Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 09, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS TO FRACTIONAL ORDER ORDINARY AND DELAY DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Abstract. In this article, we discuss the existence and uniqueness of solution to fractional order ordinary and delay differential equations. We apply our results on the single species model of Lotka Volterra type. Fixed point theorems are the main tool used here to establish the existence and uniqueness results. First we use Banach contraction principle and then Krasnoselskii's fixed point theorem to show the existence and uniqueness of the solution under certain conditions. Moreover, we prove that the solution can be extended to maximal interval of existence.

1. Introduction

Fractional differential equations is a generalization of ordinary differential equations and integration to arbitrary non integer orders. The origin of fractional calculus goes back to Newton and Leibniz in the seventieth century. It is widely and efficiently used to describe many phenomena arising in engineering, physics, economy, and science. Recent investigations have shown that many physical systems can be represented more accurately through fractional derivative formulation [29]. Fractional differential equations, therefore find numerous applications in the field of visco-elasticity, feed back amplifiers, electrical circuits, electro analytical chemistry, fractional multipoles, neuron modelling encompassing different branches of physics, chemistry and biological sciences [31]. There have been many excellent books and monographs available on this field [11, 24, 30, 31, 34, 38]. In [24], the authors gave the most recent and up-to-date developments on fractional differential and fractional integro-differential equations with applications involving many different potentially useful operators of fractional calculus. In a recent work by Jaimini et.al. [23] the authors have given the corresponding Leibnitz rule for fractional calculus. For the history of fractional calculus, interested reader may see the recent review paper by Machado et. al. [28].

Many physical processes appear to exhibit fractional order behavior that may vary with time or space. The fractional calculus has allowed the operations of integration and differentiation to any fractional order. The order may take on any

2000 *Mathematics Subject Classification.* 34K40, 34K14.
Key words and phrases. Fractional differential equation; fixed point theorems;
maximum interval of existence.

Submitted November 16, 2010. Published Janaury 16, 2011.

real or imaginary value. Recently theory of fractional differential equations attracted many scientists and mathematicians to work on [7, 18, 19, 31, 32, 33, 39]. For the existence of solutions for fractional differential equations, one can see [13, 12, 14, 3, 6, 8, 9, 10, 15, 16, 20, 21, 22, 25, 26, 40] and references therein. The results have been obtained by using fixed point theorems like Picard's, Schauder fixed-point theorem and Banach contraction mapping principle. About the development of existence theorems for fractional functional differential equations, many contribution exists [1, 14, 2, 5, 7, 27, 41]. Many applications of fractional calculus amount to replacing the time derivative in a given evolution equation by a derivative of fractional order. The results of several studies clearly stated that the fractional derivatives seem to arise generally and universally from important mathematical reasons. Recently, interesting attempts have been made to give the physical meaning to the initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives were proposed in [17, 19, 32, 33].

Ahmed et. al. [4] considered the fractional order predator-prey model and the fractional order rabies model. They have shown the existence and uniqueness of solutions of the model system and also studied the stability of equilibrium points. The motivation behind fractional order system are discussed in [4]. Lakshmikantham and Vatsala in [25, 26] and Lakshmikantham in [27] defined and proved existence of the solution of fractional initial value problems.

In this article our aim is to show the existence of the solutions of the differential equations

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = g(t, x(t)), \quad t \in [0, T] \\ x(0) = x_0, \quad 0 < \alpha < 1, \end{gathered} \tag{1.1}$$

and

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t), x(t-\tau)), \quad t \in [0, T] \\ x(t) = \phi(t), \quad t \in [-\tau, 0] \quad 0 < \alpha < 1, \end{gathered} \tag{1.2}$$

under suitable conditions on g, f and ϕ. We assume that g satisfies Lipschitz condition with Lipschitz constant L_g and $f(t,x,y)$ can be written as $f_1(t,x) + f_2(t,x,y)$, where both f_1, f_2 are Lipschitz continuous with Lipschitz constants L_{f_1} and L_{f_2} respectively. Moreover, we show the existence of maximum interval of existence for the problems (1.1) and (1.2). As far as I know these kind of results are new for fractional differential equations.

