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CONTINUOUS SOLUTIONS OF DISTRIBUTIONAL CAUCHY PROBLEMS

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ABSTRACT. Existence of the smallest, greatest, minimal, maximal and unique continuous solutions to distributional Cauchy problems, as well as their dependence on the data, are studied. The main tools are a continuous primitive integral and fixed point results in function spaces.

1. INTRODUCTION

New existence results are derived for the smallest, greatest, minimal, maximal and unique continuous solutions of the distributional Cauchy problem

$$y' = f(y), \quad y(a) = c. \tag{1.1}$$

Novel results for dependence of solutions on f and on the initial value $c \in \mathbb{R}$ are also derived. The values of f are distributions (generalized functions) on $[a, b]$, $-\infty < a < b < \infty$. Definition of such distributions and their main properties needed in this paper are presented in section 2.

In section 3, existence results are derived for the smallest and greatest continuous solutions of (1.1). Dependence of these solutions both on f and on c are also studied. A concrete example is presented.

Existence of minimal and maximal solutions of problem (1.1) with $c = 0$ is studied in section 4. In sections 5 and 6 we present conditions which ensure that problem (1.1) has a unique solution that depends continuously on the initial value c. Main tools are a continuous primitive integral introduced for distributions in [6] and fixed point results proved in [2, 4].

2. PRELIMINARIES

Distributions on a compact real interval $[a, b]$ are (cf. [7]) continuous linear functionals on the topological vector space $\mathcal{D}$ of functions $\varphi : \mathbb{R} \to \mathbb{R}$ possessing for every $j \in \mathbb{N}_0$ a derivative $\varphi^{(j)}$ of order j which is continuous on $\mathbb{R}$ and vanishes on $\mathbb{R} \setminus (a, b)$. The space $\mathcal{D}$ is endowed with the topology in which the sequence (φ_k) of $\mathcal{D}$ tends to $\varphi \in \mathcal{D}$ if and only if $\varphi_k^{(j)} \to \varphi_k^{(j)}$ uniformly on (a, b) for all $j \in \mathbb{N}_0$ as $k \to \infty$. As for the theory of distributions; see e.g. [3, 5].

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A distribution g on $[a,b]$ is called distributionally Denjoy (shortly DD) integrable on $[a,b]$ if g has a continuous primitive; i.e., if g is a distributional derivative G' of a function $G \in C[a,b]$ (cf. [6]). Thus the value $\langle g, \varphi\rangle$ of g at $\varphi \in \mathcal{D}$ is defined by

$$\langle g, \varphi\rangle = \langle G', \varphi\rangle = -\langle G, \varphi'\rangle = \int_a^b G(t)\varphi'(t)\,dt.$$

The continuous primitive integral function of g is defined by

$$ {}^{c}\!\!\int_a^t g = {}^{c}\!\!\int_a^t G' = G(t) - G(a), \quad t \in [a,b]. \tag{2.1}$$

It belongs to the function space

$$\mathcal{B}_c[a,b] = \{x : [a,b] \to \mathbb{R} : x \text{ is continuous and vanishes at } a\}.$$

Assuming that $\mathcal{B}_c[a,b]$ is ordered pointwise, it can be shown (cf. [6]) that relation $\preceq$, defined by

$$f \preceq g \text{ if and only if } {}^{c}\!\!\int_a^t f \le {}^{c}\!\!\int_a^t g \quad \text{for all } t \in [a,b], \tag{2.2}$$

is a partial ordering on the set of DD integrable distributions on $[a,b]$.

The following result reduces problem (1.1) to a fixed point equation on $C[a,b]$.

Lemma 2.1. *Assume that $f(x)$ is DD integrable on $[a,b]$ for every $x \in C[a,b]$. Then the Cauchy problem (1.1) has a continuous solution y if and only if y is a solution of the fixed point equation*

$$x(t) = F(x)(t) := c + {}^{c}\!\!\int_a^t f(x), \quad t \in [a,b]. \tag{2.3}$$

Proof. Assume first that $y \in C[a,b]$ is a solution of problem (1.1). Applying (1.1), (2.1) and (2.3) we have for each $t \in [a,b]$,

$$y(t) = c + y(t) - y(a) = c + {}^{c}\!\!\int_a^t y' = c + {}^{c}\!\!\int_a^t f(y) = F(y)(t).$$

