Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 137, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF SOLUTIONS FOR NONLOCAL ELLIPTIC SYSTEMS WITH NONSTANDARD GROWTH CONDITIONS

GUOWEI DAI

Abstract. This article concerns the existence and multiplicity of solutions for a $p(x)$-Kirchhoff-type systems with Dirichlet boundary condition. By a direct variational approach and the theory of the variable exponent Sobolev spaces, under growth conditions on the reaction terms, we establish the existence and multiplicity of solutions.

1. Introduction

In this article, we study the following nonlocal elliptic systems of gradient type with nonstandard growth conditions

$$\begin{gathered}
-M_1\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\operatorname{div}\big(|\nabla u|^{p(x)-2}\nabla u\big) = \frac{\partial F}{\partial u}(x,u,v) \quad \text{in } \Omega,\\
-M_2\Big(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\,dx\Big)\operatorname{div}\big(|\nabla v|^{q(x)-2}\nabla v\big) = \frac{\partial F}{\partial v}(x,u,v) \quad \text{in } \Omega,\\
u=0, \quad v=0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, $p(x), q(x) \in C_+(\overline{\Omega})$ with

$$\begin{gathered}
1 < p^- := \min_{\overline{\Omega}} p(x) \le p^+ := \max_{\overline{\Omega}} p(x) < +\infty,\\
1 < q^- := \min_{\overline{\Omega}} q(x) \le q^+ := \max_{\overline{\Omega}} q(x) < +\infty,
\end{gathered}$$

$M_1(t)$, $M_2(t)$ are continuous functions. We confine ourselves to the case where $M_1 = M_2$ for simplicity. Notice that the results of this paper remain valid for $M_1 \ne M_2$ by adding some slight changes in the hypothesis (H4) and (H5). The function $F : \Omega\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is assumed to be continuous in $x\in\overline{\Omega}$ and of class C^1 in $u, v\in\mathbb{R}$.

The operator $-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian, and becomes p-Laplacian when $p(x)\equiv p$ (a constant). The $p(x)$-Laplacian possesses more complicated nonlinearities than the p-Laplacian; for example, it is inhomogeneous. The study of various mathematical problems with variable exponent growth condition

---

2000 *Mathematics Subject Classification.* 35D05, 35J60, 35J70.
Key words and phrases. Variational method; nonlinear elliptic systems; nonlocal condition.

Submitted July 9, 2011. Published October 19, 2011.
Supported by grants 11061030 from the NSFC, and NWNU-LKQN-10-21.

has been received considerable attention in recent years. These problems are interesting in applications and raise many difficult mathematical problems. One of the most studied models leading to problem of this type is the model of motion of electrorheological fluids, which are characterized by their ability to drastically change the mechanical properties under the influence of an exterior electromagnetic field [1, 34, 37]. Problems with variable exponent growth conditions also appear in the mathematical modeling of stationary thermo-rheological viscous flows of non-Newtonian fluids and in the mathematical description of the processes filtration of an ideal barotropic gas through a porous medium [5, 6]. Another field of application of equations with variable exponent growth conditions is image processing [9]. The variable nonlinearity is used to outline the borders of the true image and to eliminate possible noise. We refer the reader to [13, 29, 35, 38, 39] for an overview of and references on this subject, and to [2, 20, 21, 22, 23, 24, 25, 26] for the study of the $p(x)$-Laplacian equations and the corresponding variational problems.

Problem (1.1) is related to the stationary version of a model introduced by Kirchhoff [30]. More precisely, Kirchhoff proposed the model

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx\Big)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation, by considering the effects of the changes in the length of the strings during the vibrations. A distinguishing feature of equation (1.2) is that the equation contains a nonlocal coefficient $\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx$ which depends on the average $\frac{1}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\,dx$, and hence the equation is no longer a pointwise identity. Some early classical studies of Kirchhoff equations were Bernstein [7] and Pohožaev [33]. The equation

$$\begin{gathered}-\Big(a+b\int_\Omega|\nabla u|^2\,dx\Big)\Delta u=f(x,u)\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega,\end{gathered} \tag{1.3}$$

is related to the stationary analogue of the equation (1.2). Equation (1.3) received much attention only after Lions [31] proposed an abstract framework to the problem. Some important and interesting results can be found, for example, in [3, 8, 17]. More recently Alves et al. [4] and Ma and Rivera [32] obtained positive solutions of such problems by variational methods. The study of Kirchhoff type equations has already been extended to the case involving the p-Laplacian (for details, see [10, 18, 19])and $p(x)$-Laplacian (see [12, 15, 17, 27]). In [12], by a direct variational approach, we establish conditions ensuring the existence and multiplicity of solutions for the problem

$$\begin{gathered}-M\Big(\int_\Omega\frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)=f(x,u)\quad\text{in }\Omega,\\ u=0\quad\text{on }\partial\Omega.\end{gathered}$$

In [28], the author established that existence and multiplicity results for a class of elliptic systems with nonstandard growth conditions.

