
\end{gathered}$$

Therefore, the function u in domain $D(\overline{A})$ possesses derivatives $\frac{\partial u}{\partial t}, D^p u, 0 \le |p| \le 2m$ in $L_2(Q_T)$. Meanwhile, all $D^p u, 0 <\le p \le m$ depend on t continuously which are functions in the space H for all $t \in (0,T)$. According to this fact, the image of u under $\overline{A}$ is computed as usually

$$\overline{A}u = (u_t - \mathcal{L}u; u(x,0)) \tag{3.4}$$

From (3.3) it is also valid that if Au_n converges in W then the sequence u_n also converges in H^t (even in some stronger sense). It says about that the closure of the operator A implies the closure of the range $R(A)$; i.e., $R(\overline{A}) = \overline{R(A)}$.

To complete the proof of the existence result, it is sufficient to show that the orthogonal complement to $\overline{R(A)}$ in space W consists of an element 0 solely. Let us assume the reverse; i.e., there exists an element $(f;\varphi) \neq (0;0)$ which is orthogonal to all elements from $\overline{R(A)}$, or equivalently, from $R(A)$:

$$0 \equiv \{(f;\varphi),(Au;u(x,0))\} = \int_{Q_T} (f(u_t - \mathcal{L}u))\,dx\,dt + \int_\Omega \sum_{|p|=0}^{m} D^p\varphi\overline{D^p u(x,0)}dx\,. \tag{3.5}$$

From the solvability of the variational problem, we can find basing on f the function $(\mathcal{L})^{-1}f$. Put

$$u(x,t) = -\int_0^t (\mathcal{L})^{-1} f(x,\tau)d\tau$$

in the identity (3.5) (this function obviously belongs to $D(A)$). Therefore

$$0 = \int_{Q_T} (f\big((\mathcal{L})^{-1} f\,dx\,dt + \int_0^t f d\tau\big).$$

Since $\mathcal{L}^{-1}$ is a negatively defined and self-adjoint operator, we can rewrite the last equality in the form:

$$0 = \int_{Q_T} |-\mathcal{L}^{\frac12} f|\,dx\,dt + \frac12\int_\Omega \big(\int_0^t f d\tau\big)^2,$$

from which it is clear that $-\mathcal{L}^{\frac12} f \equiv 0$ and thus $f \equiv 0$.

Consider again the equality (3.5) which now takes a simple form

$$0 \equiv [u(x,0),\varphi]_V, \forall u \in D(A).$$

From this formula we must have $\varphi \equiv 0$, since for above selected functions u the values $u(x,0)$ form a dense subset in V.

3.1. **Unbounded domains.** If the domain Ω is unbounded then the operator $\mathcal{L}$ may not have the bounded inverse. In this case the spectrum of $\mathcal{L}^{-1}$ may contain the element 0, but in the whole it is non-positive. By the substitution $v = ue^{-\gamma_0 t}, \gamma_0 > 0$ we will arrive to the system:

$$v_t + (-1)^m \mathcal{L}_1 v = f e^{-\gamma_0 t} \tag{3.6}$$

with the operator $\mathcal{L}_1 = \mathcal{L} - \gamma_0 I$ having the negative spectrum, that means the operator $\mathcal{L}_1^{-1}$ exists. The solvability for (3.6) follows in the same manner as it was carried out for the bounded domain. Note that by this argument we also have proved the uniqueness of the weak solution. The proof is complete. □

4. Behavior of solutions at infinity

We begin with the inequality:

$$\frac12 \frac{\partial}{\partial t}\|u(x,t)\|^2 + \mu\|u(x,t)\|_V^2 \le (f(x,t),u(x,t)) \tag{4.1}$$

which follows from (2.2) by multiplying both sides of the system by u and using the Garding's inequality. From the last relation it follows that

$$u\frac{\partial}{\partial t}\|u\| \le \|f\|\|u\|,$$

where $\|.\|$ denotes the norm in H. In other words, the above statement confirms that $\|u\| = 0$ or $\frac{\partial}{\partial t}\|u\| \le \|f\|$.

