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EXISTENCE OF SOLUTIONS OF SYSTEMS OF VOLTERRA INTEGRAL EQUATIONS VIA BREZIS-BROWDER ARGUMENTS

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Abstract. We consider two systems of Volterra integral equations

$$u_i(t) = h_i(t) + \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \quad 1 \le i \le n$$

where t is in the closed interval $[0,T]$, or in the half-open interval $[0,T)$. By an argument originated from Brezis and Browder [8], criteria are offered for the existence of solutions of the systems of Volterra integral equations. We further establish the existence of *constant-sign* solutions, which include *positive* solutions (the usual consideration) as a special case. Some examples are also presented to illustrate the results obtained.

1. Introduction

In this article, we shall consider the system of Volterra integral equations

$$u_i(t) = h_i(t) + \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \tag{1.1}$$

for $t \in [0,T]$, $1 \le i \le n$, where $0 < T < \infty$; and the following system on a half-open interval

$$u_i(t) = h_i(t) + \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \tag{1.2}$$

for $t \in [0,T)$, $1 \le i \le n$, where $0 < T \le \infty$. Throughout, let $u = (u_1, u_2, \dots, u_n)$. We are interested in establishing the existence of solutions u of the systems (1.1) and (1.2), in $(C[0,T])^n = C[0,T] \times C[0,T] \times \cdots \times C[0,T]$ (n times), and $(C[0,T))^n$, respectively. In addition, we shall tackle the existence of *constant-sign* solutions of (1.1) and (1.2). A solution u of (1.1) (or (1.2)) is said to be of *constant sign* if for each $1 \le i \le n$, we have $\theta_i u_i(t) \ge 0$ for all $t \in [0,T]$ (or $t \in [0,T)$), where $\theta_i \in \{-1, 1\}$ is fixed. Note that when $\theta_i = 1$ for all $1 \le i \le n$, a constant-sign solution reduces to a *positive* solution, which is the usual consideration in the literature.

System (1.1) when $h_i = 0,\ 1 \le i \le n$ reduces to

$$u_i(t) = \int_0^t g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \quad t \in [0,T],\ 1 \le i \le n. \tag{1.3}$$

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This equation when $n = 1$ has received a lot of attention in the literature [9, 10, 11, 13, 14, 15, 19], since it arises in real-world problems. For instance, astrophysical problems (e.g., the study of the density of stars) give rise to the Emden differential equation

$$\begin{gathered} y'' - t^r y^q = 0, \quad t \in [0,T] \\ y(0) = y'(0) = 0, \quad r \geq 0,\ 0 < q < 1 \end{gathered} \tag{1.4}$$

which reduces to (1.3) with $n = 1$ when $g_1(t,s) = (t-s)s^r$ and $f_1(t,y) = y^q$. Other examples occur in nonlinear diffusion and percolation problems (see [10, 11] and the references cited therein), and here we obtain (1.3) where g_i is a convolution kernel; i.e.,

$$u_i(t) = \int_0^t g_i(t-s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \quad t \in [0,T],\ 1 \leq i \leq n. \tag{1.5}$$

In particular, Bushell and Okrasiński [10] investigated a special case of the above system given by

$$y(t) = \int_0^t (t-s)^{\gamma-1} f(y(s))ds, \quad t \in [0,T] \tag{1.6}$$

where $\gamma > 1$.

Using an argument originated from Brezis and Browder [8], we shall establish the existence of solutions as well as constant-sign solutions of the systems (1.1) and (1.2). Our results extend, improve and complement the existing theory in the literature [1, 12, 17, 18, 20, 21]. We have generalized the problems to (i) *systems*, (ii) more *general* form of nonlinearities f_i, $1 \leq i \leq n$, and (iii) existence of *constant-sign* solutions. Other related work on systems of integral equations can be found in [2, 3, 4, 5, 6, 7]. Note that the technique employed in Volterra integral equations [5, 6, 7] is entirely different from the present work. The paper is outlined as follows. In Section 2, we present an existence result for a system of Fredholm integral equations which will be used in Section 3 to develop existence criteria for (1.1) and (1.2). The existence of constant-sign solutions is tackled in Section 4. Finally, some examples are included in Section 5 to illustrate the results obtained.

2. Preliminary result

We shall obtain an existence result for the following system of Fredholm integral equations which will be used later in Section 3:

$$u_i(t) = h_i(t) + \int_0^T g_i(t,s) f_i(s, u_1(s), u_2(s), \dots, u_n(s))ds, \tag{2.1}$$

for $t \in [0,T]$, $1 \leq i \leq n$. Let the Banach space $B = (C[0,T])^n$ be equipped with the norm

$$\|u\| = \max_{1 \leq i \leq n} \sup_{t \in [0,T]} |u_i(t)| = \max_{1 \leq i \leq n} |u_i|_0$$

where we let $|u_i|_0 := \sup_{t\in[0,T]} |u_i(t)|$, $1 \leq i \leq n$.

Theorem 2.1. *For each $1 \leq i \leq n$, let $1 \leq p_i \leq \infty$ be an integer and q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Assume the following conditions hold for each $1 \leq i \leq n$:*

$$h_i \in C[0,T]; \tag{2.2}$$

$$f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R} \quad \text{is an } L^{q_i}\text{-Carathéodory function;} \tag{2.3}$$

i.e.,

(i) *the map* $u \mapsto f_i(t,u)$ *is continuous for almost all* $t \in [0,T]$,

(ii) *the map* $t \mapsto f_i(t,u)$ *is measurable for all* $u \in \mathbb{R}^n$,

(iii) *for any* $r > 0$, *there exists* $\mu_{r,i} \in L^{q_i}[0,T]$ *such that* $|u| \le r$ *implies* $|f_i(t,u)| \le \mu_{r,i}(t)$ *for almost all* $t \in [0,T]$;

$$g_i^t(s) := g_i(t,s) \in L^{p_i}[0,T] \quad \text{for each } t \in [0,T] \tag{2.4}$$

and

$$\text{the map } t \mapsto g_i^t \text{ is continuous from } [0,T] \text{ to } L^{p_i}[0,T]. \tag{2.5}$$

In addition, suppose there is a constant $M > 0$, *independent of* λ, *with* $\|u\| \neq M$ *for any solution* $u \in (C[0,T])^n$ *to*

$$u_i(t) = \lambda\Big(h_i(t) + \int_0^T g_i(t,s) f_i(s,u(s))ds\Big), \quad t \in [0,T],\ 1 \le i \le n \tag{2.6}$$

for each $\lambda \in (0,1)$. *Then* (2.1) *has at least one solution in* $(C[0,T])^n$.

Proof. Let the operator S be defined by

$$Su(t) = \big(S_1u(t), S_2u(t), \dots, S_nu(t)\big), \quad t \in [0,T] \tag{2.7}$$

where

$$S_iu(t) = h_i(t) + \int_0^T g_i(t,s) f_i(s,u(s))ds, \quad t \in [0,T],\ 1 \le i \le n. \tag{2.8}$$

Clearly, system (2.1) is equivalent to $u = Su$, and (2.6) is the same as $u = \lambda Su$.

Note that S maps $(C[0,T])^n$ into $(C[0,T])^n$; i.e., $S_i : (C[0,T])^n \to C[0,T]$, $1 \le i \le n$. To see this, note that for any $u \in (C[0,T])^n$, there exits $r > 0$ such that $\|u\| < r$. Since f_i is a L^{q_i}-Carathéodory function, there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s,u)| \le \mu_{r,i}(s)$ for almost all $s \in [0,T]$. Hence, for any $t_1, t_2 \in [0,T]$, we find for $1 \le i \le n$,

$$\begin{aligned} |S_iu(t_1) - S_iu(t_2)| &\le |h_i(t_1) - h_i(t_2)| \\ &\quad + \Big[\int_0^T |g_i^{t_1}(s) - g_i^{t_2}(s)|^{p_i}\,ds\Big]^{1/p_i} \|\mu_{r,i}\|_{q_i} \to 0 \end{aligned} \tag{2.9}$$

as $t_1 \to t_2$, where we have used (2.2) and (2.4). This shows that $S : (C[0,T])^n \to (C[0,T])^n$.