Motivated by above, we consider the nonlocal elliptic systems (1.1). We establish the existence and multiplicity of solutions for system (1.1). Local elliptic systems with standard growth conditions have been the subject of a sizeable literature. We refer to the excellent survey article by De Figueiredo [14]. We also refer to [11] about nonlocal elliptic systems of p-Kirchhoff-type.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge on variable exponent Sobolev spaces. In Sections 3, we give some existence results of weak solutions of problem (1.1) and their proofs.

2. Preliminaries

To discuss problem (1.1), we need some theory on $W_0^{1,p(x)}(\Omega)$ which is called variable exponent Sobolev space. Firstly we state some basic properties of spaces $W_0^{1,p(x)}(\Omega)$ which will be used later (for details, see [25]). Denote by $\mathbf{S}(\Omega)$ the set of all measurable real functions defined on Ω. Two functions in $\mathbf{S}(\Omega)$ are considered as the same element of $\mathbf{S}(\Omega)$ when they are equal almost everywhere. Write

$$C_+(\overline{\Omega}) = \{h : h \in C(\overline{\Omega}), h(x) > 1 \text{ for any } x \in \overline{\Omega}\},$$
$$h^- := \min_{\overline{\Omega}} h(x), \quad h^+ := \max_{\overline{\Omega}} h(x) \quad \text{for every } h \in C_+(\overline{\Omega}).$$

Define

$$L^{p(x)}(\Omega) = \{u \in \mathbf{S}(\Omega) : \int_\Omega |u(x)|^{p(x)}\,dx < +\infty \text{ for } p \in C_+(\overline{\Omega})\}$$

with the norm

$$|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}\,dx \le 1\},$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.$$

Denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Proposition 2.1 ([25]). *The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.2 ([25]). *Set $\rho(u) = \int_\Omega |u(x)|^{p(x)}\,dx$. For any $u \in L^{p(x)}(\Omega)$, then*

1. *for $u \neq 0$, $|u|_{p(x)} = \lambda$ if and only if $\rho(\frac{u}{\lambda}) = 1$;*
2. *$|u|_{p(x)} < 1$ $(= 1; > 1)$ if and only if $\rho(u) < 1$ $(= 1; > 1)$;*
3. *if $|u|_{p(x)} > 1$, then $|u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$;*
4. *if $|u|_{p(x)} < 1$, then $|u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$;*
5. *$\lim_{k\to+\infty} |u_k|_{p(x)} = 0$ if and only if $\lim_{k\to+\infty} \rho(u_k) = 0$;*
6. *$\lim_{k\to+\infty} |u_k|_{p(x)} = +\infty$ if and only if $\lim_{k\to+\infty} \rho(u_k) = +\infty$.*

Proposition 2.3 ([25]). *In $W_0^{1,p(x)}(\Omega)$ the Poincaré inequality holds; that is, there exists a positive constant C_0 such that*

$$|u|_{L^{p(x)}(\Omega)} \le C_0 |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

So, $|\nabla u|_{L^{p(x)}(\Omega)}$ is a norm equivalent to the norm $\|u\|$ in the space $W_0^{1,p(x)}(\Omega)$. We will use the equivalent norm in the following discussion and write $\|u\|_p = |\nabla u|_{L^{p(x)}(\Omega)}$ for simplicity.

Proposition 2.4 ([22, 25]). *If $q \in C_+(\overline{\Omega})$ and $q(x) \leq p^*(x)$ ($q(x) < p^*(x)$) for $x \in \overline{\Omega}$, then there is a continuous (compact) embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.5 ([23, 25]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{q(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{p(x)} = 1$ holds a.e. in Ω. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have the following Hölder-type inequality*

$$\Big|\int_\Omega uv\,dx\Big| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-}\Big)|u|_{p(x)}|v|_{q(x)}.$$

We write

$$I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx.$$

Proposition 2.6 ([23]). *The functional $I : X \to \mathbb{R}$ is convex. The mapping $I' : X \to X^*$ is a strictly monotone, bounded homeomorphism, and is of (S_+) type, namely*

$$u_n \rightharpoonup u \text{ and } \limsup_{n\to+\infty} I'(u_n)(u_n - u) \leq 0 \text{ implies } u_n \to u,$$

where $X = W_0^{1,p(x)}(\Omega)$, X^* is the dual space of X.

For every (u, v) and (φ, ψ) in $W := W_0^{1,p(x)}(\Omega) \times W_0^{1,q(x)}(\Omega)$, let

$$\mathcal{F}(u, v) := \int_\Omega F(x, u, v)\,dx.$$

Then

$$\mathcal{F}'(u, v)(\varphi, \psi) = D_1\mathcal{F}(u, v)(\varphi) + D_2\mathcal{F}(u, v)(\psi),$$

where

$$D_1\mathcal{F}(u, v)(\varphi) = \int_\Omega \frac{\partial F}{\partial u}(x, u, v)\varphi\,dx,$$
$$D_2\mathcal{F}(u, v)(\psi) = \int_\Omega \frac{\partial F}{\partial v}(x, u, v)\psi\,dx.$$

