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ESTIMATES AND UNIQUENESS FOR BOUNDARY BLOW-UP SOLUTIONS OF P-LAPLACE EQUATIONS

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Abstract. We investigate boundary blow-up solutions of the p-Laplace equation $\Delta_p u = f(u)$, $p > 1$, in a bounded smooth domain $\Omega \subset R^N$. Under appropriate conditions on the growth of $f(t)$ as t approaches infinity, we find an estimate of the solution $u(x)$ as x approaches $\partial\Omega$, and a uniqueness result.

1. Introduction

Let $f(t)$ be a $C^1(0,\infty)$ function, positive, non decreasing, satisfying $f(0) = 0$ and the condition

$$\lim_{t\to\infty} \frac{t\big(f^{\frac{1}{p-1}}(t)\big)'}{f^{\frac{1}{p-1}}(t)} = \alpha, \tag{1.1}$$

with $p > 1$ and $\alpha > 1$. It is well known (see [6, page 282]) that a smooth function f which satisfies (1.1) has the following representation

$$f^{\frac{1}{p-1}}(t) = Ct^{\alpha} \exp\Big(\int_{t_0}^{t} \frac{g(\tau)}{\tau} d\tau\Big), \tag{1.2}$$

where C and t_0 are positive constants and $g(t) \to 0$ as $t \to \infty$. Functions which have this representation are said to be normalized regularly varying at ∞. More precisely, $f^{\frac{1}{p-1}}(t)$ is regularly varying of index α, and $f(t)$ is regularly varying of index $\alpha(p-1)$. Since

$$\Big(\frac{f^{\frac{1}{p-1}}(t)}{t^{\beta}}\Big)' = t^{-\beta-1} f^{\frac{1}{p-1}}(t)\Big[\frac{t\big(f^{\frac{1}{p-1}}(t)\big)'}{f^{\frac{1}{p-1}}(t)} - \beta\Big],$$

if f satisfies (1.1) then the function $\frac{f^{\frac{1}{p-1}}(t)}{t^{\beta}}$ is increasing for large t whenever $\beta < \alpha$. In particular, since $\alpha > 1$, the function $\frac{f(t)}{t^{p-1}}$ is increasing for large t. Furthermore, condition (1.1) implies the generalized Keller-Osserman condition

$$\int_1^{\infty} \frac{dt}{\big(F(t)\big)^{1/p}} < \infty, \quad F(t) = \int_0^t f(\tau) d\tau. \tag{1.3}$$

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Consider the Dirichlet problem

$$\Delta_p u = f(u) \quad \text{in } \Omega, \qquad u(x) \to \infty \quad \text{as } x \to \partial\Omega. \tag{1.4}$$

It is well known that when f satisfies condition (1.3), problem (1.4) has a solution (see for example [9]). In the present paper, assuming condition (1.1), we find a quite precise estimate for a solution near the boundary $\partial\Omega$, and we derive a result of uniqueness.

In case of $p = 2$, problems about the existence of boundary blow-up solutions have been investigated for a long time, see the classical papers [11, 17], and the recent survey [18]. We refer to the paper [14] for a description of spatial heterogeneity models, including historical hints. For the investigation of the boundary behaviour of blow-up solutions we refer to [1, 3, 4, 5, 6, 12]. The case of weighted semilinear equations has been discussed in [13, 15, 20]. The case $p > 1$, has been treated in [9, 10, 16]. In the present paper, assuming condition (1.1), we find an estimate of the solution up to the second order.

In case of $p = 2$, condition (1.1) appears in the paper [7], where the author proves a uniqueness result for problem (1.4). We emphasize that the method used in [7] is not applicable in the present case because of the nonlinearity of the p-Laplacian.