Next we apply our results on the following fractional order Lotka Volterra model for $0 < \alpha < 1$,

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = (t)\Big(r(t) - a(t)x(t-\tau)\Big), \quad t \in [0, T],\ \tau \geq 0, \\ x(t) = \phi(t), \quad t \in [-\tau, 0]. \end{gathered} \tag{1.3}$$

where $\frac{d^\alpha}{dt^\alpha}$ denotes RiemannLiouville derivative of order α, $0 < \alpha < 1$. The coefficients $r(t)$ and $a(t)$ satisfy

$$r_* \leq r(t) \leq r^*, \quad a_* \leq a(t) \leq a^*$$

which are biologically feasible. We use fixed point theory to show the existence of a solution. For the fixed point theory and many related results, interested reader may consult [37].

2. Preliminaries and Results

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided the right side exists pointwise on $\mathbb{R}^+$. Γ is the gamma function.

For instance, $I^\alpha f$ exists for all $\alpha > 0$, when $f \in C^0(\mathbb{R}^+) \cap L^1_{\text{loc}}(\mathbb{R}^+)$; note also that when $f \in C^0(\mathbb{R}_0^+)$ then $I^\alpha f \in C^0(\mathbb{R}_0^+)$ and moreover $I^\alpha f(0) = 0$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\frac{d^\alpha}{dt^\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} f(s) ds = \frac{d}{dt} I_0^{1-\alpha} h(t).$$

Using fractional calculus, the equation (1.1) can be represented by following integral form

$$x(t) = x_0 + \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds.$$

First we discuss the existence of the solution of the following ordinary fractional differential equation (1.1)

$$\begin{gathered} \frac{d^\alpha x(t)}{dt^\alpha} = g(t, x(t)), \quad t \in [0, T] \\ x(0) = x_0. \end{gathered}$$

Define the operator

$$Tx(t) = x_0 + \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} g(s, x(s)) ds.$$

Let the function $g : B(a, \beta) \to \mathbb{R}$ be bounded by M, where

$$B(a, \beta) = \{(t, x) : \ |t| \le a, \ |x - x_0| \le \beta\}.$$

We assume that our function g is Lipschitz continuous with respect to x with Lipschitz constant L_g. Denote $b = \min\{a, \frac{\beta}{M}\}$. Let C be the set of all continuous functions from $[-b, b]$ to $B(a, \beta)$. Consider

$$\begin{aligned} |Tx(t) - x_0| &\le \frac{1}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} |g(s, x(s))| ds \\ &\le \frac{M}{\Gamma\alpha} \int_0^t (t-s)^{\alpha-1} ds \le \frac{M}{\Gamma\alpha} \int_0^t s^{\alpha-1} ds \\ &\le \frac{M}{\Gamma(\alpha+1)} t^\alpha \le \frac{M}{\Gamma(\alpha+1)} T^\alpha \end{aligned}$$

Thus for x bounded, continuous, Tx is also bounded, continuous. We denote $B(a,\beta)$ by B in short. For $x, y \in B$, we have

$$\begin{aligned}
|Tx(t) - Ty(t)| &\le \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|g(s,x(s)) - g(s,y(s))|ds \\
&\le \frac{L_g}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds \\
&\le \frac{L_g}{\Gamma\alpha}\Big(\int_0^t (t-s)^{\alpha-1}ds\Big)\sup_{x\in[0,T]}|x(s)-y(s)| \\
&\le \frac{L_g}{\Gamma\alpha}\|x-y\|\int_0^t s^{\alpha-1}ds \\
&\le \frac{L_g}{\Gamma(\alpha+1)}\|x-y\|T^\alpha
\end{aligned}$$

Thus for

$$\frac{L_g T^\alpha}{\Gamma(\alpha+1)} < 1,$$

we have $\|Tx - Ty\| < \|x - y\|$.

By the contraction mapping principle, we therefore know that T has a unique fixed point in B. This implies that our problem has a unique solution in B. Hence we summarize our result in the following theorem.

Theorem 2.3. *Problem* (1.1) *has a unique solution in B provided that*

$$\frac{L_g T^\alpha}{\Gamma(\alpha+1)} < 1.$$

Now we prove the existence of maximal interval of existence for the fractional differential equation (1.1). The analysis is similar to analysis done by [36] for ordinary differential equation. Let Ω be the open, connected subset of $[0,T]\times\mathbb{R}$.