However, we know beforehand that u is a continuous function with respect to t (see the proof of Theorem 3.1), thus for all t and τ we have

$$\|u(x,t)\| \le \|u(x,\tau)\| + \int_\tau^t \|f(x,\xi)\| d\xi\,. \tag{4.2}$$

Differentiating both sides of inequality (4.1) with respect to t from τ to t and using (4.2) we can conclude that

$$\begin{aligned}\|u(x,t)\|^2 + 2\mu\int_\tau^t dt &\le \|u(x,t)\|^2 + 2\int_\tau^t \|f\|dt\big(\|u(x,\tau)\| + \int_\tau^t \|f\|dt\big)\\ &\le 2\|u(x,\tau)\|^2 + 3\Big(\int_\tau^t (f(.,\xi)d\xi\Big)^2\,.\end{aligned} \tag{4.3}$$

We consider two cases:

Bounded Ω. If the integral $\int_0^\infty \|f\|dt$ converges, then from (4.1) it follows that the integral $\int_0^\infty \|u\|_V dt$ also converges according to the Gronwall-Bellman's inequality. The direct consequence from it is the existence of a sequence $\{t_k\}$ tending to ∞, for which the norm $\|u(x,t_k)\|_V$ tends to zero.

However, for the bounded domains: $\|u(x,t)\| \le C\|u(x,t)\|_V$, hence the norm $\|u(x,t_k)\|$ also tends to 0 when $\{t_k\} \to \infty$. Using (4.2) with $t \ge \tau = t_k$ we get that $\|u(x,t)\| \to 0$ as $t \to \infty$.

Unbounded Ω. We cannot apply the Friedrichs's inequality now, therefore we have to use both inequalities (3.3), 4.3 simultaneously. If we assume the convergence of

$$\int_0^\infty (\|f\| + \|f\|^2)dt,$$

then from (4.3) there exists a sequence $\{t_k\} \to \infty$, for which $\|u(x,t_k)\|_V \to 0$. By (3.3) and the Gronwall-Bellman's inequality,

$$\|u(x,t)\|_V^2 \le \|u(x,t_k)\|_V^2 + \frac{1}{\mu}\int_\tau^t \|f(x,\xi)\|^2 d\xi, \quad \text{for all} t \ge t_k\,.$$

Consequently, the norm $\|u(x,t)\|_V$ can be made arbitrarily small for sufficiently large t. By this argument we have proved the following result.

Theorem 4.1. *If Ω is a bounded domain and the integral $\int_0^\infty \|f\|dt$ converges then $\|u(x,t)\| \to 0$ when $t \to \infty$. If Ω is an arbitrary domain then $\|u(x,t)\|_V \to 0$ when $t \to \infty$ if the integral $\int_0^\infty (\|f\| + \|f\|^2)dt$ converges.*

References

[1] Ladyzhenskaya, O. A.; *Mathematical theory of viscous imcompressible flow*. Rev. English ed. New York: Gordon and Breach, 1969.

[2] Kondratev, V. A.; Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat.Obshch.* 16: 209292; *English transl. in Trans. Moscow Math. Soc.* 16 (1967): 227313.

[3] Hung, N. M.; On the smoothness of solution of Dirichlet problem for hyperbolic systems in domains with conical or angular points. DAN. 362(2) (1998): 161-164.

[4] Hung, N. M., Duong, P. T.; On the smoothness with respect to time variable of generalized solution with respect to time variable of the first initial boundary-value problem for strongly parabolic systems in the cylinder with non-smooth base. *Ucrainian Mathematical Journal*. 56(1) (2004): 78-87.

[5] Hung, N. M., Duong, P. T.; On the smoothness of generalized solution for parabolic systems in domain with conic point on boundary. *Ucrainian Mathematical Journal*. V.56(6) (2004): 857-864.

Pham Trieu Duong
Department of Mathematics, Hanoi National University of Education, Hanoi, Vietnam
E-mail address: duongptmath@hnue.edu.vn

Do Van Loi
Department of Mathematics, Hong Duc University, Thanhhoa, Vietnam
E-mail address: 37loilinh@gmail.com