Next, we shall prove that $S : (C[0,T])^n \to (C[0,T])^n$ is continuous. Let $u^m = (u_1^m, u_2^m, \dots, u_n^m) \to u$ in $(C[0,T])^n$; i.e., $u_i^m \to u_i$ in $C[0,T]$, $1 \le i \le n$. We need to show that $Su^m \to Su$ in $(C[0,T])^n$, or equivalently $S_iu^m \to S_iu$ in $C[0,T]$, $1 \le i \le n$. There exists $r > 0$ such that $\|u^m\|, \|u\| < r$. Since f_i is a L^{q_i}-Carathéodory function, there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s,u^m)|, |f_i(s,u)| \le \mu_{r,i}(s)$ for almost all $s \in [0,T]$. Using a similar argument as in (2.9), we obtain for any $t_1, t_2 \in [0,T]$ and $1 \le i \le n$,

$$|S_iu^m(t_1) - S_iu^m(t_2)| \to 0 \quad \text{and} \quad |S_iu(t_1) - S_iu(t_2)| \to 0 \tag{2.10}$$

as $t_1 \to t_2$. Furthermore, $S_iu^m(t) \to S_iu(t)$ pointwise on $[0,T]$, since, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} |S_iu^m(t) - S_iu(t)| &\le \sup_{t\in[0,T]} \|g_i^t\|_{p_i} \Big[\int_0^T |f_i(s,u^m(s)) - f_i(s,u(s))|^{q_i} ds\Big]^{1/q_i} \\ &\to 0 \end{aligned} \tag{2.11}$$

as $m \to \infty$. Combining (2.10) and (2.11) and using the fact that $[0,T]$ is compact, gives for all $t \in [0,T]$,

$$\begin{aligned} |S_i u^m(t) - S_i u(t)| \le |S_i u^m(t) - S_i u^m(t_1)| + |S_i u^m(t_1) - S_i u(t_1)| \\ + |S_i u(t_1) - S_i u(t)| \to 0 \end{aligned} \tag{2.12}$$

as $m \to \infty$. Hence, we have proved that $S : (C[0,T])^n \to (C[0,T])^n$ is continuous.

Finally, we shall show that $S : (C[0,T])^n \to (C[0,T])^n$ is completely continuous. Let Ω be a bounded set in $(C[0,T])^n$ with $\|u\| \le r$ for all $u \in \Omega$. We need to show that $S_i\Omega$ is relatively compact for $1 \le i \le n$. Clearly, $S_i\Omega$ is uniformly bounded, since there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(s,u)| \le \mu_{r,i}(s)$ for all $u \in \Omega$ and *a.e.* $s \in [0,T]$, and hence

$$|S_i u|_0 \le |h_i|_0 + \sup_{t\in[0,T]} \|g_i^t\|_{p_i} \cdot \|\mu_{r,i}\|_{q_i} \equiv K_i, \quad u \in \Omega.$$

Further, using a similar argument as in (2.9), we see that $S_i\Omega$ is equicontinuous. It follows from the Arzéla-Ascoli theorem [21, Theorem 1.2.4] that $S_i\Omega$ is relatively compact.

We now apply the Nonlinear Alternative [21, Theorem 1.2.1] with $\tilde{N} = S$, $U = \{u \in (C[0,T])^n : \|u\| < M\}$, $C = E = (C[0,T])^n$ and $p^* = 0$ to obtain the conclusion of the theorem. □

3. Existence of solutions

In this section, we shall establish the existence of solutions of the systems (1.1) and (1.2), in $(C[0,T])^n$ and $(C[0,T))^n$ respectively. We shall first apply Theorem 2.1 to obtain an existence result for (1.1).

Theorem 3.1. *For each $1 \le i \le n$, let $1 \le p_i \le \infty$ be an integer and q_i be such that $\frac{1}{p_i} + \frac{1}{q_i} = 1$. Assume the following conditions hold for each $1 \le i \le n$:*

$$h_i \in C[0,T]; \tag{3.1}$$

$$f_i : [0,T] \times \mathbb{R}^n \to \mathbb{R} \text{ is an } L^{q_i}\text{-Carathéodory function;} \tag{3.2}$$

$$\begin{gathered} g_i^t(s) := g_i(t,s) \in L^{p_i}[0,t] \text{ for each } t \in [0,T], \\ \sup_{t\in[0,T]} \int_0^t |g_i^t(s)|^{p_i}\,ds < \infty, \quad 1 \le p_i < \infty, \\ \sup_{t\in[0,T]} \operatorname{ess\,sup}_{s\in[0,t]} |g_i^t(s)| < \infty, \quad p_i = \infty \end{gathered} \tag{3.3}$$

and for any $t, t' \in [0,T]$ with $t^* = \min\{t,t'\}$, we have

$$\begin{gathered} \int_0^{t^*} |g_i^t(s) - g_i^{t'}(s)|^{p_i}\,ds \to 0 \quad \text{as } t \to t',\ 1 \le p_i < \infty \\ \operatorname{ess\,sup}_{s\in[0,t^*]} |g_i^t(s) - g_i^{t'}(s)| \to 0 \quad \text{as } t \to t',\ p_i = \infty. \end{gathered} \tag{3.4}$$

In addition, suppose there is a constant $M > 0$, independent of λ, with $\|u\| \ne M$ for any solution $u \in (C[0,T])^n$ to

$$u_i(t) = \lambda\Big(h_i(t) + \int_0^t g_i(t,s) f_i(s,u(s))ds\Big), \quad t \in [0,T],\ 1 \le i \le n \tag{3.5}$$

for each $\lambda \in (0,1)$. Then (1.1) has at least one solution in $(C[0,T])^n$.

Proof. For each $1 \le i \le n$, define

$$g_i^*(t,s) = \begin{cases} g_i(t,s), & 0 \le s \le t \le T \\ 0, & 0 \le t \le s \le T. \end{cases}$$

Then (1.1) is equivalent to

$$u_i(t) = h_i(t) + \int_0^T g_i^*(t,s) f_i(s,u(s)) ds, \quad t \in [0,T],\ 1 \le i \le n. \tag{3.6}$$

In view of (3.3) and (3.4), g_i^* satisfies (2.4) and (2.5). Hence, by Theorem 2.1 the system (3.6) (or equivalently (1.1)) has at least one solution in $(C[0,T])^n$. □

Remark 3.2. If (3.4) is changed to: for any $t, t' \in [0,T]$ with $t^* = \min\{t,t'\}$ and $t^{**} = \max\{t,t'\}$, we have

$$\begin{gathered} \int_0^{t^*} |g_i(t,s) - g_i(t',s)|^{p_i}\, ds + \int_{t^*}^{t^{**}} |g_i(t^{**},s)|^{p_i}\, ds \to 0 \quad \text{as } t \to t', \quad 1 \le p_i < \infty, \\ \operatorname{ess\,sup}_{s\in[0,t^*]} |g_i(t,s) - g_i(t',s)| + \operatorname{ess\,sup}_{s\in[t^*,t^{**}]} |g_i(t^{**},s)| \to 0 \end{gathered} \tag{3.7}$$

as $t \to t'$, $p_i = \infty$; then automatically we have the inequalities in (3.3).

Our subsequent results use an argument originated from Brezis and Browder [8].

Theorem 3.3. *Let the following conditions be satisfied: for each $1 \le i \le n$, (3.1), (3.2)–(3.4) with $p_i = \infty$ and $q_i = 1$, there exist $B_i > 0$ such that for any $u \in (C[0,T])^n$,*

$$\int_0^T [f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s)) ds] dt \le B_i; \tag{3.8}$$

and there exist $r > 0$ and $\alpha_i > 0$ with $r\alpha_i > H_i \equiv \sup_{t\in[0,T]} |h_i(t)|$ such that for any $u \in (C[0,T])^n$,

$$u_i(t) f_i(t,u(t)) \ge r\alpha_i |f_i(t,u(t))| \quad \text{for a. e. } t \in [0,T] \text{ such that } \|u(t)\| > r, \tag{3.9}$$

where we denote $\|u(t)\| := \max_{1\le i\le n} |u_i(t)|$. Then (1.1) has at least one solution in $(C[0,T])^n$.