Theorem 2.4. *Assume that $g : \Omega \to \mathbb{R}$ is continuous and let x be a solution of the problem defined on some interval I. Then x may be extended as a solution of* (1.1) *to a maximal interval of existence (ω_-, ω_+) and $(t, x(t)) \to \partial\Omega$ as $t \to \omega_\pm$.*

Proof. We need to show only the existence of a right maximal interval of existence. For the left maximal interval of existence a similar argument will work. Combining both argument together will imply the existence of a maximal interval of existence. Let x be a solution of (1.1) with the given initial condition $x(0) = x_0$ defined on an interval $I = [0, a_x)$ for $a_x > 0$. We say that two solutions x_1, x_2 of the problem (1.1) satisfy $x_1 \prec x_2$, if and only if

$$\begin{gathered}
x \equiv x_1 \equiv x_2 \quad \text{on } [0, a_x], \\
x_1 \text{ is defined on } I_{x_1} = [0, a_{x_1}),\ a_{x_1} > a_x, \\
x_2 \text{ is defined on } I_{x_2} = [0, a_{x_2}),\ a_{x_2} > a_x,
\end{gathered}$$

and $a_{x_2} \ge a_{x_1}$, also $x_1 \equiv x_2$ on I_{x_1}.

To show $\prec$ is a partial order on the set of all solutions S of (1.1) which coincide with x on I, we need to show that it is reflexive, antisymmetric and transitive. It is easy to see that $x_1 \prec x_1$ always holds. Now if $x_1 \prec x_2$ and $x_2 \prec x_1$ we have

$x_1 \equiv x_2$ on I_x by choosing $a_{x_1} = a_{x_2}$. Now, if $x_1 \prec x_2, x_2 \prec x_3$, we have

$$\begin{gathered} x \equiv x_1 \equiv x_2 \quad \text{on } [0, a_x], \\ x_1 \text{ is defined on } I_{x_1} = [0, a_{x_1}),\ a_{x_1} > a_x, \\ x_2 \text{ is defined on } I_{x_2} = [0, a_{x_2}),\ a_{x_2} > a_x, \end{gathered}$$

and $a_{x_2} \geq a_{x_1}$, also $x_1 \equiv x_2$ on I_{x_1}; and

$$\begin{gathered} x \equiv x_2 \equiv x_3 \quad \text{on } [0, a_x], \\ x_2 \text{ is defined on } I_{x_2} = [0, a_{x_2}),\ a_{x_2} > a_x, \\ x_3 \text{ is defined on } I_{x_3} = [0, a_{x_3}),\ a_{x_3} > a_x, \end{gathered}$$

and $a_{x_3} \geq a_{x_2}$, also $x_2 \equiv x_3$ on I_{x_2}.

From these two conditions, we can easily obtain

$$\begin{gathered} x \equiv x_1 \equiv x_3 \text{ on } [0, a_x], \\ x_1 \text{ is defined on } I_{x_1} = [0, a_{x_1}),\ a_{x_1} > a_x, \\ x_3 \text{ is defined on } I_{x_3} = [0, a_{x_3}),\ a_{x_3} > a_x, \end{gathered}$$

and $a_{x_3} \geq a_{x_1}$, also $x_3 \equiv x_1$ on I_{x_3}. Thus $x_1 \prec x_3$.

Thus $\prec$ is a partial order. Now we verify that the conditions of the Hausdroff maximum principle ([35]) hold and hence that S contains a maximal element, say $\bar{x}$. This maximal element $\bar{x}$ cannot be further extended to the right. Let x be a solution of (1.1) with right maximal interval of existence $[0, \omega_+)$. Now, we must show that $(t, x(t)) \to \partial\Omega$ as $t \to \omega_+$; that is, given any compact set $K \subset \Omega$, there exists t_K, such that $(t, x(t)) \notin K$, for $t > t_K$.

