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MODIFIED QUASI-BOUNDARY VALUE METHOD FOR CAUCHY PROBLEMS OF ELLIPTIC EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this article, we study a Cauchy problem for an elliptic equation with variable coefficients. It is well-known that such a problem is severely ill-posed; i.e., the solution does not depend continuously on the Cauchy data. We propose a modified quasi-boundary value regularization method to solve it. Convergence estimates are established under two a priori assumptions on the exact solution. A numerical example is given to illustrate our proposed method.

1. Introduction

In this article, we consider the following Cauchy problem for an elliptic equation with variable coefficients in a strip, as in [10],

$$\begin{gathered} u_{xx} + a(y)u_{yy} + b(y)u_y + c(y)u = 0, \quad x \in \mathbb{R},\ y \in (0,1) \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \\ u_y(x,0) = 0, \quad x \in \mathbb{R}, \end{gathered} \tag{1.1}$$

where a, b, c are given functions such that for some given positive constants $\lambda \le \Lambda$,

$$\lambda \le a(y) \le \Lambda, \quad y \in [0,1], \tag{1.2}$$

$$a(y) \in C^2[0,1], \quad b(y) \in C^1[0,1], \quad c(y) \in C[0,1], \quad c(y) \le 0. \tag{1.3}$$

Without loss of generality, in the following we suppose that $\lambda \ge 1$.

This problem is well-known to be severely ill-posed; i.e., a small perturbation in the given Cauchy data may result in a very large error on the solution [11, 13, 14, 16]. Therefore, it is very difficult to solve it using classic numerical methods. In order to overcome this difficulty, the regularization methods are required [12, 13, 15, 6].

It should be mentioned that, for the Cauchy problem of the elliptic equations, many regularization methods have been proposed: such as Tikhonov regularization method [7, 23], the modified method [3, 20], the moment method [24], the center difference method [4, 21], etc. For the Cauchy problem of elliptic equations with variable coefficients (1.1), in 2007, Hào and his group [10] applied the mollification method to solve it, and prove some stability estimates of Hölder type for the solution

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and its derivatives. In 2008, Qian [19] used a wavelet regularization method to treat it. In the present article, following Hào [10] and Qian [19], we continue to consider problem (1.1).

In 1983, Showalter presented a method called the quasi-boundary value (QBV) to regularize the linear homogeneous ill-posed problem [22]. The main idea of this method is making an appropriate modification to the final data. Recently many authors have successfully used this method to solve the backward heat conduction problem (BHCP) [1, 2, 9, 17, 18]. In [8], this method was used to solve a Cauchy problem for elliptic equation in a cylindrical domain (where the authors called it a non-local boundary value problem method). In this paper, we shall apply a modified quasi-boundary value method to solve problem (1.1). Here our idea mainly comes from Showalter's method (see Section 3).

This paper is constructed as follows. In Section 2, we give some required results for this paper. In Section 3, we present our regularization method. Section 4 is devoted to the convergence estimates. Numerical results are shown in Section 5, and some conclusions are given.

2. Some required results

For a function $f \in L^2(\mathbb{R})$, its Fourier transform is defined by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx, \quad \xi \in \mathbb{R}. \tag{2.1}$$

Let the exact data $\varphi \in L^2(\mathbb{R})$ and the measured data $\varphi^\delta \in L^2(\mathbb{R})$ satisfy

$$\|\varphi^\delta - \varphi\| \le \delta, \tag{2.2}$$

where $\|\cdot\|$ denotes the L^2-norm, the constant $\delta > 0$ denotes a noise level, and there exists a constant $E > 0$, such that the following a-priori bounds exist,

$$\|u(\cdot,1)\| \le E. \tag{2.3}$$

or

$$\|u(\cdot,1)\|_p \le E. \tag{2.4}$$

Here $\|u(\cdot,1)\|_p$ denotes the Sobolev space H^p-norm defined by

$$\|u(\cdot,1)\|_p = \Big(\int_{-\infty}^{\infty} (1+\xi^2)^p |\widehat{u}(\cdot,1)|^2 d\xi \Big)^{1/2}. \tag{2.5}$$

Now, we firstly consider the following Cauchy problem in the frequency domain,

$$\begin{gathered} -\xi^2 v(\xi,y) + a(y) v_{yy}(\xi,y) + b(y) v_y(\xi,y) + c(y) v(\xi,y) = 0, \quad \xi \in \mathbb{R},\ y \in (0,1) \\ v(\xi,0) = 1, \quad \xi \in \mathbb{R}, \\ v_y(\xi,0) = 0, \quad \xi \in \mathbb{R}. \end{gathered} \tag{2.6}$$

The following Lemma is very important to our analysis, and its proof can be found in [10].