Proof. We shall employ Theorem 3.1, so let $u = (u_1, u_2, \dots, u_n) \in (C[0,T])^n$ be any solution of (3.5) where $\lambda \in (0,1)$. For each $z \in [0,T]$, define

$$I_z = \{t \in [0,z] : \|u(t)\| \le r\}, \quad J_z = \{t \in [0,z] : \|u(t)\| > r\}. \tag{3.10}$$

Clearly, $[0,z] = I_z \cup J_z$ and hence

$$\int_0^z = \int_{I_z} + \int_{J_z}. \tag{3.11}$$

Let $1 \le i \le n$. For a.e. $t \in I_z$, by (3.2) there exists $\mu_{r,i} \in L^1[0,T]$ such that $|f_i(t,u(t))| \le \mu_{r,i}(t)$. Thus, we obtain

$$\int_{I_z} |f_i(t,u(t))| dt \le \int_{I_z} \mu_{r,i}(t) dt \le \int_0^T \mu_{r,i}(t) dt = \|\mu_{r,i}\|_1. \tag{3.12}$$

On the other hand, if $t \in J_z$, then it is clear from (3.9) that $u_i(t) f_i(t,u(t)) \ge 0$ for a.e. $t \in [0,T]$. It follows that

$$\int_{J_z} u_i(t) f_i(t,u(t)) dt = \int_{J_z} |u_i(t) f_i(t,u(t))| dt \ge r\alpha_i \int_{J_z} |f_i(t,u(t))| dt. \tag{3.13}$$

Let $z \in [0,T]$. We now multiply (3.5) by $f_i(t,u(t))$, then integrate from 0 to z, and use (3.8) to obtain

$$\begin{aligned}
&\int_0^z u_i(t) f_i(t,u(t))dt \\
&= \lambda \int_0^z h_i(t) f_i(t,u(t))dt + \lambda \int_0^z \Big[f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s))ds \Big] dt \\
&\le H_i \int_0^z |f_i(t,u(t))|dt + B_i.
\end{aligned} \tag{3.14}$$

Splitting the integrals in (3.14) using (3.11), and applying (3.13), we obtain

$$\begin{aligned}
&\int_{I_z} u_i(t) f_i(t,u(t))dt + r\alpha_i \int_{J_z} |f_i(t,u(t))|dt \\
&\le H_i \int_{I_z} |f_i(t,u(t))|dt + H_i \int_{J_z} |f_i(t,u(t))|dt + B_i
\end{aligned}$$

or

$$\begin{aligned}
(r\alpha_i - H_i) \int_{J_z} |f_i(t,u(t))|dt &\le H_i \int_{I_z} |f_i(t,u(t))|dt + \int_{I_z} |u_i(t) f_i(t,u(t))|dt + B_i \\
&\le (H_i + r)\|\mu_{r,i}\|_1 + B_i
\end{aligned}$$

where we have used (3.12) in the last inequality. It follows that

$$\int_{J_z} |f_i(t,u(t))|dt \le \frac{(H_i + r)\|\mu_{r,i}\|_1 + B_i}{r\alpha_i - H_i} \equiv c_i. \tag{3.15}$$

Now, it is clear from (3.5) that for $t \in [0,T]$ and $1 \le i \le n$,

$$\begin{aligned}
|u_i(t)| &\le H_i + \int_0^t |g_i(t,s) f_i(s,u(s))|ds \\
&= H_i + \Big(\int_{I_t} + \int_{J_t} \Big) |g_i(t,s) f_i(s,u(s))|ds \\
&\le H_i + \big(\sup_{t\in[0,T]} \operatorname{ess\,sup}_{s\in[0,t]} |g_i(t,s)| \big) (\|\mu_{r,i}\|_1 + c_i) \equiv d_i
\end{aligned}$$

where we have applied (3.12) and (3.15) in the last inequality. Thus, $|u_i|_0 \le d_i$ for $1 \le i \le n$ and $\|u\| \le \max_{1\le i\le n} d_i \equiv D$. It follows from Theorem 3.1 (with $M = D + 1$) that (1.1) has a solution $u^* \in (C[0,T])^n$. $\square$

Our next result replaces condition (3.8) with condition (3.16) which involves the integral of f_i in the right side.

Theorem 3.4. *Let the following conditions be satisfied for each $1 \le i \le n$:* (3.1), (3.2)–(3.4) *with $p_i = \infty$ and $q_i = 1$, there exist constants $a_i \ge 0$ and b_i such that for any $z \in [0,T]$,*

$$\int_0^z \Big[f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s))ds \Big] dt \le a_i \int_0^z |f_i(t,u(t))|dt + b_i; \tag{3.16}$$

and there exist $r > 0$ and $\alpha_i > 0$ with $r\alpha_i > H_i + a_i$ such that for any $u \in (C[0,T])^n$,

$$u_i(t) f_i(t,u(t)) \ge r\alpha_i |f_i(t,u(t))| \quad \text{for a.e. } t \in [0,T] \text{ such that } \|u(t)\| > r. \tag{3.17}$$

Then (1.1) *has at least one solution in $(C[0,T])^n$.*

Proof. The proof is the same as that of Theorem 3.3 until (3.13). Let $z \in [0,T]$ and $1 \le i \le n$. Multiplying (3.5) by $f_i(t,u(t))$ and then integrating from 0 to z, we use (3.16) to get

$$\begin{aligned}
&\int_0^z u_i(t) f_i(t,u(t))dt \\
&\le \int_0^z |h_i(t) f_i(t,u(t))|dt + \lambda \int_0^z \Big[f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s))ds \Big] dt \\
&\le (H_i + a_i) \int_0^z |f_i(t,u(t))|dt + |b_i|.
\end{aligned} \tag{3.18}$$

Splitting the integrals in (3.18) and applying (3.13), we obtain

$$\begin{aligned}
&(r\alpha_i - H_i - a_i) \int_{J_z} |f_i(t,u(t))|dt \\
&\le (H_i + a_i) \int_{I_z} |f_i(t,u(t))|dt + \int_{I_z} |u_i(t) f_i(t,u(t))|dt + |b_i| \\
&\le (H_i + a_i + r) \|\mu_{r,i}\|_1 + |b_i|
\end{aligned}$$

where we have used (3.12) in the last inequality. It follows that

$$\int_{J_z} |f_i(t,u(t))|dt \le \frac{(H_i + a_i + r)\|\mu_{r,i}\|_1 + |b_i|}{r\alpha_i - H_i - a_i} \equiv c_i^*. \tag{3.19}$$

The rest of the proof proceeds as in the proof of Theorem 3.3. $\square$

The next result is for general p_i, q_i (i.e., $1 \le p_i \le \infty$ and $\frac{1}{p_i} + \frac{1}{q_i} = 1$), it also replaces condition (3.9) or (3.17) with conditions (3.20) and (3.21). Note that in Theorems 3.3 and 3.4 the conditions (3.2)–(3.4) hold for $p_i = \infty$, whereas in Theorem 3.5 the conditions (3.2)–(3.4) hold for $1 \le p_i \le \infty$.

Theorem 3.5. *Let the following conditions be satisfied: for each $1 \le i \le n$: (3.1)–(3.4), (3.8), there exist $r > 0$ and $\beta_i > 0$ such that for any $u \in (C[0,T])^n$,*

$$\begin{gathered}
u_i(t) f_i(t,u(t)) \ge \beta_i |u_i|_0 \cdot |f_i(t,u(t))| \\
\text{for a.e. } t \in [0,T] \text{ such that } \|u(t)\| > r,
\end{gathered} \tag{3.20}$$

where we denote $|u_i|_0 := \max_{t\in[0,T]} |u_i(t)|$; and there exist $\eta_i > 0$, $\gamma_i \ge q_i - 1 > 0$ and $\phi_i \in L^{p_i}([0,T],\mathbb{R})$ such that for any $u \in (C[0,T])^n$,

$$|u_i|_0 \ge \eta_i |f_i(t,u(t)|^{\gamma_i} + \phi_i(t) \quad \text{for a.e. } t \in [0,T] \text{ such that } \|u(t)\| > r. \tag{3.21}$$

Then (1.1) has at least one solution in $(C[0,T])^n$.