For $s > 0$, define the function $\phi(s)$ as

$$\int_{\phi(s)}^{\infty} \frac{dt}{(qF(t))^{1/p}} = s, \tag{1.5}$$

where $q = \frac{p}{p-1}$. If u is a solution to problem (1.4), we prove the estimate

$$u(x) = \phi(\delta)[1 + O(1)\delta], \tag{1.6}$$

where $\delta = \delta(x) = operatornamedist(x, \partial\Omega)$ and $O(1)$ denotes a bounded quantity. Estimate (1.6) implies, in particular, that if u_1 and u_2 are two solutions of problem (1.4) then

$$\lim_{x\to\partial\Omega} \frac{u_1(x)}{u_2(x)} = 1.$$

By using this result, the monotonicity of $f(t)$ for $t > 0$ and the monotonicity of $\frac{f(t)}{t^{p-1}}$ for large t we prove the uniqueness of the solution to problem (1.4).

2. Main results

We have already noticed that if $f(t)$ satisfies (1.1) then the representation (1.2) holds. By (1.2) it follows that, for $\epsilon > 0$, we can find positive constants C_1 and C_2 such that for t large we have

$$C_1 t^{\alpha(p-1)+1-\epsilon} < F(t) < C_2 t^{\alpha(p-1)+1+\epsilon}, \tag{2.1}$$

where F is defined as in (1.3). Furthermore, the function ϕ defined in (1.5), for s small satisfies

$$C_1 \Big(\frac{1}{s}\Big)^{\frac{p-\epsilon}{(p-1)(\alpha-1)}} < \phi(s) < C_2 \Big(\frac{1}{s}\Big)^{\frac{p+\epsilon}{(p-1)(\alpha-1)}}. \tag{2.2}$$

Lemma 2.1. *Let $A(\rho, R) \subset \mathbb{R}^N$, $N \geq 2$, be the annulus with radii ρ and R centered at the origin. Let $f(t) > 0$ be smooth, increasing for $t > 0$ and such that* (1.1) *holds with $\alpha > 1$. If $u(x)$ is a radial solution to problem* (1.4) *in $\Omega = A(\rho, R)$ and $v(r) = u(x)$ for $r = |x|$, then*

$$v(r) < \phi(R - r)[1 + C(R - r)], \quad \tilde{r} < r < R, \tag{2.3}$$

and,

$$v(r) > \phi(r-\rho)[1 - C(r-\rho)], \quad \rho < r < \tilde{r}, \tag{2.4}$$

where ϕ is defined as in (1.5), *$\rho < \tilde{r} < R$ and C is a suitable positive constant.*

Proof. We have

$$\big(|v'|^{p-2}v'\big)' + \frac{N-1}{r}|v'|^{p-2}v' = f(v), \quad v(\rho) = v(R) = \infty. \tag{2.5}$$

It is easy to show that there is r_0 such that $v(r)$ is decreasing for $\rho < r < r_0$ and increasing for $r_0 < r < R$, with $v'(r_0) = 0$. For $r > r_0$ we have

$$\big(|v'|^{p-2}v'\big)' = \big((v')^{p-1}\big)' = (p-1)(v')^{p-2}v''.$$

Therefore, multiplying (2.5) by v' and integrating over (r_0, r) we find

$$\frac{(v')^p}{q} + (N-1)\int_{r_0}^{r}\frac{(v')^p}{s}ds = F(v) - F(v_0), \quad v_0 = v(r_0). \tag{2.6}$$

Since $F(v_0) > 0$, (2.6) implies that

$$v' < (qF(v))^{1/p}, \quad r \in (r_0, R). \tag{2.7}$$

As a consequence we have

$$\int_{r_0}^{r}\frac{(v')^p}{s}ds \le \frac{1}{r_0}\int_{r_0}^{r}(qF(v))^{1/q}v'ds < \frac{q^{1/q}}{r_0}\int_0^v (F(t))^{1/q}dt. \tag{2.8}$$

On the other hand, by (2.6) we find

$$\frac{(v')^p}{qF(v)} = 1 - \frac{(N-1)\int_{r_0}^{r}\frac{(v')^p}{s}ds + F(v_0)}{F(v)}.$$