Lemma 2.1. *There exists a unique solution of* (2.6) *such that*

(i) $v(\xi,y) \in W^{2,\infty}(0,1)$ *for all* $\xi \in \mathbb{R}$,
(ii) $v(\xi,1) \neq 0$ *for all* $\xi \in \mathbb{R}$,

(iii) *there exist positive constants* c_1, c_2, *such that for* $\xi \in \mathbb{R}$,

$$|v(\xi, y)| \le c_1 e^{|\xi| A(y)}, \quad \forall\, y \in [0,1], \tag{2.7}$$

$$|v(\xi, 1)| \ge c_2 e^{|\xi| A(1)}, \tag{2.8}$$

where,

$$A(y) = \int_0^y \frac{ds}{\sqrt{a(s)}},\ y \in [0,1]. \tag{2.9}$$

3. A modified quasi-boundary value regularization method

Taking the Fourier transform in problem (1.1) with respect to x, we have

$$\begin{gathered} a(y)\widehat{u}_{yy}(\xi, y) + b(y)\widehat{u}_y(\xi, y) + c(y)\widehat{u}(\xi, y) - \xi^2 \widehat{u}(\xi, y) = 0, \quad \xi \in \mathbb{R},\ y \in (0,1) \\ \widehat{u}(\xi, 0) = \widehat{\varphi}(\xi), \quad \xi \in \mathbb{R}, \\ \widehat{u}_y(\xi, 0) = 0, \quad \xi \in \mathbb{R}. \end{gathered} \tag{3.1}$$

It can be shown that the solution of (1.1) in the frequency domain is

$$\widehat{u}(\xi, y) = v(\xi, y)\widehat{u}(\xi, 0) = v(\xi, y)\widehat{\varphi}(\xi). \tag{3.2}$$

Then, the exact solution of (1.1) is

$$u(x, y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(\xi, y)\widehat{\varphi}(\xi) e^{i\xi x} d\xi. \tag{3.3}$$

From Lemma 2.1 and $v(\xi, 1) \ne 0$, we have

$$\widehat{\varphi}(\xi) = \widehat{u}(\xi, 0) = \frac{\widehat{u}(\xi, 1)}{v(\xi, 1)}, \tag{3.4}$$

and from (3.4), we can note that $\widehat{u}(\xi, 1) \ne 0$.

If $\widehat{\varphi}(\xi), \widehat{u}(\xi, 1) > 0$, we consider the following Cauchy problem in the frequency domain

$$\begin{gathered} a(y)\widehat{u}_{yy}(\xi, y) + b(y)\widehat{u}(\xi, y) + c(y)\widehat{u}(\xi, y) - \xi^2 \widehat{u}(\xi, y) = 0, \quad \xi \in \mathbb{R},\ y \in (0,1) \\ \widehat{u}(\xi, 0) + \alpha \widehat{u}(\xi, 1) = \widehat{\varphi}_\delta(\xi), \quad \xi \in \mathbb{R}, \\ \widehat{u}_y(\xi, 0) = 0, \quad \xi \in \mathbb{R}. \end{gathered} \tag{3.5}$$

Denoting $\widehat{u}^\delta_{\alpha 1}(\xi, y)$ as the solution of (3.5), we obtain

$$\widehat{u}^\delta_{\alpha 1}(\xi, y) = \frac{v(\xi, y)}{1 + \alpha v(\xi, 1)} \widehat{\varphi}_\delta(\xi). \tag{3.6}$$