Thus y is a solution of (2.3). Conversely, assume that $y \in C[a,b]$ satisfies the fixed point equation (2.3). Then for every $t \in [a,b]$,

$$y(t) - c = F(y)(t) - c = {}^{c}\!\!\int_a^t f(y).$$

This result implies also that $y(a) = c$, since ${}^{c}\!\!\int_a^a f(y) = 0$ by (2.1). On the other hand,

$$y(t) - c = y(t) - y(a) = {}^{c}\!\!\int_a^t y', \quad t \in [a,b].$$

The above results imply that ${}^{c}\!\!\int_a^t f(y) = {}^{c}\!\!\int_a^t y'$ for every $t \in [a,b]$, whence $y' = f(y)$ by (2.2). Thus y is a solution of problem (1.1). $\square$

3. Existence of the smallest and greatest solutions

The application of monotone methods for fixed point problems in $C[a,b]$ is complicated by the fact that the limit function of a pointwise convergent monotone sequence of $C[a,b]$ is not necessarily continuous. For instance, the functions $x_n(t)=t^n$, $t\in[0,1]$, $n\in\mathbb{N}$, form such a sequence. Therefore it is assumed in this section that $f(x)$ is a distributional derivative of a continuous function whenever x belongs to the space $L^1[a,b]$ of Lebesgue integrable functions on $[a,b]$. This space is equipped with a.e. pointwise ordering, and its a.e. equal functions are identified.

The main result of this section reads as follows.

Theorem 3.1. *The Cauchy problem* (1.1) *has the smallest solution y_* and the greatest solution y^* in $C[a,b]$ if the following hypotheses are valid.*

(A0) *$f(x)$ is DD integrable on $[a,b]$ for every $x\in L^1[a,b]$;*
(B0) *$f(x)\preceq f(y)$ whenever $x\le y$ in $L^1[a,b]$;*
(C0) *There exist distributions $f_\pm$ that are DD integrable on $[a,b]$ such that $f_-\preceq f(x)\preceq f_+$ for all $x\in L^1[a,b]$.*

Moreover, both y_* and y^* are increasing with respect to f and to c.

Proof. For each $x\in L^1[a,b]$ denote by $F(x)$ the primitive of $f(x)$, defined in (2.3), and define $y_\pm\in C[a,b]$ by

$$y_\pm(t)=c+{}^{c}\!\!\int_a^t f_\pm,\quad t\in[a,b].$$

The given hypotheses imply by (2.2) and (2.3) that F is an increasing mapping from $L^1[a,b]$ to its order interval $[y_-,y_+]=\{x\in L^1[a,b]: y_-\le x\le y_+\}$. If (x_n) is a monotone sequence in $L^1[a,b]$, then $(F(x_n))$ is a monotone sequence in $[y_-,y_+]$. Thus, by monotone convergence theorem, $(F(x_n))$ converges in $L^1[a,b]$, and the limit function belongs to $[y_-,y_+]$. It then follows from [4, Theorem 1.2.2] that F has the smallest fixed point y_* and the greatest fixed point y^* in $L^1[a,b]$, and hence also in $C[a,b]$, since $F[L^1[a,b]]\subset C[a,b]$ by the hypothesis (A0). These fixed points are by Lemma 2.1 also the smallest and greatest continuous solutions of (1.1). Moreover, according to [4, Theorem 1.2.2],

$$y_*=\min\{x\in[y_-,y_+]:F(x)\le x\},\quad y^*=\max\{x\in[y_-,y_+]:x\le F(x)\}.\tag{3.1}$$

Applying these relations it is easy to show that both y_* and y^* are increasing with respect to F, and hence, by (2.3) and (2.2), also with respect to f and to c. ☐

As noticed in [6], the distributional Denjoy integral contains the wide Denjoy integral, and hence also integrals called Riemann, Lebesgue, Denjoy and Henstock-Kurzweil. In the next corollary the Henstock-Kurzweil integral can be replaced by any of those integrals listed above.

Corollary 3.2. *The results of Theorem 3.1 are valid if $f(x)$ is for every $x\in L^1[a,b]$ the distributional derivative of a function (${}^{K}\!\!\int$ denotes the Henstock-Kurzweil integral)*

$$G(x)(t)=\sum_{i=1}^n H_i(t)\,{}^{K}\!\!\int_a^t g_i(x)+H(t),\quad t\in[a,b],\tag{3.2}$$

where for each $i=1,\dots,n$, H_i is nonnegative-valued, H_i, $H\in\mathcal{B}_c[a,b]$, and $g_i(x):[a,b]\to\mathbb{R}$ satisfies the following hypotheses.