For the case $\omega_+ = \infty$, the conclusion clearly holds. For the other case, that is if $\omega_+ < \infty$, we proceed indirectly. In the later case there exists a compact set $K \subset \Omega$, such that for every $n = 1, 2, \dots$ there exists t_n, $0 < \omega_+ - t_n < \frac{1}{n}$, and $(t_n, x(t_n)) \in K$. Since K is compact, there will be a subsequence, for the convenience call it again $\{(t_n, x(t_n))\}$ such that $\{(t_n, x(t_n))\}$ converges to (ω_+, x^*) which belongs to K. Since $(\omega_+, x^*) \in K$, it is an interior point of Ω. We may therefore choose a constant $a > 0$, such that $Q = \{(t, x) : |\omega_+ - t| \leq a,\ |x - x^*| \leq a\} \subset \Omega$. Thus for n large $(t_n, x(t_n)) \in Q$. Let $m = \max_{(t,x)\in Q} |f(t, x)|$, and let n be so large that

$$0 < \omega_+ - t_n \leq \frac{a}{2m}, \quad |x(t_n) - x^*| \leq \frac{a}{2}.$$

Then

$$|x(t_n) - x(t)| < m(\omega_+ - t_n) \leq \frac{a}{2},$$

for $t < \omega_+$, by an easy argument. Therefore, $\lim_{t\to\omega_+} x(t) = x^*$. Hence we may extend x to the right of ω_+ contradicting the maximality of x. Hence the result is proved for the fractional ordinary differential equation (1.1). $\square$

Consider the following function $g(t, x(t)) = x(t)(r(t) - a(t)x(t))$, where we assume that $r(t) \in [r_*, r^*]$ and $a(t) \in [a_*, a^*]$. The corresponding fractional differential equations represent the evolution model of a single species without delay. It is easy to see that the function f is Lipschitz and bounded for any $x \in B$. Thus from the above analysis we obtain the existence of the solution which can be extended to the maximal interval. One can easily observe that the above results can be easily extended to $\mathbb{R}^n$.

Moreover, for the fractional delay differential equation (1.2), it is easy to see that if $t \in [0, \tau]$, our function $x(t-\tau) = \phi(t-\tau)$. Thus in this interval the delay fractional differential equations behave like non-delay fractional differential equations,

$$\begin{gathered}\frac{d^\alpha x(t)}{dt^\alpha} = f(t, x(t), \phi(t-\tau)), \quad t \in [0, \tau]\\ x(t) = \phi(t), \quad t \in [-\tau, 0],\ 0 < \alpha < 1.\end{gathered}$$

A similar analysis we described above for problem (1.1) can be used to show the local existence and uniqueness of the solution of fractional delay differential equation (1.2).

Let us consider the function $f(t, x(t), x(t-\tau)) = x(t)(r(t) - a(t)x(t-\tau))$, where we assume that $r(t) \in [r_*, r^*]$ and $a(t) \in [a_*, a^*]$. These kind of function come from the modelling of interspecific competition in one species with τ as a maturity time period. The corresponding fractional differential equations for this function f is (1.3). Thus for $t \in [0, \tau]$, our function is $x(t)(r(t) - a(t)\phi(t-\tau))$. It is easy to see that the function f is Lipschitz and bounded for any $x \in B$. Hence by using similar analysis as mentioned above, we obtain local existence of the solution.

Now our next target is to use Krasnoselskii's fixed point theorem to prove the existence and uniqueness of the solution of fractional delay differential equations (1.2). A similar analysis yield the existence of solution of the problem (1.1).

By a solution $x(t)$ of (1.2) we mean that it satisfy the relation

$$x(t) = \phi(0) + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1} f(s, x(s), x(s-\tau))ds$$

for $t \in [0, T]$ and $x(t) = \phi(t)$ for $t \in [-\tau, 0]$.

First we mention statement of Krasnoselskii's fixed point theorem.

Theorem 2.5 (Krasnoselskii). *Let B be a nonempty closed convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that Λ_1 and Λ_2 map B into X such that*

- (i) *for any $x, y \in B$, $\Lambda_1 x + \Lambda_2 y \in B$,*
- (ii) *Λ_1 is a contraction,*
- (iii) *Λ_2 is continuous and $\Lambda_2(B)$ is contained in a compact set.*

Then there exists $z \in B$ such that $z = \Lambda_1 z + \Lambda_2 z$.

