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SOLUTION TO THE TRIHARMONIC HEAT EQUATION

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Abstract. In this article, we study the equation

$$\frac{\partial}{\partial t}\, u(x,t) - c^2 \circledast u(x,t) = 0$$

with initial condition $u(x,0) = f(x)$. Where x is in the Euclidean space $\mathbb{R}^n$,

$$\circledast = \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^3 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^3$$

with $p+q=n$, $u(x,t)$ is an unknown function, $(x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is a generalized function, and c is a positive constant. Under suitable conditions on f and u, we obtain a unique solution. Note that for $q = 0$, we have the triharmonic heat equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \Delta^3 u(x,t) = 0\,.$$

1. Introduction

It is well known that the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Delta u(x,t), \tag{1.1}$$

with the initial condition $u(x,0) = f(x)$, has solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\Big(-\frac{|x-y|^2}{4c^2 t}\Big) f(y)dy\,, \tag{1.2}$$

where $(x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, and $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator. It is also known that the solution can be written as the convolution $u(x,t) = E(x,t) * f(x)$, where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{4c^2 t}\Big), \tag{1.3}$$

which is called *the heat kernel* [1, pp. 208-209]. Here $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$.

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In 1996, Kananthai [3] introduced the Diamond operator

$$\diamondsuit = \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2, \quad \text{with } p+q=n\,. \tag{1.4}$$

This operator can be written in the form $\diamondsuit = \Delta\Box = \Box\Delta$, where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \tag{1.5}$$

is the Laplacian, and

$$\Box = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \dots - \frac{\partial^2}{\partial x_{p+q}^2} \tag{1.6}$$

is the ultra-hyperbolic operator. The Fourier transform and the elementary solution of the Diamond operator has been studied; see for example [3]. Nonlaopon and Kananthai [5] studied the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t),$$

and obtain the ultra-hyperbolic heat kernel

$$E(x,t) = \frac{i^q}{(4c^2\pi t)^{n/2}} \exp\Big(-\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\Big),$$

where $p+q=n$, and $i=\sqrt{-1}$.

The purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \circledast u(x,t) = 0, \tag{1.7}$$

with the initial condition $u(x,0)=f(x)$, for $x \in \mathbb{R}^n$. The operator is

$$\begin{aligned}
\circledast &= \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^3 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^3 \\
&= \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)\Big[\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^2 - \Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)\Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big) \\
&\quad + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^2\Big] \\
&= \Delta(\Delta^2 - \frac{3}{4}(\Delta+\Box)(\Delta-\Box)) \\
&= \frac{3}{4}\diamondsuit\Box + \frac{1}{4}\Delta^3
\end{aligned}$$

where $\Delta, \Box, \diamondsuit$ are defined by (1.5), (1.6) and (1.4) respectively.

Here, $p+q=n$, $u(x,t)$ is an unknown function, $(x,t)=(x_1,x_2,\dots,x_n,t)$ is in $\mathbb{R}^n \times (0,\infty)$, $f(x)$ is a generalized function, and c is a positive constant. We obtain a solution $u(x,t)=E(x,t)*f(x)$, where

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t + i(\xi,x)\Big]d\xi\,, \tag{1.8}$$

and $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ for any fixed $t>0$.

Here $E(x,t)$ is the elementary solution of (1.7), whose properties will be studied in this article. If we put $q=0$, then (1.7) reduces to the equation

$$\frac{\partial}{\partial t}u(x,t)-c^2\Delta^3u(x,t)=0$$

which is related to the triharmaoic heat equation.

2. Preliminaries

Definition 2.1. Let $f(x)\in L_1(\mathbb{R}^n)$, the space of integrable function in $\mathbb{R}^n$. Then the Fourier transform of $f(x)$ is

$$\widehat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i(\xi,x)}f(x)\,dx, \tag{2.1}$$

where $\xi=(\xi_1,\xi_2,\dots,\xi_n)$, $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $(\xi,x)=\xi_1x_1+\xi_2x_2+\dots+\xi_nx_n$, and $dx=dx_1\,dx_2\dots dx_n$. The inverse of Fourier transform is defined as

$$f(x)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{f}(\xi)\,d\xi. \tag{2.2}$$

If f is a distribution with compact support by [6, Theorem 7.4-3], we can write

$$\widehat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\langle f(x),e^{-i(\xi,x)}\rangle. \tag{2.3}$$

Definition 2.2. The spectrum of the kernel $E(x,t)$ in (1.6) is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t>0$.