(GI1) *$g_i(x)$ is Henstock-Kurzweil integrable on $[a,b]$ for all $x \in L^1[a,b]$.*
(GI2) *There exist Henstock-Kurzweil integrable functions $\underline{g}_i, \overline{g}_i : [a,b] \to \mathbb{R}$ such that*
$$ {}^K\!\!\int_a^t \underline{g}_i \le {}^K\!\!\int_a^t g_i(x) \le {}^K\!\!\int_a^t g_i(y) \le {}^K\!\!\int_a^t \overline{g}_i, $$
for $t \in [a,b]$, whenever $x \le y$ in $L^1[a,b]$.

Proof. The hypotheses imposed above ensure that (3.2) defines for every $x \in L^1[a,b]$ a continuous function $G(x) : [a,b] \to \mathbb{R}$. Moreover, the distributional derivatives $f(x)$ of $G(x)$ satisfy the hypotheses (A0), (B0) and (C0), where
$$ {}^c\!\!\int_a^t f_- = \sum_{i=1}^n H_i(t)\, {}^K\!\!\int_a^t \underline{g}_i + H(t), \quad {}^c\!\!\int_a^t f_+ = \sum_{i=1}^n H_i(t)\, {}^K\!\!\int_a^t \overline{g}_i + H(t), \quad t \in [a,b]. $$
Thus the Cauchy problem (1.1) has by Theorem 3.1 the smallest and greatest solutions in $C[a,b]$, and they are increasing with respect to f and to c. □

Remark 3.3. Under the hypotheses of Theorem 3.1 the smallest fixed point y_* of F is by [4, Theorem 1.2.1] the maximum of the chain C of $L^1[a,b]$ that is well-ordered; i.e., every nonempty subset of C has the smallest element, and that satisfies

(I) $y_- = \min C$, and if $y_- < x$, then $x \in C$ if and only if $x = \sup F[\{y \in C : y < x\}]$.

The smallest elements of C are $F^n(y_-)$, $n \in \mathbb{N}_0$, as long as $F^n(y_-) = F(F^{n-1}(y_-))$ is defined and $F^{n-1}(y_-) < F^n(y_-)$, $n \in \mathbb{N}$. If $F^{n-1}(y_-) = F^n(y_-)$ for some $n \in \mathbb{N}$, there is the smallest such an n, and $F^n(y_-) = \sup F[C] = y_*$ is the smallest fixed point of F in $C[a,b]$. If $x_\omega = \sup_{n\in\mathbb{N}} F^n(y_-)$ is defined in $L^1[a,b]$ and is a strict upper bound of $\{F^n(y_-)\}_{n\in\mathbb{N}}$, then x_ω is the next element of C. If $x_\omega = F(x_\omega)$, then $y_* = x_\omega$, otherwise the next elements of C are of the form $F^n(x_\omega)$, $n \in \mathbb{N}$, and so on.

The greatest fixed point y^* of F is by [4, Proposition 1.2.1] the minimum of the chain D of $L^1[a,b]$ that is inversely well-ordered; i.e., every nonempty subset of D has the greatest element, and that satisfies

(II) $y_+ = \max D$, and if $y_+ > x$, then $x \in C$ if and only if $x = \inf F[\{y \in D : y > x\}]$.

The greatest elements of D are n-fold iterates $F^n(y_+)$, as long as they are defined and $F^n(y_+) < F^{n-1}(y_+)$, etc.

Example 3.4. Consider the Cauchy problem
$$ y' = f(y), \quad y(0) = 0, \tag{3.3} $$
where $f(x)$ is for each $x \in L^1[0,1]$ the distributional derivative of the function $G(x) \in \mathcal{B}_c[0,1]$, defined by
$$ G(x)(t) = H_1(t)\, {}^K\!\!\int_0^t g_1(x) + H(t), \quad t \in [0,1], \tag{3.4} $$
where $H \in \mathcal{B}_c[0,1]$, H_1 is the Heaviside step function on $[0,1]$, and
$$ g_1(x)(t) = \arctan\Big([10^5 \int_0^1 (x(t) - H(t))\,dt]10^{-4}\Big)\Big(\frac{1}{t}\cos(\frac{1}{t}) - \sin(\frac{1}{t}) + 1\Big), $$

where $[\cdot]$ denotes the greatest integer function. Denote

$$y_\pm(t) := H(t) \pm 4t(1-\sin(\frac{1}{t})), \quad t\in(0,1],\ y_\pm(0)=0,$$

and let $f_\pm$ be distributional derivatives of $y_\pm$. The validity of the hypotheses (GI1) and (GI2) is easy to verify. Thus, the Cauchy problem (3.3) has by Corollary 3.2 the smallest and greatest solutions in $\mathcal{B}_c[0,1]$.