Proof. As in the proof of Theorem 3.3, we consider the sets I_z and J_z where $z \in [0,T]$ (see (3.10)). Let $1 \le i \le n$. If $t \in I_z$, then by (3.2) there exists $\mu_{r,i} \in L^{q_i}[0,T]$ such that $|f_i(t,u(t))| \le \mu_{r,i}(t)$. Consequently, we have

$$\int_{I_z} |f_i(t,u(t))|dt \le \int_{I_z} \mu_{r,i}(t)dt \le \int_0^T \mu_{r,i}(t)dt \le T^{1/p_i} \|\mu_{r,i}\|_{q_i}. \tag{3.22}$$

On the other hand, if $t \in J_z$, then noting (3.20) we have $u_i(t)f_i(t,u(t)) \geq 0$ for a.e. $t \in [0,T]$, and so

$$\begin{aligned} \int_{J_z} u_i(t)f_i(t,u(t))dt &= \int_{J_z} |u_i(t)f_i(t,u(t))|dt \\ &\geq \beta_i \int_{J_z} |u_i|_0 \cdot |f_i(t,u(t))|dt \\ &\geq \beta_i\eta_i \int_{J_z} |f_i(t,u(t))|^{\gamma_i+1}dt + \beta_i \int_{J_z} \phi_i(t)|f_i(t,u(t))|dt \end{aligned} \tag{3.23}$$

where we have used (3.21) in the last inequality.

Let $z \in [0,T]$. Multiplying (3.5) by $f_i(t,u(t))$ and then integrating from 0 to z, we use (3.8) to get (3.14). Splitting the integrals in (3.14) and applying (3.23), we find

$$\begin{aligned} &\int_{I_z} u_i(t)f_i(t,u(t))dt + \beta_i\eta_i \int_{J_z} |f_i(t,u(t))|^{\gamma_i+1}dt + \beta_i \int_{J_z} \phi_i(t)|f_i(t,u(t))|dt \\ &\leq H_i \int_{I_z} |f_i(t,u(t))|dt + H_i \int_{J_z} |f_i(t,u(t))|dt + B_i \end{aligned}$$

or

$$\begin{aligned} &\beta_i\eta_i \int_{J_z} |f_i(t,u(t))|^{\gamma_i+1}dt \\ &\leq \beta_i \int_{J_z} |\phi_i(t)| \cdot |f_i(t,u(t))|dt + H_i \int_{J_z} |f_i(t,u(t))|dt + B_i \\ &\quad + \int_{I_z} (|u_i(t)| + H_i)|f_i(t,u(t))|dt \\ &\leq \beta_i \int_{J_z} |\phi_i(t)| \cdot |f_i(t,u(t))|dt + H_i \int_{J_z} |f_i(t,u(t))|dt + B_i \\ &\quad + (r + H_i)T^{1/p_i}\|\mu_{r,i}\|_{q_i} \\ &= \beta_i \int_{J_z} |\phi_i(t)| \cdot |f_i(t,u(t))|dt + H_i \int_{J_z} |f_i(t,u(t))|dt + B_i' \end{aligned} \tag{3.24}$$

where (3.22) has been used in the last inequality and $B_i' \equiv B_i+(r+H_i)T^{1/p_i}\|\mu_{r,i}\|_{q_i}$.

Next, an application of Hölder's inequality gives

$$\begin{aligned} &\int_{J_z} |\phi_i(t)| \cdot |f_i(t,u(t))|dt \\ &\leq \Big[\int_0^T |\phi_i(t)|^{(\gamma_i+1)/\gamma_i}dt\Big]^{\gamma_i/(\gamma_i+1)} \cdot \Big[\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1}dt\Big]^{1/\gamma_i+1}. \end{aligned} \tag{3.25}$$

Another application of Hölder's inequality yields

$$\int_0^T |\phi_i(t)|^{\frac{\gamma_i+1}{\gamma_i}}dt \leq T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i\gamma_i}}\Big[\int_0^T |\phi_i(t)|^{p_i}dt\Big]^{\frac{\gamma_i+1}{\gamma_i p_i}},$$

which upon substituting into (3.25) leads to

$$\int_{J_z} |\phi_i(t)|\cdot|f_i(t,u(t))|dt \leq T^{\frac{\gamma_i p_i-\gamma_i-1}{p_i(\gamma_i+1)}}\|\phi_i\|_{p_i}\Big[\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1}dt\Big]^{1/(\gamma_i+1)}. \tag{3.26}$$

Similarly, we have

$$\int_{J_z} |f_i(t,u(t))|dt \le T^{\frac{\gamma_i p_i - \gamma_i - 1}{p_i(\gamma_i+1)} + \frac{1}{p_i}} \Big[\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \Big]^{1/(\gamma_i+1)}. \tag{3.27}$$

Substituting (3.26) and (3.27) into (3.24), we obtain

$$\beta_i \eta_i \int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \le A_i \Big[\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \Big]^{1/(\gamma_i+1)} + B_i' \tag{3.28}$$

where

$$A_i = T^{\frac{\gamma_i p_i - \gamma_i - 1}{p_i(\gamma_i+1)}} \big(\beta_i \|\phi_i\|_{p_i} + H_i T^{1/p_i} \big).$$

Since $\frac{1}{\gamma_i+1} < 1$, from (3.28) there exists a constant c_i^{**} such that

$$\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \le c_i^{**}. \tag{3.29}$$

Now, it is clear from (3.5) that for $t \in [0,T]$ and $1 \le i \le n$,

$$\begin{aligned}
|u_i(t)| &\le H_i + \int_0^t |g_i(t,s) f_i(s,u(s))| ds \\
&= H_i + \int_{I_t} |g_i(t,s) f_i(s,u(s))| ds + \int_{J_t} |g_i(t,s) f_i(s,u(s))| ds \\
&\le H_i + \big(\sup_{t\in[0,T]} \|g_i^t\|_{p_i} \big) \|\mu_{r,i}\|_{q_i} \\
&\quad + T^{\frac{\gamma_i p_i - \gamma_i - 1}{p_i(\gamma_i+1)}} \big(\sup_{t\in[0,T]} \|g_i^t\|_{p_i} \big) \Big[\int_{J_t} |f_i(s,u(s))|^{\gamma_i+1} ds \Big]^{1/(\gamma_i+1)} \\
&\le d_i^* \quad \text{(a constant)},
\end{aligned}$$

where in the second last inequality a similar argument as in (3.26) is used and in the last inequality we have used (3.29). Thus, $|u_i|_0 \le d_i^*$ for $1 \le i \le n$ and $\|u\| \le \max_{1\le i\le n} d_i^* \equiv D^*$. It follows from Theorem 3.1 (with $M = D^* + 1$) that (1.1) has a solution $u^* \in (C[0,T])^n$. $\square$

The next result is also for general p_i, q_i, and here the condition (3.8) is replaced by (3.16).

Theorem 3.6. *Let the following conditions be satisfied for each $1 \le i \le n$: (3.1)–(3.4), (3.16), (3.20) and (3.21). Then (1.1) has at least one solution in $(C[0,T])^n$.*

Proof. The proof is similar to that of Theorem 3.5 until (3.23). Let $z \in [0,T]$ and $1 \le i \le n$. Multiplying (3.5) by $f_i(t,u(t))$ and then integrating from 0 to z, we use (3.16) to get (3.18).