Definition 2.3. Let $\xi=(\xi_1,\xi_2,\dots,\xi_n)$ be a point in $\mathbb{R}^n$ and let

$$\Gamma_+=\{\xi\in\mathbb{R}^n:\xi_1^2+\xi_2^2+\dots+\xi_p^2-\xi_{p+1}^2-\xi_{p+2}^2-\dots-\xi_{p+q}^2>0\text{ and }\xi_1>0\}$$

be the interior of the forward cone, and $\overline{\Gamma}_+$ denote the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by Definition 2.2 for any fixed $t>0$, and $\Omega\subset\overline{\Gamma}_+$. Let the Fourier transform of $E(x,t)$ be

$$\widehat{E(\xi,t)}=\begin{cases}\frac{1}{(2\pi)^{n/2}}\exp\Big[-c^2\Big(\Big(\sum_{i=1}^p\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big] & \text{for }\xi\in\Gamma_+,\\ 0 & \text{for }\xi\notin\Gamma_+.\end{cases} \tag{2.4}$$

Lemma 2.4. *The Fourier transform of* $\circledast\delta$ *is*

$$\mathcal{F}\circledast\delta=\frac{(-1)^3}{(2\pi)^{n/2}}[(\xi_1^2+\xi_2^2+\dots+\xi_p^2\Big)^3+(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2)^3]$$

where $\mathcal{F}$ *is defined by* (2.1). *Let the norm of* ξ *be* $\|\xi\|=(\xi_1^2+\xi_2^2+\dots+\xi_n^2)^{1/2}$. *Then*

$$|\mathcal{F}\circledast\delta|\le\frac{M}{(2\pi)^{n/2}}\|\xi\|^6,$$

where M *is a positive constant. That is,* $\mathcal{F}\circledast$ *is bounded and continuous on the space* $\mathcal{S}'$ *of the tempered distribution. Moreover, by* (2.2),

$$\circledast\delta=\mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}[(\xi_1^2+\xi_2^2+\dots+\xi_p^2)^3+(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2)^3]$$

Proof. By (2.3),

$$
\begin{aligned}
\mathcal{F}\circledast\delta &= \frac{1}{(2\pi)^{n/2}}\langle\circledast\delta, e^{-i(\xi,x)}\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, \circledast e^{-i(\xi,x)}\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, \Big(\frac{3}{4}\Diamond\Box+\frac{1}{4}\Delta^3\Big)e^{-i(\xi,x)}\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\langle\delta, \frac{3}{4}\Diamond\Box e^{-i(\xi,x)}\rangle + \frac{1}{(2\pi)^{n/2}}\langle\delta, \frac{1}{4}\Delta^3 e^{-i(\xi,x)}\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\Big\langle\delta, \frac{3}{4}(-1)^3\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^2-\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^2\Big]\\
&\quad\times\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)-\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)\Big]e^{-i(\xi,x)}\Big\rangle\\
&\quad+\frac{1}{(2\pi)^{n/2}}\Big\langle\delta, \frac{1}{4}(-1)^3\Big[\Big(\sum_{i=1}^{n}\xi_i^2\Big)\Big]^3 e^{-i(\xi,x)}\Big\rangle\\
&= \frac{1}{(2\pi)^{n/2}}\Big[\frac{3}{4}(-1)^3\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^2-\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^2\Big]\Big]\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)-\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)\Big]\\
&\quad+\frac{1}{(2\pi)^{n/2}}\Big(\frac{1}{4}(-1)^3\Big[\Big(\sum_{i=1}^{n}\xi_i^2\Big)\Big]^3\Big)\\
&= \frac{(-1)^3}{(2\pi)^{n/2}}\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]\\
&= \frac{(-1)^3}{(2\pi)^{n/2}}\Big[\Big(\xi_1^2+\xi_2^2+\dots+\xi_p^2\Big)^3+\Big(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2\Big)^3\Big].
\end{aligned}
$$

Then

$$
\begin{aligned}
&|\mathcal{F}\circledast\delta|\\
&= \frac{1}{(2\pi)^{n/2}}\big|\big(\xi_1^2+\xi_2^2+\dots+\xi_p^2\big)^3+\big(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2\big)^3\big|\\
&\le \frac{1}{(2\pi)^{n/2}}|\xi_1^2+\dots+\xi_n^2|\big|(\xi_1^2+\dots+\xi_n^2)^2+(\xi_1^2+\dots+\xi_n^2)^2+(\xi_1^2+\dots+\xi_n^2)^2\big|\\
&\le \frac{M}{(2\pi)^{n/2}}\|\xi\|^6,
\end{aligned}
$$

where $\|\xi\| = \big(\xi_1^2+\xi_2^2+\dots+\xi_n^2\big)^{1/2}$, $\xi_i\in\mathbb{R}$ $(i=1,2,\dots,n)$. Hence we obtain $\mathcal{F}\circledast\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is a one-to-one transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, by (2.2), we have

$$\circledast\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}[(\xi_1^2+\xi_2^2+\dots+\xi_p^2)^3+(\xi_{p+1}^2+\xi_{p+2}^2+\dots+\xi_{p+q}^2)^3].$$