Calculating the successive approximations

$$y_{n+1} = G(y_n),\ y_0 = y_- \quad \text{and} \quad z_{n+1} = G(z_n),\ z_0 = y_+,$$

we see, that $(y_n)_{n=0}^{24}$ is strictly increasing, that $(z_n)_{n=0}^{24}$ is strictly decreasing, that $y_{24} = G(y_{24})$, and that $z_{24} = G(z_{24})$.

Thus $y_* = y_{24}$ and $y^* = z_{24}$ are by Remark 3.3 and Lemma 2.1 the smallest and greatest solutions of (3.3) in $\mathcal{B}_c[0,1]$ when $f(x)$ is for each $x\in L^1[a,b]$ the distributional derivative of $G(x)$ defined by (3.3). The exact formulas of y_* and y^* are

$$y_*(t) = \arctan\left(\frac{8693}{10000}\right) t(\sin(\frac{1}{t})-1) + H(t),\ t\in(0,1], \quad y_*(0)=0,$$
$$y^*(t) = \arctan\left(\frac{869}{1000}\right) t(1-\sin(\frac{1}{t})) + H(t),\ t\in(0,1], \quad y^*(0)=0.$$

4. Existence of minimal and maximal solutions

In this section existence results are derived for local and global minimal and maximal continuous solutions of the distributional Cauchy problem

$$y' = f(y), \quad y(a) = 0. \tag{4.1}$$

The space $L^1[a,b]$, ordered a.e. pointwise and normed by L^1-norm: $\|x\|_1 = \int_a^b |x(s)|\,ds$, is an ordered normed space $E := (L^1[a,b], \|\cdot\|_1)$ having the following properties (θ denotes the zero-element of $L^1[a,b]$).

- (E0) Bounded and monotone sequences of E converge.
- (E1) $x^+ = \sup\{\theta, x\}$ exists, and $\|x^+\|_1 \le \|x\|_1$ for every $x\in E$.

Denote

$$B(\theta,R) = \{x\in L^1[a,b] : \|x\|_1 \le R\}. \tag{4.2}$$

Because of the properties (E0) and (E1) we obtain the following result as a consequence of [2, Theorem 2.44].

Lemma 4.1. *Given a subset P of $L^1[a,b]$, assume that $F: P\to P$ is increasing, and that $F[P]\subseteq B(\theta,R)\subseteq P$ for some $R>0$. Then F has*

- (a) *minimal and maximal fixed points;*
- (b) *smallest and greatest fixed points y_* and y^* in the order interval $[\underline{y},\overline{y}]$ of P, where $\underline{y}$ is the greatest solution of $y = -(-F(y))^+$, and $\overline{y}$ is the smallest solution of $y = F(y)^+$.*

Moreover, y^*, y_*, $\underline{y}$ and $\overline{y}$ are all increasing with respect to F.

As an application of Lemma 4.1 we obtain the following result.

Proposition 4.2. *Assume that the hypotheses (A0) and (B0) hold, and that the primitives $F(x)$ of $f(x)$ in $\mathcal{B}_c[a,b]$ satisfy the following hypothesis for some $R>0$.*

- (C1) *$\|F(x)\| \le R$ for all $x\in L^1[a,b]$, $\|x\|_1\le R$.*

Then the Cauchy problem (4.1) *has*

(a) *minimal and maximal solutions in* $B(\theta, R)$*;*
(b) *smallest and greatest solutions* y_* *and* y^* *in the order interval* $[\underline{y}, \overline{y}]$ *of* $B(\theta, R)$*, where* $\underline{y}$ *is the greatest solution of* $y = -(-F(y))^+$*, and* $\overline{y}$ *is the smallest solution of* $y = F(y)^+$*.*

Moreover, y^**,* y_**,* $\underline{y}$ *and* $\overline{y}$ *are all increasing with respect to* f*.*

Proof. For each $x \in L^1[a, b]$ the distribution $f(x)$ has by (A0) the primitive $F(x)$ in $\mathcal{B}_c[a, b] \subset L^1[a, b]$. The hypotheses (B0) and (C1) imply that F satisfies the hypotheses of Lemma 4.1 when $P = B(\theta, R)$. Thus, by Lemma 4.1(a), F has in $B(\theta, R)$ minimal and maximal fixed points, which are by Lemma 2.1 also minimal and maximal solutions of (4.1) in $B(\theta, R)$. The results of (b) and the last result of theorem follow from the corresponding results of Lemma 4.1 and from (2.2). $\square$

As for the existence of minimal and maximal solutions of (4.1) in the whole $\mathcal{B}_c[a, b]$, we have the following result.