If $\widehat{\varphi}(\xi) > 0$, $\widehat{u}(\xi, 1) < 0$, we consider the following Cauchy problem in the frequency domain

$$\begin{gathered} a(y)\widehat{u}_{yy}(\xi, y) + b(y)\widehat{u}(\xi, y) + c(y)\widehat{u}(\xi, y) - \xi^2 \widehat{u}(\xi, y) = 0, \quad \xi \in \mathbb{R},\ y \in (0,1) \\ \widehat{u}(\xi, 0) - \alpha \widehat{u}(\xi, 1) = \widehat{\varphi}_\delta(\xi), \quad \xi \in \mathbb{R}, \\ \widehat{u}_y(\xi, 0) = 0, \quad \xi \in \mathbb{R}. \end{gathered} \tag{3.7}$$

Denoting by $\widehat{u}^\delta_{\alpha 2}(\xi, y)$ the solution of (3.7), we have

$$\widehat{u}^\delta_{\alpha 2}(\xi, y) = \frac{v(\xi, y)}{1 - \alpha v(\xi, 1)} \widehat{\varphi}_\delta(\xi). \tag{3.8}$$

If $\widehat{\varphi}(\xi) > 0$, $\widehat{u}(\xi, 1)$ can be positive or negative, we define the following modified regularization solution to (1.1) in the frequency domain:

$$\widehat{u}_\alpha^\delta(\xi, y) = \frac{v(\xi, y)}{1 + \alpha|v(\xi, 1)|}\widehat{\varphi}_\delta(\xi). \tag{3.9}$$

By the above analysis, for $\widehat{\varphi}(\xi) > 0$, we define a modified regularization solution of form (3.9) to problem (1.1) in the frequency domain.

Equivalently, the regularization solution of (1.1) is given by

$$u_\alpha^\delta(x, y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{v(\xi, y)}{1 + \alpha|v(\xi, 1)|}\widehat{\varphi}_\delta(\xi)e^{i\xi x}d\xi. \tag{3.10}$$

Adopting similar analysis, when $\widehat{\varphi}(\xi) < 0$, we can also define the modified regularization solution of form (3.10).

In the following section, we will prove that the regularization solution $u_\alpha^\delta(x, y)$ given by (3.10) is a stable approximation to the exact solution $u(x, y)$ given by (3.3), and the regularization solution $u_\alpha^\delta(x, y)$ depends continuously on the measured data φ^δ for a fixed parameter $\alpha > 0$.

4. Convergence Estimates

In this section, we give the convergence estimates for $0 < y < 1$ and $y = 1$ under two different a-priori assumptions for the exact solution u, respectively.

Theorem 4.1. *Suppose that u is defined by (3.3) with the exact data φ and u_α^δ is defined by (3.10) with the measured data φ^δ. Let the measured data φ^δ satisfy (2.2), and let the exact solution u at $y = 1$ satisfy (2.3). If the regularization parameter α is chosen as*

$$\alpha = \frac{\delta}{E}, \tag{4.1}$$

then for fixed $0 < y < 1$ we have the following convergence estimate

$$\|u_\alpha^\delta(\cdot, y) - u(\cdot, y)\| \le 2C_y E^{\frac{A(y)}{A(1)}}\delta^{1-\frac{A(y)}{A(1)}}. \tag{4.2}$$

Proof. From (3.2), (3.9), (2.2), (2.3), we have

$$\begin{aligned}
\|u_\alpha^\delta(\cdot, y) - u(\cdot, y)\| &= \|\widehat{u}_\alpha^\delta(\xi, y) - \widehat{u}(\xi, y)\| \\
&= \Big\|\frac{v(\xi, y)\widehat{\varphi}(\xi)(1 + \alpha|v(\xi, 1)|) - v(\xi, y)\widehat{\varphi}_\delta(\xi)}{1 + \alpha|v(\xi, 1)|}\Big\| \\
&= \Big\|\frac{v(\xi, y)(\widehat{\varphi}_\delta(\xi) - \widehat{\varphi}(\xi)) + \alpha|v(\xi, 1)|v(\xi, y)\widehat{\varphi}(\xi)}{1 + \alpha|v(\xi, 1)|}\Big\| \\
&\le \delta\sup_{\xi\in\mathbb{R}}\frac{|v(\xi, y)|}{1 + \alpha|v(\xi, 1)|} + \alpha E\frac{|v(\xi, y)|}{1 + \alpha|v(\xi, 1)|} \\
&:= \delta\sup_{\xi\in\mathbb{R}} I_1 + \alpha E\sup_{\xi\in\mathbb{R}} I_1.
\end{aligned} \tag{4.3}$$