The above equation yields

$$\frac{v'}{(qF(v))^{1/p}} = 1 - \Gamma(r), \tag{2.9}$$

where,

$$\Gamma(r) = 1 - \Big(1 - \frac{(N-1)\int_{r_0}^{r}\frac{(v')^p}{s}ds + F(v_0)}{F(v)}\Big)^{1/p}.$$

By using the inequality $1 - (1-t)^{1/p} < t$ (true for $0 < t < 1$), and (2.8) we find, for some constant M,

$$\Gamma(r) \le \frac{(N-1)\int_{r_0}^{r}\frac{(v')^p}{s}ds + F(v_0)}{F(v)} \le M\frac{\int_0^v (F(t))^{1/q}dt}{F(v)}.$$

Since

$$\int_0^v (F(t))^{1/q}dt \le (F(v))^{1/q}v,$$

we have

$$\Gamma(r) < \frac{Mv(r)}{(F(v(r)))^{1/p}}. \tag{2.10}$$

By using (2.1) (with ϵ small enough) one finds that $\Gamma(r) \to 0$ as $r \to R$. Furthermore, using (1.2) one proves that

$$\lim_{t\to\infty}\frac{F(t)}{tf(t)} = \frac{1}{\alpha(p-1)+1}.$$

Hence, since

$$\Big(\frac{t}{(F(t))^{1/p}}\Big)' = \frac{tf(t)}{(F(t))^{\frac{p+1}{p}}}\Big[\frac{F(t)}{tf(t)} - \frac{1}{p}\Big],$$

and $\frac{1}{\alpha(p-1)+1} < \frac{1}{p}$, the function $\frac{t}{(F(t))^{1/p}}$ is decreasing for large t. As a consequence, the function $\frac{Mv(r)}{(F(v(r)))^{1/p}}$ tends to zero monotonically as r tends to R.

The inverse function of ϕ is the following

$$\psi(s) = \int_s^\infty \frac{1}{(qF(t))^{1/p}}dt.$$

Integration of (2.9) over (r, R) yields

$$\psi(v) = R - r - \int_r^R \Gamma(s)ds, \tag{2.11}$$

from which we find

$$v(r) = \phi(R-r) - \phi'(\omega)\int_r^R \Gamma(s)ds, \tag{2.12}$$

with

$$R - r > \omega > R - r - \int_r^R \Gamma(s)ds.$$

Since

$$-\phi'(\omega) = (qF(\phi(\omega))^{1/p},$$

and since the function $t \to F(\phi(t))$ is decreasing we have

$$-\phi'(\omega) < \Big(qF\big(\phi\big(R - r - \int_r^R \Gamma(s)ds\big)\big)\Big)^{1/p} = (qF(v))^{1/p},$$

where (2.11) has been used in the last step. Hence, by (2.12) and (2.10) we find

$$v(r) < \phi(R-r) + (qF(v))^{1/p}\int_r^R \frac{Mv(s)}{(F(v(s)))^{1/p}}ds.$$

Recalling that the function $\frac{Mv(r)}{(F(v(r)))^{1/p}}$ is decreasing for r close to R, the latter estimate implies

$$v(r) < \phi(R-r) + q^{1/p}Mv(r)(R-r),$$

and

$$v(r) < \frac{\phi(R-r)}{1 - q^{1/p}M(R-r)},$$

from which inequality (2.3) follows.

For $r < r_0$ we have $v' < 0$ and, instead of equation (2.6), we find

$$\frac{|v'|^p}{q} = F(v) - F(v_0) + (N-1)\int_r^{r_0} \frac{|v'|^p}{s}ds, \tag{2.13}$$

with $\rho < r < r_0$. Note that, since $|v'(r)|^p \to \infty$ as $r \to \rho$ and $v'' > 0$, we have (Lemma 2.1 of [12])

$$\lim_{r\to\rho} \frac{\int_r^{r_0} \frac{|v'|^p}{t}dt}{|v'|^p} = 0.$$

Hence, (2.13) implies $|v'| < q(F(v))^{1/p}$ for r near to ρ. Using equation (2.13) again we find

$$\frac{|v'|^p}{qF(v)} = 1 + \frac{(N-1)\int_r^{r_0} \frac{|v'|^p}{s} ds - F(v_0)}{F(v)}.$$