This completes the proof. □

Lemma 2.5. *Let*

$$L = \frac{\partial}{\partial t} - c^2\Big[\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^3 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^3\Big], \tag{2.5}$$

where

$$\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^3 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^3 = \frac{3}{4}\Diamond\Box + \frac{1}{4}\Delta^3,$$

$p+q=n$, $(x,t) = (x_1, x_2, \dots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, *and* c *is a positive constant. Then*

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t + i(\xi,x)\Big]d\xi. \tag{2.6}$$

is an elementary solution of (2.5).

Proof. Let $E(x,t)$ be an elementary solution of operator L. Then

$$LE(x,t) = \delta(x,t),$$

where δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2\Big[\Big(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\Big)^3 + \Big(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\Big)^3\Big]E(x,t) = \delta(x)\delta(t).$$

Taking the Fourier transform on both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} + c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big]$$

where $H(t)$ is the Heaviside function. Since $H(t)=1$ for $t>0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big]$$

which by (2.3), we obtain

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi = \frac{1}{(2\pi)^{n/2}}\int_\Omega e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.2),

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t + i(\xi,x)\Big]d\xi.$$

This completes the proof. $\square$

3. Main Results

Theorem 3.1. *Consider the equation*

$$\frac{\partial}{\partial t}u(x,t)-c^2\circledast u(x,t)=0 \tag{3.1}$$

with initial condition

$$u(x,0)=f(x) \tag{3.2}$$

and the operator

$$\begin{aligned}
\circledast &= \Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^3+\Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^3\\
&= \Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}+\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)\Big[\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^2-\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)\Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)\\
&\quad+\Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^2\Big]\\
&= \frac{3}{4}\Diamond\Box+\frac{1}{4}\Delta^3
\end{aligned}$$

where $p+q=n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\dots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is a given generalized function, and c is a positive constant. Then

$$u(x,t)=E(x,t)*f(x)$$

as a solution of (3.1)-(3.2)*, where $E(x,t)$ is given by* (2.5)*.*

Proof. Taking the Fourier transform on both sides of (3.1), we obtain

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t)+c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]\widehat{u}(\xi,t)=0,$$

(see Lemma 2.4). Thus

$$\widehat{u}(\xi,t)=K(\xi)\exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big] \tag{3.3}$$

where $K(\xi)$ is constant and $\widehat{u}(\xi,0)=K(\xi)$. By (3.2) we have

$$K(\xi)=\widehat{u}(\xi,0)=\widehat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i(\xi,x)}f(x)\,dx \tag{3.4}$$

and by the inversion in (2.2), (3.3) and (3.4) we obtain

$$\begin{aligned}
&u(x,t)\\
&=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{u}(\xi,t)\,d\xi\\
&=\frac{1}{(2\pi)^{n}}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(\xi,x)}e^{-i(\xi,y)}f(y)\exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big]\,dy\,d\xi.
\end{aligned}$$

Thus

$$u(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(\xi,x-y)}\exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)t\Big]f(y)\,dy\,d\xi$$

or

$$u(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t+i(\xi,x-y)\Big]f(y)\,dy\,d\xi. \tag{3.5}$$

Set

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t+i(\xi,x)\Big]\,d\xi. \tag{3.6}$$

We choose $\Omega\subset\mathbb{R}^n$ be the spectrum of $E(x,t)$ and by (2.5), we have

$$\begin{aligned}E(x,t)&=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t+i(\xi,x)\Big]\,d\xi\\&=\frac{1}{(2\pi)^n}\int_{\Omega}\exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3+\Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t+i(\xi,x)\Big]\,d\xi.\end{aligned} \tag{3.7}$$