From Lemma 2.1, we can derive that

$$I_1 = \frac{|v(\xi, y)|}{1 + \alpha|v(\xi, 1)|} \le \frac{c_1 e^{|\xi|A(y)}}{1 + \alpha c_2 e^{|\xi|A(1)}} \le \frac{c_1}{\min\{1, c_2\}}\cdot\frac{e^{|\xi|A(y)}}{1 + \alpha e^{|\xi|A(1)}}. \tag{4.4}$$

Let $f(s) = e^{sA(y)}/(1 + \alpha e^{sA(1)})$, $s \ge 0$, then

$$f'(s) = f(s)\frac{A(y) - \alpha(A(1) - A(y))e^{A(1)s}}{1 + \alpha e^{|s|A(1)}}. \tag{4.5}$$

Setting $f'(s) = 0$, we have

$$\alpha(A(1) - A(y))e^{A(1)s} = A(y). \tag{4.6}$$

Note that $A(1) \geq 0$, $A(1) \geq A(y) \geq 0$ for $0 \leq y \leq 1$, it is easy to see that $f(s)$ has a unique maximal value point s^* such that

$$\alpha e^{A(1)s^*} = \frac{A(y)}{A(1) - A(y)}. \tag{4.7}$$

Thus,

$$f(s) \leq f(s^*) = c_y \alpha^{-\frac{A(y)}{A(1)}}, \tag{4.8}$$

where

$$c_y = \frac{(A(y))^{\frac{A(y)}{A(1)}}}{A(1)} (A(1) - A(y))^{\frac{A(y)}{A(1)} - 1}.$$

Then

$$I_1 \leq \frac{c_1}{\min\{1, c_2\}} \cdot \frac{e^{|\xi| A(y)}}{1 + \alpha e^{|\xi| A(1)}} \leq \frac{c_1 c_y}{\min\{1, c_2\}} \alpha^{-\frac{A(y)}{A(1)}} := C_y \alpha^{-\frac{A(y)}{A(1)}}, \tag{4.9}$$

By (4.1), (4.3), (4.9), for fixed $0 < y < 1$, we obtain

$$\|u_\alpha^\delta(\cdot, y) - u(\cdot, y)\| \leq 2C_y E^{\frac{A(y)}{A(1)}} \delta^{1 - \frac{A(y)}{A(1)}}.$$

□

From Theorem 4.1, we note that u_α^δ defined by (3.10) is an effective approximation to the exact solution u for the fixed $0 < y < 1$. But the estimate (4.2) gives no information about the error estimate at $y = 1$ as the constraint (2.3) is too weak for this purpose. To retain the continuity, as common, we suppose that $u(x, y)$ satisfies a stronger a-priori assumption (2.4) at $y = 1$.

Theorem 4.2. *Let the exact solution u and the regularization solution u_α^δ be defined by* (3.3), (3.10)*, respectively. Assume that the measured data φ^δ satisfies $\|\varphi^\delta - \varphi\| \leq \delta$, and let the exact solution u satisfy* (2.4)*. If the regularization parameter α is chosen as*

$$\alpha = \sqrt{\delta/E}, \tag{4.10}$$

then we have the following convergence estimate at $y = 1$,

$$\|u(\cdot, 1) - u_\alpha^\delta(\cdot, 1)\| \leq \sqrt{\delta E} + CE \max\big\{\big(\frac{\delta}{E}\big)^{1/3}, \big(\frac{1}{6} \ln \frac{E}{\delta}\big)^{-p}\big\}. \tag{4.11}$$