The above equation yields

$$\frac{-v'}{(qF(v))^{1/p}} = 1 + \tilde{\Gamma}(r), \tag{2.14}$$

where

$$\tilde{\Gamma}(r) = \Big(1 + \frac{(N-1)\int_r^{r_0} \frac{|v'|^p}{s} ds - F(v_0)}{F(v)}\Big)^{1/p} - 1.$$

Since $(1+t)^{1/p} - 1 < t$ (true for $t > 0$), we have

$$\tilde{\Gamma}(r) < \frac{(N-1)\int_r^{r_0} \frac{|v'|^p}{s} ds - F(v_0)}{F(v)}.$$

Using the estimate $|v'| < q(F(v))^{1/p}$ we find $|v'|^p < q^{p-1}(F(v))^{\frac{p-1}{p}}(-v')$. Therefore, $\tilde{\Gamma}(r)$ satisfies

$$\tilde{\Gamma}(r) \le \frac{Mv(r)}{(F(v(r)))^{1/p}}, \tag{2.15}$$

where M is a suitable constant (possible different from that of (2.10)). It follows that $\tilde{\Gamma}(r) \to 0$ as $r \to \rho$.

Integration of (2.14) over (ρ, r) yields

$$\psi(v) = r - \rho + \int_\rho^r \tilde{\Gamma}(s) ds,$$

from which we find

$$v(r) = \phi(r-\rho) + \phi'(\omega_1)\int_\rho^r \tilde{\Gamma}(s) ds, \tag{2.16}$$

with

$$r - \rho < \omega_1 < r - \rho + \int_\rho^r \tilde{\Gamma}(s) ds.$$

Since $\phi'(s)$ is increasing we have

$$\phi'(\omega_1) > \phi'(r-\rho) = -\big(qF(\phi(r-\rho))\big)^{1/p}.$$

This estimate, (2.15) and (2.16) imply

$$v(r) > \phi(r-\rho) - \big(qF(\phi(r-\rho))\big)^{1/p}\int_\rho^r \frac{Mv(s)}{\big(F(v(s))\big)^{1/p}} ds.$$

Since the function $\frac{t}{(F(t))^{1/p}}$ is decreasing for t large and the function $v(r)$ is decreasing for r close to ρ, it follows that $\frac{v(r)}{(F(v(r)))^{1/p}}$ is increasing. Therefore,

$$v(r) > \phi(r-\rho) - \big(qF(\phi(r-\rho))\big)^{1/p}\frac{Mv(r)}{\big(F(v(r))\big)^{1/p}}(r-\rho). \tag{2.17}$$

On the other hand, by (2.14) we have

$$\frac{-v'}{(qF(v))^{1/p}} < 2, \quad \rho < r < \tilde{r}.$$

Integrating over (ρ, r) we find

$$\psi(v) < 2(r-\rho),$$

whence,

$$v(r) > \phi(2(r-\rho)). \tag{2.18}$$

We claim that, for some $M > 1$ and δ small, we have

$$\frac{1}{M}\phi(\delta) \le \phi(2\delta). \tag{2.19}$$

Indeed, putting $\phi(\delta) = t$, we can write (2.19) as

$$\frac{t}{M} \le \phi(2\psi(t)),$$

or

$$\psi(t) \le \frac{1}{2}\psi\big(\frac{t}{M}\big)$$

for t large. To prove this inequality, we write

$$\psi(t) = \int_t^\infty (qF(\tau))^{-1/p} d\tau = M\int_{\frac{t}{M}}^\infty (qF(M\tau))^{-1/p} d\tau.$$