Splitting the integrals in (3.18) and applying (3.23), we find

$$
\begin{aligned}
&\beta_i\eta_i\int_{J_z}|f_i(t,u(t))|^{\gamma_i+1}dt\\
&\le \beta_i\int_{J_z}|\phi_i(t)|\cdot|f_i(t,u(t))|dt+(H_i+a_i)\int_{J_z}|f_i(t,u(t))|dt+|b_i|\\
&\quad+\int_{I_z}(|u_i(t)|+H_i+a_i)|f_i(t,u(t))|dt\\
&\le \beta_i\int_{J_z}|\phi_i(t)|\cdot|f_i(t,u(t))|dt+(H_i+a_i)\int_{J_z}|f_i(t,u(t))|dt+|b_i|\\
&\quad+(r+H_i+a_i)T^{1/p_i}\|\mu_{r,i}\|_{q_i}\\
&=\beta_i\int_{J_z}|\phi_i(t)|\cdot|f_i(t,u(t))|dt+(H_i+a_i)\int_{J_z}|f_i(t,u(t))|dt+B_i''
\end{aligned} \tag{3.30}
$$

where $B_i''\equiv|b_i|+(r+H_i+a_i)T^{1/p_i}\|\mu_{r,i}\|_{q_i}$. Substituting (3.26) and (3.27) into (3.30) then leads to

$$
\beta_i\eta_i\int_{J_z}|f_i(t,u(t))|^{\gamma_i+1}dt\le A_i'\Big[\int_{J_z}|f_i(t,u(t))|^{\gamma_i+1}dt\Big]^{1/(\gamma_i+1)}+B_i'' \tag{3.31}
$$

where

$$
A_i'=T^{\frac{\gamma_ip_i-\gamma_i-1}{p_i(\gamma_i+1)}}\big[\beta_i\|\phi_i\|_{p_i}+(H_i+a_i)T^{1/p_i}\big].
$$

Since $\frac{1}{\gamma_i+1}<1$, from (3.31) we obtain

$$
\int_{J_z}|f_i(t,u(t))|^{\gamma_i+1}dt\le\bar{c}_i \tag{3.32}
$$

where $\bar{c}_i$ is a constant. The rest of the proof proceeds as in that of Theorem 3.5. $\square$

We shall now tackle the system (1.2). Our next theorem is a variation of an existence principle of Lee and O'Regan [16].

Theorem 3.7. *For each $1\le i\le n$, let $1\le p_i\le\infty$ be an integer and q_i be such that $\frac{1}{p_i}+\frac{1}{q_i}=1$. Assume the following conditions hold for each $1\le i\le n$:* (3.1), (3.3), (3.4) *and*

$$
f_i:[0,T)\times\mathbb{R}^n\to\mathbb{R}\ \text{is a locally } L^{q_i}\text{-Carathéodory function;} \tag{3.33}
$$

i.e., the conditions (i)–(iii) in (2.3) *hold when f_i is restricted to $I\times\mathbb{R}^n$, where I is any compact subinterval of $[0,T)$. Also let $\{t_k\}$ be a positive and increasing sequence such that $\lim_{k\to\infty}t_k=T$. For each $k=1,2,\dots$, suppose there exists $u^k=(u_1^k,u_2^k,\dots,u_n^k)\in(C[0,t_k])^n$ that satisfies*

$$
u_i^k(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\dots,u_n^k(s))ds, \tag{3.34}
$$

for $t\in[0,t_k]$, $1\le i\le n$. Further, for $1\le i\le n$ and $\ell=1,2,\dots$, there are bounded sets $B_\ell\subseteq\mathbb{R}$ such that $k\ge\ell$ implies $u_i^k(t)\in B_\ell$ for each $t\in[0,t_\ell]$. Then (1.2) *has a solution $u^*\in(C[0,T))^n$ such that for $1\le i\le n$, $u_i^*(t)\in\overline{B}_\ell$ for each $t\in[0,t_\ell]$.*

Proof. First we shall show that for each $1\le i\le n$ and $\ell=1,2,\dots$,

$$
\text{the sequence } \{u_i^k\}_{k\ge\ell} \text{ is uniformly bounded and equicontinuous on } [0,t_\ell]. \tag{3.35}
$$

The uniform boundedness of $\{u_i^k\}_{k\ge\ell}$ follows immediately from the hypotheses, therefore we only need to prove that $\{u_i^k\}_{k\ge\ell}$ is equicontinuous. Let $1\le i\le n$. Since for all $k\ge\ell$, $u_i^k(t)\in B_\ell$ for each $t\in[0,t_\ell]$, there exists $\mu_{B_\ell}\in L^{q_i}[0,t_\ell]$ such that $|f_i(s,u^k(s))|\le \mu_{B_\ell}(s)$ for almost every $s\in[0,t_\ell]$. Fix $t,t'\in[0,t_\ell]$ with $t<t'$. Then, noting (3.4), from (3.34) we find

$$
\begin{aligned}
&|u_i^k(t)-u_i^k(t')|\\
&\le |h_i(t)-h_i(t')|+\int_0^t |g_i^t(s)-g_i^{t'}(s)|\cdot|f_i(s,u^k(s))|ds\\
&\quad+\int_t^{t'}|g_i^{t'}(s)|\cdot|f_i(s,u^k(s))|ds\\
&\le |h_i(t)-h_i(t')|+\Big[\int_0^t|g_i^t(s)-g_i^{t'}(s)|^{p_i}ds\Big]^{1/p_i}\Big[\int_0^t\big(\mu_{B_\ell}(s)\big)^{q_i}ds\Big]^{1/q_i}\\
&\quad+\Big[\int_t^{t'}|g_i^{t'}(s)|^{p_i}ds\Big]^{1/p_i}\Big[\int_t^{t'}\big(\mu_{B_\ell}(s)\big)^{q_i}ds\Big]^{1/q_i}\to 0
\end{aligned}
$$

as $t\to t'$. Therefore, $\{u_i^k\}_{k\ge\ell}$ is equicontinuous on $[0,t_\ell]$.

Let $1\le i\le n$. Now, (3.35) and the Arzéla-Ascoli Theorem yield a subsequence N_1 of $\mathbb{N}=\{1,2,\dots\}$ and a function $z_i^1\in C[0,t_1]$ such that $u_i^k\to z_i^1$ uniformly on $[0,t_1]$ as $k\to\infty$ in N_1. Let $N_2^*=N_1\backslash\{1\}$. Then (3.35) and the Arzéla-Ascoli Theorem yield a subsequence N_2 of N_2^* and a function $z_i^2\in C[0,t_2]$ such that $u_i^k\to z_i^2$ uniformly on $[0,t_2]$ as $k\to\infty$ in N_2. Note that $z_i^2=z_i^1$ on $[0,t_1]$ since $N_2\subseteq N_1$. Continuing this process, we obtain subsequences of integers $N_1,N_2,\dots$ with

$$
N_1\supseteq N_2\supseteq\dots\supseteq N_\ell\supseteq\dots,\quad \text{where } N_\ell\subseteq\{\ell,\ell+1,\dots\},
$$

and functions $z_i^\ell\in C[0,t_\ell]$ such that $u_i^k\to z_i^\ell$ uniformly on $[0,t_\ell]$ as $k\to\infty$ in N_ℓ, and $z_i^{\ell+1}=z_i^\ell$ on $[0,t_\ell]$, $ell=1,2,\dots$.