Proof. By (3.2), (3.9), (2.2), (2.4), we have

$$\begin{aligned}
\|u_\alpha^\delta(\cdot, 1) - u(\cdot, 1)\| &= \|u_\alpha^\delta(\xi, 1) - u(\xi, 1)\| \\
&= \|\frac{v(\xi,1)\widehat{\varphi}(\xi)(1 + \alpha|v(\xi,1)|) - v(\xi,1)\widehat{\varphi}_\delta(\xi)}{1 + \alpha|v(\xi,1)|}\| \\
&= \|\frac{v(\xi,1)(\widehat{\varphi}_\delta(\xi) - \widehat{\varphi}(\xi)) + \alpha|v(\xi,1)|v(\xi,1)\widehat{\varphi}(\xi)}{1 + \alpha|v(\xi,1)|}\| \\
&\leq \delta \sup_{\xi \in \mathbb{R}} \frac{|v(\xi,1)|}{1 + \alpha|v(\xi,1)|} + E \sup_{\xi \in \mathbb{R}} \frac{\alpha(1 + \xi^2)^{-\frac{p}{2}}|v(\xi,1)|}{1 + \alpha|v(\xi,1)|} \\
&:= \delta \sup_{\xi \in \mathbb{R}} I_2 + E \sup_{\xi \in \mathbb{R}} I_3.
\end{aligned} \tag{4.12}$$

It is easy to know that
$$I_2 = \frac{|v(\xi,1)|}{1+\alpha|v(\xi,1)|} \le \frac{1}{\alpha}, \tag{4.13}$$
then by (4.10), we know
$$\delta \sup_{\xi\in\mathbb{R}} I_2 \le \sqrt{\delta E}. \tag{4.14}$$
In the following, we estimate I_3. From Lemma 2.1, we obtain
$$I_3 = \frac{\alpha(1+\xi^2)^{-\frac{p}{2}}|v(\xi,1)|}{1+\alpha|v(\xi,1)|} \le \frac{c_1}{\min\{1,c_2\}} \cdot \frac{\alpha(1+\xi^2)^{-\frac{p}{2}} e^{|\xi|A(1)}}{(1+\alpha e^{|\xi|A(1)})}. \tag{4.15}$$

Case 1: For the large values with $|\xi| \ge \ln \frac{1}{\sqrt[3]{\alpha}}$, we have
$$\frac{c_1}{\min\{1,c_2\}} \cdot \frac{\alpha(1+\xi^2)^{-\frac{p}{2}} e^{|\xi|A(1)}}{(1+\alpha e^{|\xi|A(1)})} \le \frac{c_1}{\min\{1,c_2\}}\big(\ln\frac{1}{\sqrt[3]{\alpha}}\big)^{-p} := C\big(\ln\frac{1}{\sqrt[3]{\alpha}}\big)^{-p}. \tag{4.16}$$

Case 2: For $|\xi| < \ln \frac{1}{\sqrt[3]{\alpha}}$, since $1 \le \lambda \le a(y) \le \Lambda$, $A(1) = \int_0^1 \frac{1}{\sqrt{a(s)}} ds \le 1$, then
$$\frac{c_1}{\min\{1,c_2\}} \cdot \frac{\alpha(1+\xi^2)^{-\frac{p}{2}} e^{|\xi|A(1)}}{(1+\alpha e^{|\xi|A(1)})} \le \frac{c_1}{\min\{1,c_2\}} \alpha e^{|\xi|A(1)} \le C\alpha^{\frac{2}{3}}. \tag{4.17}$$
By (4.16), (4.17), we obtain
$$I_3 \le C \max\big\{\alpha^{2/3}, \big(\ln\frac{1}{\sqrt[3]{\alpha}}\big)^{-p}\big\}. \tag{4.18}$$
Then, from (4.10), (4.12), (4.14), (4.18), for $y=1$, we have
$$\|u(\cdot,1) - u_\alpha^\delta(\cdot,1)\| \le \sqrt{\delta E} + CE\max\big\{\big(\frac{\delta}{E}\big)^{1/3}, \big(\frac{1}{6}\ln\frac{E}{\delta}\big)^{-p}\big\}.$$
□

Remark 4.3. In the convergence estimate (4.11), we can see that the logarithmic term with respect to δ is the dominating term. Asymptotically this yields a convergence rate of order $O(\ln\frac{E}{\delta})^{-p}$. The first term is asymptotically negligible compared to this term.