Since $f(t)$ is regularly varying with index $\alpha(p-1)$, $F(t)$ is regularly varying with index $\alpha(p-1)+1$, and (see [6])

$$\lim_{t\to\infty} \frac{F(Mt)}{F(t)} = M^{\alpha(p-1)+1}.$$

Therefore, for t large we have

$$(F(M\tau))^{-1/p} \le \frac{(F(\tau))^{-1/p}}{M^{\frac{\alpha(p-1)+1}{p}} - 1}.$$

Hence,

$$\psi(t) \le \frac{M}{M^{\frac{\alpha(p-1)+1}{p}} - 1}\int_{\frac{t}{M}}^\infty (qF(\tau))^{-1/p} d\tau = \frac{M}{M^{\frac{\alpha(p-1)+1}{p}} - 1}\psi\big(\frac{t}{M}\big).$$

The claim follows with M such that

$$\frac{M}{M^{\frac{\alpha(p-1)+1}{p}} - 1} = \frac{1}{2}.$$

Using (2.18), (2.19), and recalling that $F(t)$ is regularly varying with index $\alpha(p-1)+1$ we find, for r close to ρ,

$$\frac{F(\phi(r-\rho))}{F(v(r))} \le \frac{F(\phi(r-\rho))}{F(\phi(2(r-\rho)))} \le \frac{F(\phi(r-\rho))}{F\Big(\frac{1}{M}\phi(r-\rho)\Big)} < M^{\alpha(p-1)+1} + 2.$$

Insertion of the latter estimate into (2.17) yields

$$v(r) > \phi(r-\rho) - \tilde{M}v(r)(r-\rho),$$

from which (2.4) follows. The lemma is proved. $\square$

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain and let $f(t) > 0$ be smooth, increasing and satisfying* (1.1) *with $\alpha > 1$. If $u(x)$ is a solution to problem* (1.4) *then we have*

$$\phi(\delta)\big[1 - C\delta\big] < u(x) < \phi(\delta)\big[1 + C\delta\big], \tag{2.20}$$

where ϕ is defined as in (1.5)*, δ denotes the distance from x to $\partial\Omega$ and C is a suitable positive constant.*

Proof. If $P \in \partial\Omega$ we consider a suitable annulus of radii ρ and R contained in Ω and such that its external boundary is tangent to $\partial\Omega$ in P. If $v(x)$ is the solution of problem (1.4) in this annulus, by using the comparison principle for elliptic equations [8, Theorem 10.1] we have $u(x) \leq v(x)$ for x belonging to the annulus. Choose the origin in the center of the annulus and put $v(x) = v(r)$ for $r = |x|$. By (2.3), for r near to R we have

$$v(r) < \phi(\delta)\big[1 + C\delta\big].$$

The latter estimate together with the inequality $u(x) \leq v(x)$ yield the right hand side of (2.20).

Consider a new annulus of radii ρ and R containing Ω and such that its internal boundary is tangent to $\partial\Omega$ in P. If $v(x)$ is the solution of problem (1.4) in this annulus, by using the comparison principle for elliptic equations we have $u(x) \geq v(x)$ for x belonging to Ω. Choose the origin in the center of the annulus and put again $v(x) = v(r)$ for $r = |x|$. By (2.4), for r near to ρ we have

$$v(r) > \phi(\delta)\big[1 - C\delta\big].$$

The latter estimate together with the inequality $u(x) \geq v(x)$ yield the left hand side of (2.20). The theorem is proved. □

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain and let $f(t) > 0$ be smooth, increasing and satisfying* (1.1) *with $\alpha > 1$. If $u(x)$ is a solution to problem* (1.4) *then, $|\nabla u| \to \infty$ as $x \to \partial\Omega$.*