Now we prove existence of the solutions for the delay fractional differential equations (1.2) using Krasnoselskii's fixed point theorem. We begin with the assumption that our function f can be written as the sum of two functions of the following form

$$f(t, x(t), y(t)) = f_1(t, x(t)) + f_2(t, x(t), y(t)),$$

where f_i, $i = 1, 2$ are Lipschitz continuous functions with Lipschitz constants L_{f_i} for $i = 1, 2$. Define the operators F_1 and F_2 by

$$\begin{gathered}F_1 x(t) = \phi(0) + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1} f_1(s, x(s))ds,\\ F_2 x(t) = \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1} f_2(s, x(s), x(s-\tau))ds.\end{gathered}$$

It is easy to see that

$$
\begin{aligned}
|F_1x(t) - F_1y(t)| &\le \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|f_1(s,x(s)) - f_1(s,y(s))|ds \\
&\le \frac{L_{f_1}}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|x(s)-y(s)|ds \\
&\le \frac{L_{f_1}}{\Gamma\alpha}\|x-y\|\int_0^t s^{\alpha-1}ds \\
&\le \frac{L_{f_1}}{\Gamma(\alpha+1)}\|x-y\|T^\alpha.
\end{aligned}
$$

We obtain

$$
\|F_1x - F_1y\| \le \frac{L_{f_1}T^\alpha}{\Gamma(\alpha+1)}\|x-y\|.
$$

Thus F_1 is a contraction provided $\frac{L_{f_1}T^\alpha}{\Gamma(\alpha+1)} < 1$.

Further assume that the functions f_i, $i=1,2$ satisfy the relations

$$
\begin{gathered}
|f_1(t,x(t))| \le M_1|x(t)|, \\
|f_2(t,x(t),y(t))| \le M_2|x(t)| \times |y(t)|.
\end{gathered}
$$

Let $BC([-\tau,T],\mathbb{R})$ denote the collection of all bounded and continuous function from $[-\tau,T]$ to $\mathbb{R}$. Consider the set

$$
D = \{x \in BC([-\tau,T],\mathbb{R}) : |x| \le r\}
$$

where r satisfies

$$
|\phi(0)| + \frac{M_1r + M_2r^2}{\Gamma(\alpha+1)}T^\alpha \le r.
$$

For $x \in D$, calculating the norm of the function $F = F_1 + F_2$, we have

$$
\begin{aligned}
&|F_1x(t) + F_2x(t)| \\
&\le |\phi(0)| + \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|f_1(s,x(s)) + f_2(s,x(s),x(s-\tau))|ds \\
&\le |\phi(0)| + \frac{M_1\|x\| + M_2\|x\|^2}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}ds \\
&\le |\phi(0)| + \frac{M_1r + M_2r^2}{\Gamma(\alpha+1)}T^\alpha.
\end{aligned}
$$

Thus $F_1x + F_2x \in D$. Moreover for $x \in D$, we obtain

$$
\begin{aligned}
|F_2x(t)| &\le \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|f_2(s,x(s),x(s-\tau))|ds \\
&\le \frac{M_2\|x\|^2}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}ds \\
&\le \frac{M_2r^2}{\Gamma(\alpha+1)}T^\alpha \le r.
\end{aligned}
$$

To prove the continuity of F_2, let us consider a sequence x_n converging to x. Taking the norm of $F_2x_n(t) - F_2x(t)$, we have

$$
\begin{aligned}
&|F_2x_n(t) - F_2x(t)|\\
&\le \frac{1}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}|f_2(s,x_n(s),x_n(s-\tau)) - f_2(s,x(s),x(s-\tau))|ds\\
&\le \frac{L_{f_2}}{\Gamma\alpha}\int_0^t (t-s)^{\alpha-1}(|x_n(s)-x(s)| + |x_n(s-\tau)-x(s-\tau)|)ds\\
&\le \frac{2L_{f_2}}{\Gamma\alpha}\Big(\int_0^t s^{\alpha-1}ds\Big)\|x_n - x\|\\
&\le \frac{2L_{f_2}}{\Gamma(\alpha+1)}T^\alpha\|x_n - x\|.
\end{aligned}
$$

From the above analysis we obtain

$$
\|F_2x_n - F_2x\| \le \frac{2L_{f_2}}{\Gamma(\alpha+1)}T^\alpha\|x_n - x\|
$$

and hence whenever $x_n \to x$, $Fx_n \to Fx$. This proves the continuity of F_2.