Let $1\le i\le n$. Define a function $u_i^*:[0,T)\to\mathbb{R}$ by

$$
u_i^*(t)=z_i^\ell(t),\quad t\in[0,t_\ell]. \tag{3.36}
$$

Clearly, $u_i^*\in C[0,T)$ and $u_i^*(t)\in\overline{B}_\ell$ for each $t\in[0,t_\ell]$. It remains to prove that $u^*=(u_1^*,u_2^*,\dots,u_n^*)$ solves (1.2). Fix $t\in[0,T)$. Then choose and fix ℓ such that $t\in[0,t_\ell]$. Take $k\ge\ell$. Now, from (3.34) we have

$$
u_i^k(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\dots,u_n^k(s))ds,\quad t\in[0,t_\ell]. \tag{3.37}
$$

Since f_i is a locally L^{q_i}-Carathéodory function and $u_i^k(t)\in B_\ell$ for each $t\in[0,t_\ell]$, there exists $\mu_{B_\ell}\in L^{q_i}[0,t_\ell]$ such that $|f_i(s,u^k(s))|\le\mu_{B_\ell}(s)$ for almost every $s\in[0,t_\ell]$. Hence, we have

$$
|g_i(t,s)f_i(s,u_1^k(s),u_2^k(s),\dots,u_n^k(s))|\le|g_i^t(s)|\mu_{B_\ell}(s),\ a.e.\ s\in[0,t]
$$

and $|g_i^t|\mu_{B_\ell}\in L^1[0,t]$. Let $k\to\infty$ in (3.37). Since $u_i^k\to z_i^\ell$ uniformly on $[0,t_\ell]$, an application of Lebesgue Dominated Convergence Theorem gives

$$
z_i^\ell(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,z_1^\ell(s),z_2^\ell(s),\dots,z_n^\ell(s))ds,\quad t\in[0,t_\ell]
$$

or equivalently (noting (3.36))

$$u_i^*(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,u_1^*(s),u_2^*(s),\dots,u_n^*(s))ds,\quad t\in[0,t_\ell]. \tag{3.38}$$

Finally, letting $\ell\to\infty$ in (3.38) yields

$$u_i^*(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,u_1^*(s),u_2^*(s),\dots,u_n^*(s))ds,\quad t\in[0,T).$$

Hence, $u^*=(u_1^*,u_2^*,\dots,u_n^*)$ is a solution of (1.2). □

Our subsequent results make use of Theorem 3.7 and an argument originated from Brezis and Browder [8].

Theorem 3.8. *Let the following conditions be satisfied for each $1\le i\le n$: (3.1), (3.3), (3.4) and (3.33) with $p_i=\infty$ and $q_i=1$. Moreover, suppose the following conditions hold for each $1\le i\le n$ and any $w\in(0,T)$: there exist $B_i>0$ such that for any $u\in(C[0,w])^n$,*

$$\int_0^w\Big[f_i(t,u(t))\int_0^t g_i(t,s)f_i(s,u(s))ds\Big]dt\le B_i\,; \tag{3.39}$$

and there exist $r>0$ and $\alpha_i>0$ with $r\alpha_i>H_i(w)\equiv\sup_{t\in[0,w]}|h_i(t)|$ such that for any $u\in(C[0,w])^n$,

$$u_i(t)f_i(t,u(t))\ge r\alpha_i|f_i(t,u(t))|\quad\text{for a.e. } t\in[0,w] \text{ such that } \|u(t)\|>r, \tag{3.40}$$

where we denote $\|u(t)\|:=\max_{1\le i\le n}|u_i(t)|$. Then (1.2) has at least one solution in $(C[0,T))^n$.

Proof. We shall establish the existence of 'local' solutions before we can apply Theorem 3.7. Indeed, we shall show that the system

$$u_i(t)=h_i(t)+\int_0^t g_i(t,s)f_i(s,u(s))ds,\quad t\in[0,w],\ 1\le i\le n \tag{3.41}$$

has a solution for any $w\in(0,T)$. Let $w\in(0,T)$ be fixed. From the hypotheses, we see that (3.1)–(3.4) are satisfied with T replaced by w. We shall employ a similar technique as in the proof of Theorem 3.3, with T replaced by w. Let $u=(u_1,u_2,\dots,u_n)\in(C[0,w])^n$ be any solution of

$$u_i(t)=\lambda\Big(h_i(t)+\int_0^t g_i(t,s)f_i(s,u(s))ds\Big),\quad t\in[0,w],\ 1\le i\le n \tag{3.42}$$

where $\lambda\in(0,1)$. We define for each $z\in[0,w]$,

$$I_z=\{t\in[0,z]:\|u(t)\|\le r\},\quad J_z=\{t\in[0,z]:\|u(t)\|>r\}.$$

Following the proof of Theorem 3.3, we obtain, corresponding to (3.15),

$$\int_{J_z}|f_i(t,u(t))|dt\le\frac{[H_i(w)+r]\int_0^w\mu_{r,i}(s)ds+B_i}{r\alpha_i-H_i(w)}\equiv c_i(w),\quad 1\le i\le n. \tag{3.43}$$

Consequently, from (3.42) it follows that for $t\in[0,w]$ and $1\le i\le n$,

$$\begin{aligned}|u_i(t)|&\le H_i(w)+[\sup_{t\in[0,w]}\operatorname{ess\,sup}_{s\in[0,t]}|g_i(t,s)|]\Big[\int_0^w\mu_{r,i}(s)ds+c_i(w)\Big]\\&\equiv d_i(w).\end{aligned} \tag{3.44}$$

Thus, $|u_i|_0 = \sup_{t\in[0,w]} |u_i(t)| \le d_i(w)$ for $1 \le i \le n$ and $\|u\| = \max_{1\le i\le n} |u_i|_0 \le \max_{1\le i\le n} d_i(w) \equiv D(w)$. It follows from Theorem 3.1 (with $M = D(w) + 1$) that (3.41) has a solution $u^* \in (C[0, w])^n$. Hence, we have shown that (3.41) has a solution for any $w \in (0, T)$.

Now, let $\{t_k\}$ be a positive and increasing sequence such that $\lim_{k\to\infty} t_k = T$. For each $k = 1, 2, \dots$, let $u^k = (u_1^k, u_2^k, \dots, u_n^k) \in (C[0, t_k])^n$ be a solution of (3.34). If we restrict $z \in [0, t_\ell]$ and $k \ge \ell$, then using the same arguments as before, we can obtain (3.43) and (3.44) with $w = t_\ell$ and $u = u^k$. So for $k \ge \ell$ we have

$$|u_i^k(t)| \le d_i(t_\ell), \quad t \in [0, t_\ell],\ 1 \le i \le n.$$

All the conditions of Theorem 3.7 are satisfied and hence it follows that (1.2) has at least one solution in $(C[0, T))^n$. $\square$

Our next result replaces condition (3.39) with condition (3.45) which involves the integral of f_i in the right side.

Theorem 3.9. *Let the following conditions be satisfied for each $1 \le i \le n$: (3.1), (3.3), (3.4) and (3.33) with $p_i = \infty$ and $q_i = 1$. Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$: there exist constants $a_i \ge 0$ and b_i such that for any $z \in [0, w]$,*

$$\int_0^z \Big[f_i(t, u(t)) \int_0^t g_i(t, s) f_i(s, u(s))ds\Big]dt \le a_i \int_0^z |f_i(t, u(t))|dt + b_i; \tag{3.45}$$

and there exist $r > 0$ and $\alpha_i > 0$ with $r\alpha_i > H_i(w) + a_i$ such that for any $u \in (C[0, w])^n$,

$$u_i(t) f_i(t, u(t)) \ge r\alpha_i |f_i(t, u(t))| \quad \text{for a.e. } t \in [0, w] \text{ such that } \|u(t)\| > r. \tag{3.46}$$

Then (1.2) has at least one solution in $(C[0, T))^n$.

Proof. As in the proof of Theorem 3.8, we shall first show that the system (3.41) has a solution for any $w \in (0, T)$. Let $w \in (0, T)$ be fixed and let $u = (u_1, u_2, \dots, u_n) \in (C[0, w])^n$ be any solution of (3.42). Using a similar argument as in the proof of Theorem 3.4, with T replaced by w, we obtain, corresponding to (3.19),

$$\int_{J_z} |f_i(t, u(t))|dt \le \frac{[H_i(w) + a_i + r] \int_0^w \mu_{r,i}(s)ds + |b_i|}{r\alpha_i - H_i(w) - a_i} \equiv c_i^*(w), \tag{3.47}$$

for $1 \le i \le n$, and subsequently $\|u\| \le D^*(w)$ (a constant). Then, it follows from Theorem 3.1 that (3.41) has a solution for any $w \in (0, T)$. The rest of the proof proceeds as in the proof of Theorem 3.8. $\square$

The next result is for general p_i, q_i, it also replaces condition (3.40) or (3.46) with conditions (3.48) and (3.49).