5. Numerical implementations

In this section, we use a numerical example to verify the stability of our proposed regularization method. For simplicity, we consider the following Cauchy problem for the Laplace equation,
$$\begin{gathered} u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R},\ y \in (0,1) \\ u(x,0) = \varphi(x), \quad x \in \mathbb{R}, \\ u_y(x,0) = 0, \quad x \in \mathbb{R}. \end{gathered} \tag{5.1}$$
It is easy to verify that
$$u(x,y) = e^{y^2-x^2}\cos(2xy), \tag{5.2}$$
is the exact solution of problem (5.1), with initial data
$$\varphi(x) = e^{-x^2}. \tag{5.3}$$
In this case, the solution of (2.6) becomes
$$v(\xi,y) = \cosh(|\xi|y). \tag{5.4}$$

We define all functions to be zero for $x \in (-\infty, -3\pi) \cup (3\pi, \infty)$, so we choose the interval $[-3\pi, 3\pi]$ to complete our numerical experiment by using the discrete Fourier transform and inverse Fourier transform (FFT and IFFT).

The measured data φ_δ is given by $\varphi^\delta(x_i) = \varphi(x_i) + \varepsilon\,\mathrm{rand}(i)$, where ε is the error level,

$$\varphi(x) = (\varphi(x_1), \dots, \varphi(x_N)), \tag{5.5}$$

$$x_j = -3\pi + \frac{6\pi(j-1)}{N-1}, \quad j = 1, 2, \dots, N, \tag{5.6}$$

$$\delta = \|\varphi_\delta - \varphi\|_{l_2} = \Big(\frac{1}{N}\sum_{j=1}^{N} |\varphi_\delta(x_j) - \varphi(x_j)|\Big)^{1/2}. \tag{5.7}$$

the function $\mathrm{rand}(\cdot)$ denotes arrays of random numbers whose elements are uniformly distributed in the interval $[0,1]$. The relative root mean square error between the exact and approximate solution is given by

$$\epsilon(u) = \frac{\sqrt{\frac{1}{N}\sum_{j=1}^{N}\left(u_j - (u_\alpha^\delta)_j\right)^2}}{\sqrt{\frac{1}{N}\sum_{j=1}^{N}(u_j)^2}}. \tag{5.8}$$

Then we obtain the regularization solution u_α^δ computed by (3.10).

Numerical results are shown in Figures 1-2. The numerical result for $u(\cdot, y)$ and $u_\alpha^\delta(\cdot, y)$ at $x = 0.2$, $x = 0.5$, and $x = 0.8$ with $\varepsilon = 1 \times 10^{-4}, 10^{-3}$ are shown in Figure 1. In Figure 1, we choose the a-priori bound $E = 1$ and the regularization parameter α is chosen by (4.1). The numerical results for $u(\cdot, 1)$ and $u_\alpha^\delta(\cdot, 1)$ with $\varepsilon = 1 \times 10^{-4}, \varepsilon = 10^{-3}$ are shown in Fig.2, where the regularization parameter α is chosen by (4.10) and the a-priori bound $E = 1$. The relative root mean square errors at $y = 0.6, y = 1$ for the computed solution versus the error levels ε are shown in Tables $1-2$.

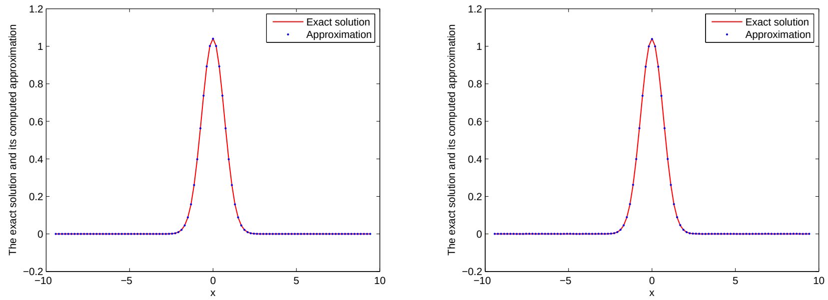
From Figures 1-2, we find the stability of our proposed method. From Tables 1–2, we note that the smaller the ε is, the better the computed solution is, which means that our proposed regularization method is sensitive to the noise level ε. In addition, we can note that numerical results become worse when y approaches to 1.