Now for $t_1 \le t_2 \le T$, we have

$$
\begin{aligned}
&|F_2x(t_2) - F_2x(t_1)|\\
&\le \frac{1}{\Gamma\alpha}\Big|\int_0^{t_2}(t_2-s)^{\alpha-1}f_2(s,x(s),x(s-\tau)ds\\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha-1}f_2(s,x(s),x(s-\tau)ds\Big|\\
&\le \frac{1}{\Gamma\alpha}\Big|\int_0^{t_1}(t_2-s)^{\alpha-1}f_2(s,x(s),x(s-\tau)ds\\
&\quad + \int_{t_1}^{t_2}(t_2-s)^{\alpha-1}f_2(s,x(s),x(s-\tau)ds\\
&\quad - \int_0^{t_1}(t_1-s)^{\alpha-1}f_2(s,x(s),x(s-\tau)ds\Big|\\
&\le \frac{1}{\Gamma\alpha}\int_0^{t_1}|((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1})||f_2(s,x(s),x(s-\tau)|ds\\
&\quad + \frac{1}{\Gamma\alpha}\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}| \times |f_2(s,x(s),x(s-\tau)|ds\\
&\le \frac{M_1r^2}{\Gamma\alpha}\int_0^{t_1}|(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}|ds + \frac{M_2r^2}{\Gamma\alpha}\int_{t_1}^{t_2}|(t_2-s)^{\alpha-1}|ds\\
&\le \frac{r^2}{\Gamma(\alpha+1)}\max\{M_1,M_2\}\Big| -2(t_2-t_1)^\alpha + t_2^\alpha - t_1^\alpha\Big|\\
&\le \frac{r^2}{\Gamma(\alpha+1)}\max\{M_1,M_2\}(t_2-t_1)^\alpha
\end{aligned}
$$

The right-hand side of above expression does not depends on x. Thus we conclude that $F_2(D)$ is relatively compact and hence F_2 is compact by Arzela-Ascoli theorem. Using Krasnoselskiis fixed point theorem, we obtain that there exists $z \in D$ such that $Fz = F_1z + F_2z = z$, which is a fixed point of F. Hence the problem (1.2) has at least one solution in D. We summarize the above results in the form of the following theorem.

Theorem 2.6. *Model* (1.2) *has a solution in the set* D *provided* $\frac{L_{f_1}T^\alpha}{\Gamma(\alpha+1)} < 1$ *and*

$$|\phi(0)| + \frac{M_1 r + M_2 r^2}{\Gamma(\alpha+1)} T^\alpha \le r\,.$$

Consider the function $f(t, x(t), x(t-\tau)) = x(t)(r(t) - a(t)x(t-\tau))$ and let us denote

$$f_1(t, x(t)) = x(t)r(t), \quad f_2(t, x(t), x(t-\tau)) = -a(t)x(t)x(t-\tau).$$

It is easy to see that

$$|f_1(t, x(t))| \le r^*|x(t)|,$$
$$|f_2(t, x(t), x(t-\tau))| \le a^*|x(t)||x(t-\tau)|.$$

Using fractional calculus, (1.3) can be representable as an integral form of the type

$$x(t) = \phi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\Big(r(s) - a(s)x(s-\tau)\Big)ds$$
$$x(t) = \phi(t), \quad t \in [-\tau, 0].$$

Define a mapping Λ by

$$\Lambda x(t) = \Lambda_1 x(t) + \Lambda_2 x(t),$$

where

$$\Lambda_1 x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) r(s) ds,$$
$$\Lambda_2 x(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) a(s) x(s-\tau) ds.$$

One can easily see that in this case our operator F_1 coincide with Λ_1 and F_2 coincides with Λ_2. Thus our model systems (1.3) have at least one solution. We summarize the result for problem (1.3) in the form of the following theorem.