Theorem 3.10. *Let the following conditions be satisfied for each $1 \le i \le n$: (3.1), (3.3), (3.4) and (3.33). Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$: (3.39), there exist $r > 0$ and $\beta_i > 0$ such that for any $u \in (C[0, w])^n$,*

$$u_i(t) f_i(t, u(t)) \ge \beta_i |u_i|_0 \cdot |f_i(t, u(t))| \quad \text{for a.e. } t \in [0, w] \text{ such that } \|u(t)\| > r, \tag{3.48}$$

where we denote $|u_i|_0 := \max_{t\in[0,w]} |u_i(t)$—; *and there exist* $\eta_i > 0$, $\gamma_i \geq q_i - 1 > 0$ *and* $\phi_i \in L^{p_i}([0,w],\mathbb{R})$ *such that for any* $u \in (C[0,w])^n$,

$$|u_i|_0 \geq \eta_i |f_i(t,u(t)|^{\gamma_i} + \phi_i(t) \quad \text{for a.e. } t \in [0,w] \text{ such that } \|u(t)\| > r. \tag{3.49}$$

Then (1.2) *has at least one solution in* $(C[0,T))^n$.

Proof. Once again we shall employ Theorem 3.1 to show the existence of 'local' solutions; i.e., the system (3.41) has a solution for any $w \in (0,T)$. For this, we use a similar argument as in the proof of Theorem 3.5, with T replaced by w, to get an analog of (3.29), viz.,

$$\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \leq c_i^{**}(w), \quad 1 \leq i \leq n \tag{3.50}$$

which leads to $\|u\| \leq D^*(w)$ (a constant). The rest of the proof follows as in the proof of Theorem 3.8. ☐

The next result is also for general p_i, q_i, and here the condition (3.39) is replaced by (3.45).

Theorem 3.11. *Let the following conditions be satisfied for each* $1 \leq i \leq n$*:* (3.1)*,* (3.3)*,* (3.4) *and* (3.33)*. Moreover, suppose the following conditions hold for each* $1 \leq i \leq n$ *and any* $w \in (0,T)$*:* (3.45)*,* (3.48) *and* (3.49)*. Then* (1.2) *has at least one solution in* $(C[0,T))^n$.

Proof. To prove that the system (3.41) has a solution for any $w \in (0,T)$, we use a similar argument as in the proof of Theorem 3.6, with T replaced by w, to get an analog of (3.32), viz.,

$$\int_{J_z} |f_i(t,u(t))|^{\gamma_i+1} dt \leq \bar{c}_i(w),\ 1 \leq i \leq n \tag{3.51}$$

and subsequently $\|u\| \leq D^*(w)$ (a constant). The rest of the proof proceeds as in the proof of Theorem 3.8. ☐

4. Existence of constant-sign solutions

In this section, we shall establish the existence of *constant-sign* solutions of the systems (1.1) and (1.2), in $(C[0,T])^n$ and $(C[0,T))^n$ respectively. Once again we shall employ an argument originated from Brezis and Browder [8].

Throughout, let $\theta_i \in \{-1,1\}$, $1 \leq i \leq n$ be fixed. For each $1 \leq j \leq n$, we define

$$[0,\infty)_j = \begin{cases} [0,\infty), & \theta_j = 1 \\ (-\infty,0], & \theta_j = -1. \end{cases}$$

Our first result is for the system (1.1) and is 'parallel' to Theorem 3.3.

Theorem 4.1. *Let the following conditions be satisfied for each* $1 \leq i \leq n$*:* (3.1)*,* (3.2)–(3.4) *with* $p_i = \infty$ *and* $q_i = 1$*,* (3.8)*,* (3.9)*,*

$$\theta_i h_i(t) \geq 0, \quad t \in [0,T]; \tag{4.1}$$

$$g_i(t,s) \geq 0, \quad 0 \leq s \leq t \leq T; \tag{4.2}$$

$$\theta_i f_i(t,u) \geq 0, \quad (t,u) \in [0,T] \times \prod_{j=1}^n [0,\infty)_j. \tag{4.3}$$

Then (1.1) *has at least one constant-sign solution in* $(C[0,T])^n$.

Proof. First, we shall show that the system

$$u_i(t) = h_i(t) + \int_0^t g_i(t,s) f_i^*(s, u(s))ds, \quad t \in [0,T],\ 1 \le i \le n \tag{4.4}$$

has a solution in $(C[0,T])^n$. Here,

$$f_i^*(t, u_1, \dots, u_n) = f_i(t, v_1, \dots, v_n), \quad t \in [0,T],\ 1 \le i \le n \tag{4.5}$$

where

$$v_j = \begin{cases} u_j, & \theta_j u_j \ge 0 \\ 0, & \theta_j u_j \le 0. \end{cases}$$

Clearly, $f_i^*(t,u) : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ and f_i^* satisfies (3.2).

We shall employ Theorem 3.1, so let $u = (u_1, u_2, \dots, u_n) \in (C[0,T])^n$ be any solution of

$$u_i(t) = \lambda\Big(h_i(t) + \int_0^t g_i(t,s) f_i^*(s, u(s))ds\Big), \quad t \in [0,T],\ 1 \le i \le n \tag{4.6}$$

where $\lambda \in (0,1)$. Using (4.1)–(4.3), we have for $t \in [0,T]$ and $1 \le i \le n$,

$$\theta_i u_i(t) = \lambda\Big(\theta_i h_i(t) + \int_0^t g_i(t,s)\theta_i f_i^*(s, u(s))ds\Big) \ge 0.$$

Hence, u is a *constant-sign* solution of (4.6), and it follows that

$$f_i^*(t, u(t)) = f_i(t, u(t)), \quad t \in [0,T],\ 1 \le i \le n. \tag{4.7}$$

For each $z \in [0,T]$, define I_z and J_z as in (3.10). Noting (4.7), we see that (4.6) is the same as (3.5). Therefore, using a similar technique as in the proof of Theorem 3.3, we obtain (3.12)–(3.15) and subsequently $|u_i|_0 \le d_i$ for $1 \le i \le n$. Thus, $\|u\| \le \max_{1 \le i \le n} d_i \equiv D$. It now follows from Theorem 3.1 (with $M = D + 1$) that (4.4) has a solution $u^* \in (C[0,T])^n$.

Noting (4.1)–(4.3), we have for $t \in [0,T]$ and $1 \le i \le n$,

$$\theta_i u_i^*(t) = \theta_i h_i(t) + \int_0^t g_i(t,s)\theta_i f_i^*(s, u^*(s))ds \ge 0.$$

So u^* is of *constant sign*. From (4.5), it is then clear that

$$f_i^*(t, u^*(t)) = f_i(t, u^*(t)), \quad t \in [0,T],\ 1 \le i \le n.$$

Hence, the system (4.4) is actually (1.1). This completes the proof. □

Based on the proof of Theorem 4.1, we can develop parallel results to Theorems 3.4–3.6 as follows.

Theorem 4.2. *Let the following conditions be satisfied for each* $1 \le i \le n$*:* (3.1), (3.2)–(3.4) *with* $p_i = \infty$ *and* $q_i = 1$, (3.16), (3.17) *and* (4.1)–(4.3)*. Then* (1.1) *has at least one constant-sign solution in* $(C[0,T])^n$.

Theorem 4.3. *Let the following conditions be satisfied for each* $1 \le i \le n$*:* (3.1)–(3.4), (3.8), (3.20), (3.21) *and* (4.1)–(4.3)*. Then* (1.1) *has at least one constant-sign solution in* $(C[0,T])^n$.

Theorem 4.4. *Let the following conditions be satisfied for each* $1 \le i \le n$*:* (3.1)–(3.4), (3.16), (3.20), (3.21) *and* (4.1)–(4.3)*. Then* (1.1) *has at least one constant-sign solution in* $(C[0,T])^n$.

We shall now establish the existence of constant-sign solutions of the system (1.2). The next result is 'parallel' to Theorem 3.8.