Theorem 4.3. *The distributional Cauchy problem* (4.1) *has minimal and maximal solutions in* $\mathcal{B}_c[a, b]$*, and they are increasing with respect to* f*, if the hypotheses (A0) and (B0) hold, and if the primitives* $F(x)$ *of* $f(x)$ *in* $\mathcal{B}_c[a, b]$ *satisfy the following hypothesis.*

(C2) $\|F(x)\|_1 \le Q(\|x\|_1)$ *for all* $x \in L^1[a, b]$*, where* $Q : \mathbb{R}_+ \to \mathbb{R}_+$ *is increasing,* $R = Q(R)$ *for some* $R > 0$*, and* $r \le Q(r)$ *implies* $r \le R$*.*

Proof. Hypothesis (C2) implies that

$$\|F(x)\|_1 \le Q(\|x\|_1) \le Q(R) = R \quad \text{for every } x \in B(\theta, R).$$

Thus hypothesis (C1) holds, whence (4.1) has the by Proposition 4.2 minimal and maximal solutions in $B(\theta, R)$, and they are increasing with respect to f.

If $y \in B(\theta, r)$ is a solution of (1.1), then y is a fixed point of F by Lemma 2.1. Hypothesis (C2) with $r = \|y\|_1$ implies that

$$\|y\|_1 = \|F(y)\|_1 \le Q(\|y\|_1) \le Q(R) = R.$$

Thus all the solutions of (4.1) are in $B(\theta, R)$. The assertion follows from the above results. $\square$

5. Existence and uniqueness results

In this section, conditions are presented for distributions $f(x)$, $x \in C[a, b]$, which ensure that (1.1) has for each $c \in \mathbb{R}$ a unique solution. Denoting $\lceil x \rceil = |x(\cdot)|$, $x \in C[a, b]$, we have the following fixed point result that is basis of our main existence and uniqueness theorem.

Proposition 5.1 ([4, Theorem 1.4.9])**.** *Let* $F : C[a, b] \to C[a, b]$ *satisfy the hypothesis:*

(F0) *There exists a* $v \in C_+[a, b] = \{u \in C[a, b] : \theta \le u\}$ *and an increasing mapping* $Q : [\theta, v] \to C_+[a, b]$ *satisfying* $Qv(t) < v(t)$ *and* $Q^n v(t) \to 0$ *for each* $t \in [a, b]$*, such that*
$$\lceil F(x) - F(z) \rceil \le Q\lceil x - z \rceil \tag{5.1}$$
for all $x, z \in C[a, b]$*,* $\lceil x - z \rceil \le v$*.*

Then for each $y_0 \in C[a,b]$ the sequence $(F^n(y_0))_{n=0}^\infty$ converges uniformly on $[a,b]$ to a unique fixed point of F.

In our main existence and uniqueness theorem we rewrite the inequality (5.1) in terms of distributions. The modulus $|g|$ of a distribution g on $(0,b)$ that is DD integrable on $[a,b]$ is defined by

$$|g| := \sup\{g, -g\}, \tag{5.2}$$

where the supremum is taken in the partial ordering $\preceq$ defined by (2.2). $|g|$ exists because $\preceq$ is a lattice ordering (cf. [6, Sect. 9]).

Now we are able to prove an existence and uniqueness theorem for the solution of the Cauchy problem (1.1).

Theorem 5.2. *Assume that distributions $f(x)$, $x \in C[a,b]$, and $h(w)$, $w \in [\theta, v]$, $v \in C_+[a,b]$, are DD integrable on $[a,b]$, and that*

$$|f(x) - f(z)| \preceq h(\lceil x - z \rceil) \tag{5.3}$$

for all x, $z \in C[a,b]$ with $\lceil x - z \rceil \le v$, and that $Q : [\theta, v] \to C_+[a,b]$, defined by

$$Q(w)(t) = {}^c\!\!\int_a^t h(w), \quad \theta \le w \le v,\ a \le t \le b, \tag{5.4}$$

is increasing, $Q(v)(t)) < v(t)$ and $Q^n(v)(t) \to 0$ for each $t \in [a,b]$. Then the Cauchy problem (1.1) has a unique solution y in $C[a,b]$. Moreover, y is for each choice of $y_0 \in C[a,b]$ the uniform limit of the sequence $(y_n)_{n=0}^\infty$ of the successive approximations

$$y_{n+1}(t) = c + {}^c\!\!\int_a^t f(y_n), \quad t \in [a,b],\ n \in \mathbb{N}_0. \tag{5.5}$$