Proof. By Theorem 2.2 we have

$$\lim_{x \to \partial\Omega} \frac{u(x)}{\phi(\delta(x))} = 1.$$

In particular, for $\delta < \delta_0$, δ_0 small, we have

$$\frac{1}{2} < \frac{u(x)}{\phi(\delta(x))} < 2.$$

Now we follow the argument described in [2, page 105], using the same notation (with $\beta = \rho$ and $\rho < \rho_0$). For $\xi \in \check{D}(\rho)$, define

$$v(\xi) = \frac{u(\rho\xi)}{\phi(\rho)}.$$

For $\xi \in \check{D}(\rho)$ we have

$$\frac{1}{2} \leq v(\xi) \leq 2. \tag{2.21}$$

We find

$$\nabla v = \frac{\rho}{\phi(\rho)} \nabla u(\rho\xi),$$

and

$$\Delta_p v = \frac{\rho^p}{(\phi(\rho))^{p-1}} \Delta_p u(\rho\xi) = \frac{\rho^p}{(\phi(\rho))^{p-1}} f(u(\rho\xi)) = \frac{\rho^p}{(\phi(\rho))^{p-1}} f(v(\xi)\phi(\rho)).$$

With $\psi(t) = \rho$ we have

$$\Delta_p v = \frac{(\psi(t))^p}{t^{p-1}} f(v(\xi)t) = \Big(\frac{\psi(t)}{t^{\frac{p-1}{p}}(f(t))^{-1/p}}\Big)^p \frac{f(v(\xi)t)}{f(t)}. \tag{2.22}$$

Since $f(t)$ is regularly varying with index $\alpha(p-1)$ we have

$$\lim_{t\to\infty} \frac{f(v(\xi)t)}{f(t)} = (v(\xi))^{\alpha(p-1)}. \tag{2.23}$$

Furthermore, we have

$$\frac{\psi(t)}{t^{\frac{p-1}{p}}(f(t))^{-1/p}} = \frac{\psi(t)}{t(F(t))^{-\frac{1}{p}}} \Big(\frac{tf(t)}{F(t)}\Big)^{1/p}.$$

We have already observed that (1.2) implies

$$\lim_{t\to\infty} \frac{tf(t)}{F(t)} = \alpha(p-1)+1.$$

Using de l'Hospital rule and the latter estimate we get

$$\lim_{t\to\infty} \frac{\psi(t)}{t(F(t))^{-\frac{1}{p}}} = \frac{q^{1/q}}{\alpha-1}.$$

Hence,

$$\lim_{t\to\infty} \frac{\psi(t)}{t^{\frac{p-1}{p}}(f(t))^{-1/p}} = \frac{q^{1/q}}{\alpha-1}\big(\alpha(p-1)+1\big)^{1/p}. \tag{2.24}$$

By (2.24), (2.23) and (2.21), (2.22) implies that

$$C_1 \le \Delta_p v \le C_2, \quad \xi \in \check{D}(\rho) \tag{2.25}$$

where C_1 and C_2 are suitable positive constants independent of ρ.

Let $x_i \in \Omega$, $x_i \to \partial\Omega$, and let $\rho_i = \operatorname{dist}(x_i, \partial\Omega)$. By (2.25) with $v_i(\xi) = \frac{u(\rho_i\xi)}{\phi(\rho_i)}$, and standard regularity results (see [19]), we find that the $C^{1,\beta}(\check{D}(\rho_i))$ norm of the sequence $v_i(\xi)$ is bounded far from zero. In particular,

$$|\nabla v_i(\xi)| \ge c,$$

with $c > 0$ independent of i. Hence,

$$|\nabla u(x_i)| = |\nabla v_i(\xi)| \frac{\phi(\rho_i)}{\rho_i} \ge c\frac{\phi(\rho_i)}{\rho_i}.$$

Since $\frac{\phi(\rho_i)}{\rho_i} \to \infty$ as $i \to \infty$, the theorem follows. $\square$

Let us discuss now the uniqueness of problem (1.4). Observe that if $\alpha > 1 + \frac{p}{p-1}$ then

$$\lim_{\delta\to 0} \phi(\delta)\delta = \lim_{t\to\infty} t\psi(t) = \lim_{t\to\infty} \frac{t^2}{(qF(t))^{1/p}} = 0,$$

where (2.1) with $\epsilon < (\alpha-1)(p-1) - p$ has been used in the last step. Hence, if $u(x)$ and $v(x)$ are solutions to problem (1.4) in case of $\alpha > 1 + \frac{p}{p-1}$, by Theorem 2.2 we have

$$\lim_{x\to\partial\Omega} [u(x) - v(x)] = 0.$$

Since $f(t)$ is non decreasing, the comparison principle yields $u(x) = v(x)$ in Ω.