Table 1. The relative root mean square errors at $y = 0.6$ for various noisy levels

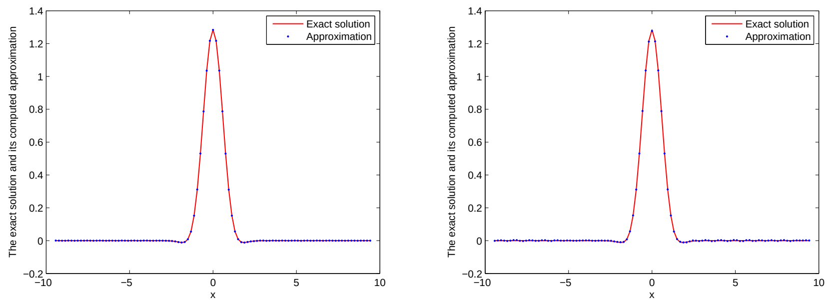
ε	0.00001	0.0001	0.001	0.01
α	0.0032	0.01	0.0316	0.1
$\epsilon(u)$	0.0118	0.0345	0.0914	0.2098

Table 2. The relative root mean square errors at $y = 1$ for various noisy levels

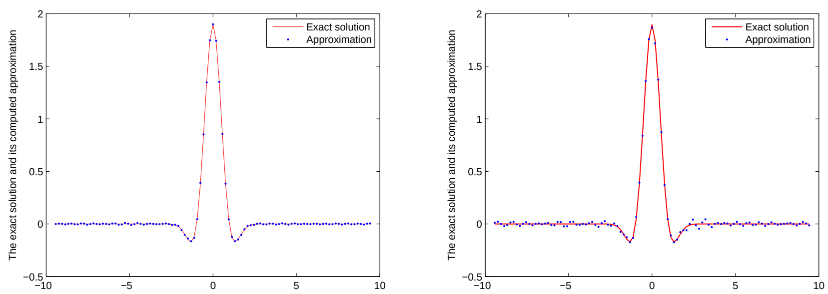
ε	0.00001	0.0001	0.001	0.01
α	0.0032	0.01	0.0316	0.1
$\epsilon(u)$	0.0269	0.0727	0.1721	0.3424



$y = 0.2$, $\varepsilon = 1 \times 10^{-4}$ $y = 0.2$, $\varepsilon = 1 \times 10^{-3}$



$y = 0.5$, $\varepsilon = 1 \times 10^{-4}$ $y = 0.8$, $\varepsilon = 1 \times 10^{-3}$.



$y = 0.8$, $\varepsilon = 1 \times 10^{-4}$ $y = 0.8$, $\varepsilon = 1 \times 10^{-3}$

FIGURE 1. Graph of $u(\cdot, y)$ and $u_\alpha^\delta(\cdot, y)$

Conclusions. In this article, a modified quasi-boundary value regularization method is used to solve a Cauchy problem for the elliptic equation with variable coefficients. The convergence estimates for $0 < y < 1$ and $y = 1$ have been obtained under two different a-priori bound assumptions for the exact solution. Some numerical results show that our proposed regularization method is feasible.

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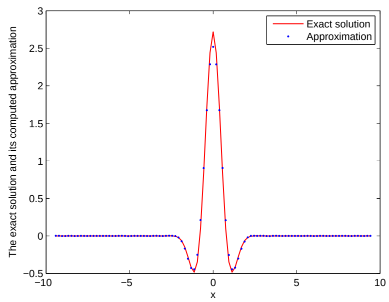
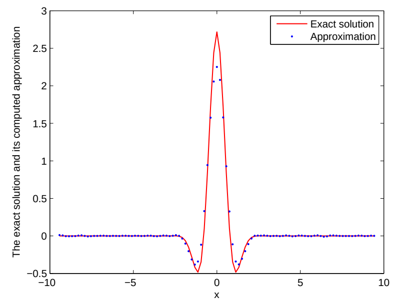


$\varepsilon = 1 \times 10^{-4}$



$\varepsilon = 1 \times 10^{-3}$

FIGURE 2. Graph of $u(\cdot, 1)$ and $u_\alpha^\delta(\cdot, 1)$

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