Thus (3.5) can be written in the convolution form

$$u(x,t)=E(x,t)*f(x).$$

Since $E(x,t)$ exists,

$$\lim_{t\to 0}E(x,t)=\frac{1}{(2\pi)^n}\int_{\Omega}e^{i(\xi,x)}\,d\xi=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}e^{i(\xi,x)}\,d\xi=\delta(x), \tag{3.8}$$

for $x\in\mathbb{R}^n$; see [3, Eq. (10.2.19b)]. Thus for the solution $u(x,t)=E(x,t)*f(x)$ of (3.1),

$$\lim_{t\to 0}u(x,t)=u(x,0)=\delta*f(x)=f(x)$$

which satisfies (3.2). □

Theorem 3.2. *The kernel $E(x,t)$ defined by* (3.7) *has the following properties:*

1. *$E(x,t)\in\mathcal{C}^\infty$ for $x\in\mathbb{R}^n$ and $t>0$, the space of function with infinitely many continuous derivatives.*
2. *For $t>0$,*
$$\Big(\frac{\partial}{\partial t}-c^2\Big[\Big(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\Big)^3+\Big(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\Big)^3\Big]\Big)E(x,t)=0\,.$$
3. *$E(x,t)>0$ for $t>0$.*
4. *For $t>0$,*
$$|E(x,t)|\le\frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})},$$
where $M(t)$ is a function of t in the spectrum Ω, and Γ denotes the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t>0$.
5. *$\lim_{t\to 0}E(x,t)=\delta$.*

Proof. (1) From (3.7),

$$\frac{\partial^n}{\partial x^n}E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \frac{\partial^n}{\partial x^n}\exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t + i(\xi,x)\Big]\,d\xi.$$

Thus $E(x,t) \in \mathcal{C}^\infty$ for $x \in \mathbb{R}^n$ and $t > 0$.

(2) By a computation,

$$\Big(\frac{\partial}{\partial t} - c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]\Big)E(x,t) = 0.$$

(3) $E(x,t) > 0$ for $t > 0$ is obvious by (3.7).

(4) We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_\Omega \exp\Big[-c^2\Big[\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big]t + i(\xi,x)\Big]\,d\xi,$$

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp\Big[-c^2\Big(\Big(\sum_{i=1}^{p}\xi_i^2\Big)^3 + \Big(\sum_{j=p+1}^{p+q}\xi_j^2\Big)^3\Big)\Big]\,d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \quad \xi_2 = r\omega_2, \quad \dots, \quad \xi_p = r\omega_p,$$
$$\xi_{p+1} = s\omega_{p+1}, \quad \xi_{p+2} = s\omega_{p+2}, \quad \dots, \quad \xi_{p+q} = s\omega_{p+q},$$

where $\sum_{i=1}^{p}\omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q}\omega_j^2 = 1$. Thus

$$|E(x,t)| \le \frac{1}{(2\pi)^n}\int_\Omega \exp[-c^2(s^6 + r^6)t]r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}\,dr\,ds\,d\Omega_p\,d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \le r \le R$ and $0 \le s \le T$ where R and T are constants. Thus we obtain

$$\begin{aligned}
|E(x,t)| &\le \frac{\Omega_p\,\Omega_q}{(2\pi)^n}\int_0^R\int_0^T \exp[-c^2(s^6 + r^6\Big)t]r^{p-1}s^{q-1}\,ds\,dr \\
&= \frac{\Omega_p\,\Omega_q}{(2\pi)^n}M(t) \\
&= \frac{2^{2-n}}{\pi^{n/2}}\frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}
\end{aligned}$$

for any fixed $t > 0$ in the spectrum Ω, where

$$M(t) = \int_0^R\int_0^T \exp[-c^2(s^6 + r^6)t]r^{p-1}s^{q-1}\,ds\,dr$$

is a function of t, $\Omega_p = 2\pi^{p/2}/\Gamma(\frac{p}{2})$ and $\Omega_q = 2\pi^{p/2}/\Gamma(\frac{q}{2})$. Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

(5) This statement follows from (3.8). □

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References

[1] R. Haberman; *Elementary Applied Partial Differential Equations*, 2-nd Edition, Prentice-Hall International, Inc. (1983).

[2] F. John; *Partial Differential Equations*, 4-th Edition, Springer-Verlag, New York, (1982).

[3] A. Kananthai; *On the Fourier Transform of the Diamond Kernel of Marcel Riesz*, Applied Mathematics and Computation 101:151-158 (1999).

[4] A. Kananthai; *On the Solution of the n-Dimensional Diamond Operator*, Applied Mathematics and Computational 88:27-37 (1997).

[5] K. Nonlaopon, A. Kananthai; *On the Ultra-hyperbolic heat kernel*, Applied Mathematics Vol. 13 No. 2 (2003), 215-225.

[6] A. H. Zemanian; *Distribution Theory and Transform Analysis*, McGraw-Hill, New York, 1965.

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