Proof. It follows from (2.2) and (5.2) that (5.3) holds if and only if

$$\Big| {}^c\!\!\int_a^t f(x) - {}^c\!\!\int_a^t f(z) \Big| \le {}^c\!\!\int_a^t h(\lceil x - z \rceil), \quad \text{for all } t \in [a,b]. \tag{5.6}$$

Equation (2.3) defines a mapping $F : C[a,b] \to C[a,b]$. The given hypotheses and the equivalence of (5.3) and (5.6) imply that the operators F and Q, defined by (2.3) and (5.4), satisfy the hypotheses of Proposition 5.1. Thus the iteration sequence $(F^n(y_0))_{n=0}^\infty$, which equals to the sequence $(y_n)_{n=0}^\infty$ of successive approximations (5.5), converges for every choice of $y_0 \in C[a,b]$ uniformly on $[a,b]$ to a unique fixed point y of F. This result and Lemma 2.1 imply that y is the uniquely determined continuous solution of the Cauchy problem (1.1). □

The following result will be applied to obtain a special case of Theorem 5.2.

Lemma 5.3 ([2, Lemma 6.11])**.** *Assume that the function $q : [a,b] \times [0,r] \to \mathbb{R}_+$, $r > 0$, satisfies the condition.*

(Q0) *$q(\cdot, x)$ is measurable for all $x \in [0,r]$, $q(\cdot, r) \in L^1([a,b], \mathbb{R}_+)$, $q(t, \cdot)$ is increasing and right-continuous for a.e. $t \in [a,b]$, and the zero-function is the only absolutely continuous (AC) solution with $u_0 = 0$ of the Cauchy problem*

$$u'(t) = q(t, u(t)) \ \text{a.e. on } [a,b], \quad u(a) = u_0. \tag{5.7}$$

Then there exists an $r_0 > 0$ such that the Cauchy problem (5.7) *has for every $u_0 \in [0, r_0]$ the smallest AC solution $u = u(\cdot, u_0)$, which is increasing with respect to u_0. Moreover, $u(t, u_0) \to 0$ uniformly over $t \in [a, b]$ when $u_0 \to 0$.*

The next result is an application of Lemma 5.3 and Theorem 5.2.

Proposition 5.4. *The results of Theorem 5.2 are valid if the distributions $f(x)$, $x \in C[a, b]$, are DD integrable on $[a, b]$ and satisfy the following hypothesis.*

(F0) *There exists an $r > 0$ such that* (5.3) *holds for all $x, z \in C[a, b]$ with $\|x - z\|_\infty \le r$ and for all $t \in [a, b]$, where h is the Nemytskij operator defined by*

$$h(w) = q(\cdot, w(\cdot)), \quad w \in C_+[a, b], \ \|w\|_\infty \le r, \tag{5.8}$$

and $q : [a, b] \times [0, r] \to \mathbb{R}_+$ satisfies the hypothesis (q) of Lemma 5.3.

Proof. According to Lemma 5.3 the Cauchy problem (5.7) has for some $u_0 = r_0 > 0$ the smallest AC solution $v = u(\cdot, r_0)$, and $r_0 \le v(t) \le r$ for each $t \in [a, b]$. Since $q(s, \cdot)$ is increasing and right-continuous in $[0, r]$ for a.e. $s \in [a, b]$, and because $q(\cdot, x)$ is measurable for all $x \in [0, r]$ and $q(\cdot, r)$ is Lebesgue integrable, it follows from [1, Theorem 2.1.1 and Remarks 2.1.1] that $q(\cdot, u(\cdot))$ is Lebesgue integrable whenever u belongs to the order interval $[\theta, v]$ of $C_+[a, b]$. Thus the equation (5.4), where h is the Nemytskij operator defined by (5.8), defines a mapping $Q : [\theta, v] \to C_+[a, b]$. Condition (Q0) ensures that Q is increasing, and the choices of r_0 and v imply that

$$r_0 + Q(v) = v. \tag{5.9}$$

Thus $v(t) - Q(v)(t) = r_0 > 0$ for each $t \in [a, b]$. The sequence $(Q^n(v))_{n=0}^\infty$ is decreasing because $q(t, \cdot)$ is increasing for a.e. $t \in [a, b]$. Noticing that the functions $Q^n(v)$ are also continuous, the reasoning similar to that applied in the proof of Lemma 5.3 shows that $(Q^n(v))_{n=0}^\infty$ converges uniformly on $[a, b]$ to the zero function. The above proof shows that the hypotheses of Theorem 5.2 hold. □