Theorem 2.7. *The model* (1.3) *has a solution in the set* D *provided* $\frac{L_{f_1}T^\alpha}{\Gamma(\alpha+1)} < 1$ *and*

$$|\phi(0)| + \frac{r^* r + a^* r^2}{\Gamma(\alpha+1)} T^\alpha \le r\,.$$

Remark 2.8. The above result can be extended for n species competitive system of the form

$$\frac{d^\alpha x_i(t)}{dt^\alpha} = x_i(t)\Big(r_i(t) - \sum_{j=1}^n a_{ij}(t) x_j(t-\tau_{ij})\Big), \quad t \in [0, T],\ i = 1, 2, \dots, n.$$
$$x_i(t) = \phi_i(t), \quad t \in [-\tau, 0],$$

where α, $0 < \alpha < 1$, $r_i(t) \in [r_*, r^*]$, and $a_{ij} \in [a_*, a^*]$.

Acknowledgments. The author would like to thank the anonymous referee for his/her valuable comments and suggestions, and Prof. Julio G. Dix for his kind support which help me to improve the manuscript considerably.

References

[1] Abbas, S.; Pseudo almost automorphic solutions of fractional order neutral differential equation, *Semigroup Forum*, Volume 81, Number 3 (2010), 393-404.

[2] Agarwal, R. P.; Benchohra, M.; Hamani, S.; Boundary value problems for fractional differential equations, *Georgian Math. J.,* 16, 3 (2009), 401-411.

[3] Agarwal, R.P.; Zhou, Y.; He, Y.; Existence of fractional neutral functional differential equations *Comp. Math. Appl.*, 59, 3 (2010), 1095-1100.

[4] Ahmed, E.; El-Sayed, A. M. A.; El-Saka, H. A. A.; Equilibrium points, stability and numerical solutions of fractional-order predator-prey and rabies models. *J. Math. Anal. Appl.*, Vol. 325 (2007), no. 1, 542-553.

[5] Banas, J.; Zajac, T.; A new approach to the theory of functional integral equations of fractional order *J. Math. Anal. Appl.,* 375, 2 (2011), 375-387.

[6] Daftardar-Gdjji, Varsha; Babakhani, A.; Analysis of a system of fractional differential equations, *J. Math. Anal. Appl.* 293 (2004), 511-522.

[7] Das, Shantanu; Functional Fractional Calculus for System Identification and Controls, *Springer*, 2007.

[8] Delbosco, D.; Rodino, L.; Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.,* 204 (1996), 609-625.

[9] Deng, J.; Ma, L.; Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations, *Appl. Math. Lett.*, 23, 6 (2010), 676-680.

[10] Diethelm, K.; Neville J. Ford; Analysis of fractional differential equations, *J. Math. Anal. Appl.,* 265 (2002), 229-248.

[11] Diethelm, K.; The analysis of fractional differential equations, *Springer*, 2010.

[12] El-Sayed,A. M. A.; Fractional order differential equation, *Kyungpook Math. J.*, Vol. 28, 2 (1988), 18-22.

[13] El-Sayed, A. M. A.; On the fractional differential equations. *Appl. Math. Comput.*, Vol. 49 (1992), no. 2-3, 205-213,

[14] El-Sayed, Ahmed M. A.; Nonlinear functional-differential equations of arbitrary orders. *Nonlinear Anal.* 33 (1998), no. 2, 181-186.

[15] El-Sayed, A. M. A.; El-Mesiry, A. E. M.; El-Saka, H. A. A.; On the fractional-order logistic equation, *Appl. Math. Lett.*, 20, 7 (2007), 817-823.

[16] Furati, Khaled M.; Tatar, Nasser-eddine; Long time behavior for a nonlinear fractional model, *J. Math. Anal. Appl.*, 332, 1 (2007), 441-454.

[17] Giannantoni, C.; The problem of the initial conditions and their physical meaning in linear differential equations of fractional order, *Appl. Math. Comp.*, 141, 1 (2003), 87-102.

[18] Hilfer, R. (ed.); Applications of fractional calculus in physics. *World Scientific Publishing Co., Inc.*, River Edge, NJ, 2000.

[19] Heymans, N.; Podlubny, I.; Physical interpretation of initial conditions for fractional differential equations with Riemann-Liouville fractional derivatives. *Rheol Acta*, 45 (2006), 765-771.

[20] Hadid, S. B.; Local and global existence theorems on differential equations of non-integer order, *J. Fract. Calc.,* 7 (1995), 101-105.