Theorem 4.5. *Let the following conditions be satisfied for each $1 \le i \le n$:* (3.1), (3.3), (3.4) *and* (3.33) *with $p_i = \infty$ and $q_i = 1$, and* (4.1)–(4.3). *Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$:* (3.39) *and* (3.40). *Then* (1.2) *has at least one constant-sign solution in $(C[0, T))^n$.*

Proof. To apply Theorem 3.7, we should show the existence of 'local' solutions by considering the following analog to (3.41),

$$u_i(t) = h_i(t) + \int_0^t g_i(t, s) f_i^*(s, u(s)) ds, \quad t \in [0, w],\ 1 \le i \le n \tag{4.8}$$

where $w \in (0, T)$ and f_i^* is given in (4.5). The rest of the proof models that of Theorems 4.1 and 3.8. $\square$

Based on the proof of Theorem 4.5, parallel results to Theorems 3.9–3.11 are established as follows.

Theorem 4.6. *Let the following conditions be satisfied for each $1 \le i \le n$:* (3.1), (3.3), (3.4) *and* (3.33) *with $p_i = \infty$ and $q_i = 1$, and* (4.1)–(4.3). *Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$:* (3.45) *and* (3.46). *Then* (1.2) *has at least one constant-sign solution in $(C[0, T))^n$.*

Theorem 4.7. *Let the following conditions be satisfied for each $1 \le i \le n$:* (3.1), (3.3), (3.4), (3.33) *and* (4.1)–(4.3). *Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$:* (3.39), (3.48) *and* (3.49). *Then* (1.2) *has at least one constant-sign solution in $(C[0, T))^n$.*

Theorem 4.8. *Let the following conditions be satisfied for each $1 \le i \le n$:* (3.1), (3.3), (3.4), (3.33) *and* (4.1)–(4.3). *Moreover, suppose the following conditions hold for each $1 \le i \le n$ and any $w \in (0, T)$:* (3.45), (3.48) *and* (3.49). *Then* (1.2) *has at least one constant-sign solution in $(C[0, T))^n$.*

5. Examples

We shall now illustrate the results obtained through some examples.

Example 5.1. Consider system (1.1) where for $1 \le i \le n$,

$$\begin{aligned} &f_i(t, u_1(t), u_2(t), \dots, u_n(t)) \\ &= \begin{cases} \rho_i(t, u_1(t), u_2(t), \dots, u_n(t)), & u_1(t), u_2(t), \dots, u_n(t) > \delta \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{5.1}$$

Here, $\delta > 0$ is a given constant, and ρ_i is such that

(a) the map $u \mapsto f_i(t, u)$ is continuous for almost all $t \in [0, T]$;
(b) the map $t \mapsto f_i(t, u)$ is measurable for all $u \in \mathbb{R}^n$;
(c) $\rho_i(t, u(t)) \in L^1[0, T]$ and $u_i(t)\rho_i(t, u(t)) \ge 0$ for any $u \in K$ where
$$K = \{u \in (C[0, T])^n : u_1(t), u_2(t), \dots, u_n(t) > \delta,\ t \in [0, T]\}.$$

Moreover, suppose $h_i \in C[0, T],\ 1 \le i \le n$ fulfills

$$H_i \equiv \sup_{t \in [0, T]} |h_i(t)| < \delta. \tag{5.2}$$

Clearly, conditions (3.1) and (3.2) with $q_i = 1$ are fulfilled. We shall check that condition (3.9) is satisfied. Pick $r > \delta$ and $\alpha_i = \frac{\delta}{r}$, $1 \le i \le n$. Then, from (5.2), we have $r\alpha_i = \delta > H_i$.

Let $u \in K$. Then, from (5.1), we have $f_i(t,u) = \rho_i(t,u)$. Consider $\|u(t)\| > r$ where $t \in [0,T]$. If $\|u(t)\| = |u_i(t)|$, then

$$\begin{aligned} u_i(t)f_i(t,u(t)) = |u_i(t)| \cdot |f_i(t,u(t))| &= \|u(t)\| \cdot |f_i(t,u(t))| \\ &> r|f_i(t,u(t))| \\ &> r \cdot \frac{\delta}{r} \cdot |f_i(t,u(t))| \\ &= r\alpha_i |f_i(t,u(t))|. \end{aligned} \tag{5.3}$$

If $\|u(t)\| = |u_k(t)|$ for some $k \ne i$, then

$$\begin{aligned} u_i(t)f_i(t,u(t)) = |u_i(t)| \cdot |f_i(t,u(t))| &= r \cdot \frac{|u_i(t)|}{r} \cdot |f_i(t,u(t))| \\ &> r \cdot \frac{\delta}{r} \cdot |f_i(t,u(t))| \\ &= r\alpha_i |f_i(t,u(t))|. \end{aligned} \tag{5.4}$$

Therefore, from (5.3) and (5.4) we see that condition (3.9) holds for $u \in K$.

For $u \in (C[0,T])^n \backslash K$, we have $f_i(t,u) = 0$ and (3.9) is trivially true. Hence, we have shown that condition (3.9) is satisfied.

The next example considers an $g_i(t,s)$ of which the particular case when $n = 1$ (see (1.6)) has been investigated by Bushell and Okrasiński [10].

Example 5.2. Consider system (1.1) with (5.1), (5.2), and for $1 \le i \le n$,

$$g_i(t,s) = (t-s)^{\gamma_i - 1} \tag{5.5}$$

where $\gamma_i > 1$.

Clearly, g_i satisfies (3.3) and (3.4) with $p_i = \infty$. Next, for $u \in K$ (K is given in Example 5.1) we have

$$\begin{aligned} &\int_0^T \Big[f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s)) ds \Big] dt \\ &= \int_0^T \Big[\rho_i(t,u(t)) \int_0^t (t-s)^{\gamma_i - 1} \rho_i(s,u(s)) ds \Big] dt \\ &\le T^{\gamma_i - 1} \int_0^T \Big[\rho_i(t,u(t)) \int_0^t \rho_i(s,u(s)) ds \Big] dt \le B_i \end{aligned} \tag{5.6}$$

since $\rho_i(t,u(t)) \in L^1[0,T]$ for any $u \in K$. This shows that condition (3.8) holds for $u \in K$. For $u \in (C[0,T])^n \backslash K$, we have $f_i(t,u) = 0$ and (3.8) is trivially true. Therefore, condition (3.8) is satisfied. It now follows from Theorem 3.3 that the system (1.1) with (5.1), (5.2) and (5.5) has at least one solution in $(C[0,T])^n$.

The next example considers an $g_i(t,s)$ of which the particular case when $n = 1$ comes from the Emden differential equation (1.4).

Example 5.3. Consider system (1.1) with (5.1), (5.2), and for $1 \le i \le n$,

$$g_i(t,s) = (t-s)s^{r_i} \tag{5.7}$$

where $r_i \ge 0$.

Clearly, g_i satisfies (3.3) and (3.4) with $p_i = \infty$. Next, for $u \in K$ (K is given in Example 5.1), corresponding to (5.6) we have

$$\begin{aligned} &\int_0^{[} f_i(t,u(t)) \int_0^t g_i(t,s) f_i(s,u(s))ds]dt \\ &= \int_0^T [\rho_i(t,u(t)) \int_0^t (t-s)s^{r_i} \rho_i(s,u(s))ds]dt \\ &\le T^{r_i+1} \int_0^T [\rho_i(t,u(t)) \int_0^t \rho_i(s,u(s))ds]dt \le B_i. \end{aligned} \tag{5.8}$$

Hence, by Theorem 3.3 the system (1.1) with (5.1), (5.2) and (5.7) has at least one solution in $(C[0,T])^n$.

Example 5.4. Let $\theta_i = 1,\ 1 \le i \le n$. Consider system (1.1) with (5.1), (5.2), and for $1 \le i \le n$,

$$h_i(t) \ge 0, \quad t \in [0,T]. \tag{5.9}$$

Clearly, conditions (4.1) and (4.3) are fulfilled. Moreover, both $g_i(t,s)$ in (5.5) and (5.7) satisfy condition (4.2). From Examples 5.1–5.3, we see that all the conditions of Theorem 4.1 are met. Hence, we conclude that system (1.1) with (5.1), (5.2), (5.5) and (5.9) and system (1.1) with (5.1), (5.2), (5.7) and (5.9) each has at least one *positive* solution in $(C[0,T])^n$.

We remark that Examples 5.1–5.4 can easily be extended to the system (1.2).

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