6. Dependence on the Initial Value

We shall first prove that under the hypotheses of Proposition 5.4 the difference of solutions y of (1.1) belonging to initial values c and $\hat{c}$, respectively, can be estimated by the *smallest solution* of the comparison problem (5.7) with initial value $u_0 = |c - \hat{c}|$. This estimate implies by Lemma 5.3 the continuous dependence of y on c.

Proposition 6.1. *Let the distributions $f(x)$, $x \in C[a, b]$, satisfy the hypotheses of Proposition 5.4. If $y = y(\cdot, c)$ denotes the solution of the Cauchy problem* (1.1) *and $u = u(\cdot, u_0)$ the smallest solution of the Cauchy problem* (5.7)*, then for all c, $\hat{c} \in \mathbb{R}$, with $|c - \hat{c}|$ small enough,*

$$|y(t, c) - y(t, \hat{c}))| \le u(t, |c - \hat{c}|), \quad t \in [a, b]. \tag{6.1}$$

In particular, $y(\cdot, c)$ depends continuously on c in the sense that $y(t, \hat{c}) \to y(t, c)$ uniformly over $t \in [a, b]$ as $\hat{c} \to c$.

Proof. Assume that $c, \hat{c} \in \mathbb{R}$, and that $|c - \hat{c}| \le r_0$, where r_0 is chosen as in Lemma 5.3. The solutions $y = y(\cdot, c)$ and $\hat{y} = y(\cdot, \hat{c})$ exist by Proposition 5.4, and they satisfy by Lemma 2.1 the fixed point equations

$$y(t) = F(y)(t) = c + {}^c\!\!\int_a^t f(y), \text{ and } \hat{y}(t) = \hat{F}(\hat{y})(t) = \hat{c} + {}^c\!\!\int_a^t f(\hat{y}), \quad t \in [a,b].$$

Moreover, F satisfies by the proof of Proposition 5.4 the hypotheses of Proposition 5.1 with Q defined by (5.4), or equivalently, by

$$Q(w)(t) = \int_a^t q(s, w(s))\, ds, \quad t \in [a,b],$$

and $u = u(\cdot, |c - \hat{c}|)$ is the smallest AC solution of

$$u = |c - \hat{c}| + Q(u).$$

Denote

$$V = \{y \in C[a,b] : \lceil y - \hat{y} \rceil \le u\}.$$

Since Q is increasing, and since

$$F(\hat{y})(t) - \hat{y}(t) = F(\hat{y})(t) - \hat{F}(\hat{y})(t) = c - \hat{c}$$

for all $t \in [a,b]$, we have for every $y \in V$,

$$\begin{aligned} \lceil F(y) - \hat{y} \rceil &\le \lceil F(\hat{y}) - \hat{y} \rceil + \lceil F(y) - F(\hat{y}) \rceil \\ &\le \lceil F(\hat{y}) - \hat{y} \rceil + Q(\lceil y - \hat{y} \rceil) \\ &\le |c - \hat{c}| + Q(u) = u. \end{aligned}$$

Thus $F[V] \subseteq V$. Since $\hat{y} \in V$, then $(F^n(\hat{y})) \in V$ for every $n \in \mathbb{N}_0$. The uniform limit $y = \lim_n F^n(\hat{y})$ exists by Theorem 5.2 and is the solution of (1.1). Because V is closed, then $y \in V$, so that $\lceil y - \hat{y} \rceil \le u$. This proves (6.1). According to Lemma 5.3, $u(t, |c - \hat{c}|) \to 0$ uniformly over $t \in [a,b]$ as $|c - \hat{c}| \to 0$. This result and (6.1) imply that the last assertion of the proposition is true. $\square$

Remark 6.2. If in condition (Q0), $r = \infty$ and $q(\cdot, z) \le \overline{q} \in L^1([a,b]$ for each $z \in \mathbb{R}_+$, then (6.1) holds for all c, $\hat{c} \in \mathbb{R}$.