[21] Hadid, S. B.; Masaedeh S. Momani; On the existence of maximal and minimal solutions of differential equations of non-integer order, *J. Fract. Calc.* 9 (1996), 41-44.

[22] Ibrahim, Rabha W.; Momani, Shaher; On the existence and uniqueness of solutions of a class of fractional differential equations, *J. Math. Anal. Appl.* 334 (2007), 1-10.

[23] Jaimini, B. B.; Shrivastava, N.; Srivastava, H. M.; The integral analogue of the Leibniz rule for fractional calculus and its applications involving functions of several variables, *Comput. Math. Appl.*, 41, 1-2 (2001), 149-155.

[24] Kilbas, A. A.; Srivastava, H. M.; Trujillo, J. J.; Theory and Applications of Fractional Differential Equations, *North-Holland Mathematics Studies*, Volume 204, 2006.

[25] Lakshmikantham, V.; Vatsala, A. S.; Basic theory of fractional differential equations, *Nonlinear Analysis: TMA*, 69, 8 (2008), 2677-2682.

[26] Lakshmikantham, V.; Vatsala, A. S.; Theory of fractional differential inequalities and applications. *Communications in Applied Analysis*, Vol-II (2007), 395-402.

[27] Lakshmikantham, V.; Theory of fractional functional differential equations, *Nonlinear Analysis: TMA*, 69, 10 (2008), 3337-3343.

[28] Machado, J. A. Tenreiro; Kiryakova, V.; Mainardi, F.; Recent history of fractional calculus, *Commun. in Nonl. Sci. Numerical Simul.,* 16, 3 (2011), 1140-1153.

[29] Mainardi, F.; Fractional Calculus: Some basic problems in continuum and statistical mechanics, in Fractals and Fractional Calculus in Continuum Mechanics, Carpinteri, A. and Mainardi, F. (eds), Springer, New York, 1997.

[30] Miller, K. S.; Ross, B.; An Introduction to the Fractional Calculus and Fractional Differential Equations, *John Wiley and Sons,* Inc., New, York, 1993.

[31] Podlubny, I.; Fractional Differential Equations. Academic Press, London, 1999.

[32] Podlubny, I.; Geometric and physical interpretation of fractional integration and frac- tional differentiation. Dedicated to the 60th anniversary of Prof. Francesco Mainardi. *Fract. Calc. Appl. Anal.,* 5, 4 (2002), 367-386.

[33] Podlubny, I.; Petras, I.; Vinagre, B. M.; O'Leary, P.; Dorcak, L.; Analogue realizations of fractional-order controllers. Fractional order calculus and its applications. *Nonlinear Dynam.* 29, 1-4 (2002), 281-296.

[34] Rubanik, V. P.; Oscillations of quasilinear systems with retardation. *Nauka, Moscow*, 1969.

[35] Rudin, W.; Real and Complex Analysis, McGraw Hill, New York, 1966.

[36] Schmitt, K.; Nonlinear analysis and differential equations, an introduction, notes, 2004.

[37] Smart, D. R.; Fixed point theorems. *Cambridge Uni. Press., Cambridge*, 1980.

[38] Samko, S. G.; Kilbas, A. A.; Marichev, O.I.; Fractional Integrals and Derivatives: Theory and Applications, *Gordon and Breach,* 1993.

[39] Sabatier, J.; Agrawal, O. P.; Tenreiro Machado, J. A.; Advances in Fractional Calculus: Theoretical Developments and Applications in Physics and Engineering, *Springer*, 2007.

[40] Yu, C.; Gao, G.; Existence of fractional differential equations. *J. Math. Anal. Appl.,* 310, 1 (2005), 26-29.

[41] Yu, C.; Guozhu Gao, G.; Some results on a class of fractional functional differential equations, *Commun. Appl. Nonlinear Anal.,* 11, 3 (2004), 67-75.

[42] Zhang, X.; Some results of linear fractional order time-delay system *Appl. Math. Comp.*, 197, 1 (2008), 407-411.

[43] Zhang, S.; Positive solutions for boundary-value problems of nonlinear fractional differential equations. *Electron. J. Diff. Eqns.* 2006 (2006), No. 36, 1-12.

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