For general $\alpha > 1$, we have the following result.

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth domain and let $f(t) > 0$ be smooth, increasing and satisfying* (1.1) *with $\alpha > 1$. If $u(x)$ and $v(x)$ are positive large solutions to problem* (1.4) *then $u(x) = v(x)$.*

Proof. Theorem 2.2 implies

$$\lim_{x \to \partial\Omega} \frac{u(x)}{v(x)} = 1.$$

Let t_0 large enough so that $\frac{f(t)}{t^{p-1}}$ is increasing for $t > t_0$, and let $\eta > 0$ such that $u(x) > t_0$ in $\Omega_\eta = \{x \in \Omega : \delta(x) < \eta\}$. For $\epsilon > 0$ define

$$D_{\epsilon,\eta} = \{x \in \Omega_\eta : (1+\epsilon)u(x) < v(x)\}.$$

If $D_{\epsilon,\eta}$ is empty for any $\epsilon > 0$ then we have $u(x) \geq v(x)$ in Ω_η. Define $\Omega^\eta = \{x \in \Omega : \delta(x) > \eta\}$. Using the equations for u and v in Ω^η and the monotonicity of $f(t)$ one proves that $u(x) \geq v(x)$ in Ω^η. Hence, in this case, $u(x) \geq v(x)$ in Ω. Changing the roles of u and v we get $u(x) = v(x)$.

Suppose $D_{\epsilon,\eta}$ is not empty for $\epsilon < \epsilon_0$. In this open set, since $\frac{f(t)}{t^{p-1}}$ is increasing for large t, we have

$$\begin{gathered}\Delta_p\big((1+\epsilon)u\big) = (1+\epsilon)^{p-1} f(u) \leq f\big((1+\epsilon)u\big),\\ \Delta_p v = f(v).\end{gathered}$$

By the comparison principle we have

$$v(x) - (1+\epsilon)u(x) \leq \max_{\delta(x)=\eta} [v(x) - (1+\epsilon)u(x)] \quad \text{in } D_{\epsilon,\eta}.$$

Letting $\epsilon \to 0$ we find

$$v(x) - u(x) \leq \max_{\delta(x)=\eta} [v(x) - u(x)] \quad \text{in } \Omega_\eta.$$

Put

$$\max_{\delta(x)=\eta} [v(x) - u(x)] = v(x_1) - u(x_1) = C.$$

Using the equations for u and v in Ω^η and the monotonicity of $f(t)$ one proves that $v(x) - u(x) \leq C$ in Ω^η. Then, $v(x) - u(x) \leq C$ in Ω. We observe that decreasing η and arguing as before we find $x_\eta \to \partial\Omega$ such that

$$v(x) - u(x) \leq v(x_\eta) - u(x_\eta) \quad \text{in } \Omega,$$

with $v(x_\eta) - u(x_\eta) = \textit{constant}$. In other words, $v(x) - u(x)$ attains its maximum value in the set described by x_η (which approaches $\partial\Omega$). By Theorem 2.3, ∇u and ∇v do not vanish in Ω_η for η small. Hence, the strong comparison principle applies (see [8]) and we must have $v(x) - u(x) = C$ in Ω_η.

Since

$$\Delta_p v = f(v) = f(u + C)$$

and

$$\Delta_p v = \Delta_p u = f(u),$$

we must have $f(u) = f(u + C)$ in Ω_η. Since $f(t)$ is strictly increasing for t large, we find $C = 0$. The theorem follows. $\square$

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