The hypotheses imposed on $q : [a,b] \times [0,r] \to \mathbb{R}_+$ in (Q0) hold if $q(t, \cdot)$ is increasing for a.e. $t \in [a,b]$, and if q is an L^1-bounded Carathéodory function such that the following local Kamke's condition holds.

$$u \in C([a,b],[0,r]) \text{ and } u(t) \le \int_a^t q(s, u(s))\, ds \text{ for all } t \in [a,b] \text{ imply } \ u(t) \equiv 0.$$

The hypotheses of (Q0) are valid also for the function $q : [a,b] \times [0,r] \to \mathbb{R}_+$, defined by

$$q(t,s) = p(t)\, \phi(s) \quad t \in [a,b],\ s \in [0,r], \tag{6.2}$$

where $p \in L^1([a,b], \mathbb{R}_+)$, $\phi : [0,r] \to \mathbb{R}_+$ is increasing and right-continuous, and $\int_0^r \frac{dv}{\phi(v)} = \infty$.

Let $\ln_n$ and $\exp_n$ denote n-fold iterated logarithm and exponential functions, respectively. The functions ϕ_n, $n \in \mathbb{N}$, defined by $\phi_n(0) = 0$, and

$$\phi_n(s) = s \prod_{j=1}^n \ln_j \frac{1}{s}, \quad 0 < s \le \exp_n(1)^{-1},$$

have properties assumed above for the function ϕ when $r = \exp_n(1)^{-1}$. These properties hold also for the function $\phi(s) = s$, $s \ge 0$. Thus the following result is a special case of Theorem 5.2 and Proposition 6.1.

Corollary 6.3. *The Cauchy problem* (1.1) *has for each* $c \in \mathbb{R}$ *a unique solution* $y = y(\cdot, c)$*, if* $f(x)$ *is DD integrable on* $[a, b]$ *for all* $x \in C[a, b]$*, and if there exists a Lebesgue integrable function* $p : [a, b] \to \mathbb{R}_+$ *such that*

$$\Big|{}^c\!\!\int_a^t f(y) - {}^c\!\!\int_a^t f(z)\Big| \le \int_a^t p(s)|y(s) - z(s)|\,ds$$

for all $y, z \in C[a, b]$ *and for all* $t \in [a, b]$*. Moreover,*

$$|y(t, c) - y(t, \hat{c}))| \le e^{\int_a^t p(s)ds}|c - \hat{c}|, \quad t \in [a, b],\ c,\ \hat{c} \in \mathbb{R}.$$

In linear case we obtain the following consequence from Corollary 6.3.

Corollary 6.4. *For each* $c \in \mathbb{R}$*, the linear Cauchy problem*

$$y' = h + py, \quad y(0) = c,$$

has a unique solution in $C[a, b]$ *whenever the distribution* h *is DD integrable on* $[a, b]$*, and* $p : [a, b] \to \mathbb{R}_+$ *is Lebesgue integrable.*

Remark 6.5. The Cauchy problem

$$y'(t) = g(t, y(t)) \text{ a.e. on } [a, b], \quad y(a) = c, \tag{6.3}$$

is a special case of problem (1.1) when f is the Nemytskij operator associated with the function $g : [a, b] \times \mathbb{R} \to \mathbb{R}$ by

$$f(x) := g(\cdot, x(\cdot)), \quad x \in L^1[a, b].$$

For instance, Theorem 3.1 implies the following result.

Corollary 6.6. *The Cauchy problem* (6.3) *has the smallest and greatest continuous solutions that are increasing with respect to* g *and* c*, if the following hypotheses are valid.*

(G0) $g(\cdot, x(\cdot))$ *is Henstock-Kurzweil integrable on* $[a, b]$ *for every* $x \in L^1[a, b]$*.*

(G1) ${}^K\!\!\int_a^t g(s, x(s))\,ds \le {}^K\!\!\int_a^t g(s, y(s))\,ds$ *for all* $t \in [a, b]$ *whenever* $x \le y$ *in* $L^1[a, b]$*.*

(G2) *There exist Henstock-Kurzweil integrable functions* $g_\pm : [a, b] \to \mathbb{R}$ *such that* ${}^K\!\!\int_a^t g_-(s)\,ds \le {}^K\!\!\int_a^t g(s, x(s))\,ds \le {}^K\!\!\int_a^t g_+(s)\,ds$ *for all* $x \in L^1[a, b]$ *and* $t \in [a, b]$*.*

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