The Euler-Lagrange functional associated to (1.1) is given by

$$J(u, v) := \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\,dx\Big) - \mathcal{F}(u, v),$$

where $\widehat{M}(t) := \int_0^t M(\tau)\,d\tau$. It is easy to verify that $J \in C^1(W, \mathbb{R})$ is weakly lower semi-continuous and $(u, v) \in W$ is a weak solution of (1.1) if and only if (u, v) is a critical point of J. Moreover, we have

$$J'(u, v)(\varphi, \psi) = D_1J(u, v)(\varphi) + D_2J(u, v)(\psi), \tag{2.1}$$

where

$$D_1J(u, v)(\varphi) = M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}\,dx\Big)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\nabla\varphi\,dx - D_1\mathcal{F}(u, v)(\varphi),$$
$$D_2J(u, v)(\psi) = M\Big(\int_\Omega \frac{1}{q(x)}|\nabla v|^{q(x)}\,dx\Big)\int_\Omega |\nabla v|^{q(x)-2}\nabla v\nabla\psi\,dx - D_2\mathcal{F}(u, v)(\psi).$$

Let us choose on W the norm $\|\cdot\|$ defined by

$$\|(u,v)\| := \max\{\|u\|_p, \|v\|_q\}.$$

The dual space of W will be denoted by W^* and $\|\cdot\|_*$ will stand for its norm. Therefore

$$\|J'(u,v)\|_* = \|D_1 J(u,v)\|_{*,p} + \|D_2 J(u,v)\|_{*,q}$$

where $\|\cdot\|_{*,p}$ (respectively $\|\cdot\|_{*,q}$) is the norm of $(W_0^{1,p(x)}(\Omega))^*$ (respectively $(W_0^{1,q(x)}(\Omega))^*$).

3. Existence of solutions

In this section we discuss the existence of weak solutions of (1.1). For simplicity, we use c, c_i, $i = 1,2,\dots$ to denote the general positive constant (the exact value may change from line to line).

Before stating our results, we introduce some natural growth hypotheses on the right-hand side of (1.1) and the nonlocal coefficient $M(t)$. These hypotheses will ensure the mountain pass geometry and the Palais-Smale condition for the Euler-Lagrange functional J.

(H1) For all $(x,s,t) \in \Omega \times \mathbb{R}^2$, we assume

$$|F(x,s,t)| \le c\Big(1 + |s|^{p_1(x)} + |t|^{q_1(x)} + |s|^{\alpha(x)}|t|^{\beta(x)}\Big),$$

where c is a positive constant, $(p_1(x), q_1(x), \alpha(x), \beta(x)) \in (C_+(\overline{\Omega}))^4$ such that

$$p_1(x) < p^*(x), \quad q_1(x) < q^*(x), \quad \frac{2\alpha(x)}{p^*(x)} + \frac{2\beta(x)}{q^*(x)} < 1 \quad \text{in } \overline{\Omega},$$
$$p_1^-, \quad 2\alpha^- > p^+, \quad q_1^-, \quad 2\beta^- > q^+.$$

(H2) There exist $M > 0$, $\theta_1 > \frac{p^+}{1-\mu}$, $\theta_2 > \frac{q^+}{1-\mu}$ such that for all $x \in \Omega$, and all $(s,t) \in \mathbb{R}^2$ with $|s|^{\theta_1} + |t|^{\theta_2} \ge 2M$, one has

$$0 < F(x,s,t) \le \frac{s}{\theta_1}\frac{\partial F}{\partial s}(x,s,t) + \frac{t}{\theta_2}\frac{\partial F}{\partial t}(x,s,t),$$

where μ comes from (H5) below.

(H3) $F(x,s,t) = o(|s|^{p^+} + |t|^{q^+})$ as $(s,t) \to (0,0)$ uniformly with respect to to $x \in \Omega$.

(H4) There exists $m_0 > 0$, such that $M(t) \ge m_0$.

(H5) There exists $0 < \mu < 1$ such that $\widehat{M}(t) \ge (1-\mu)M(t)t$.

As an example, we let $M(t) = a + bt : \mathbb{R}^+ \to \mathbb{R}$ with a, b are two positive constants. It is clear that $M(t) \ge a > 0$. Taking $\mu = 1/2$, we have

$$\widehat{M}(t) = \int_0^t M(s)\,ds = at + \frac{1}{2}bt^2 \ge \frac{1}{2}(a+bt)t = (1-\mu)M(t)t.$$

So conditions (H4), (H5) are satisfied.

Theorem 3.1. *If M satisfies* (H4) *and*

$$|F(x,s,t)| \le c_1(1 + |s|^{\alpha_1} + |t|^{\beta_1}),$$

where α_1, β_1 are two constants with $1 \le \alpha_1 < \min\{p^-, q^-\}$, $1 \le \beta_1 < \min\{p^-, q^-\}$ then (1.1) *has a weak solution.*

Proof. From (H4) we have $\widehat{M}(t) \geq m_0 t$. For $(u_n, v_n) \in W$ such that $\|(u_n, v_n)\| \to +\infty$, we have

$$
\begin{aligned}
&J(u_n, v_n)\\
&= \widehat{M}(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx) + \widehat{M}(\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx) - \int_\Omega F(x, u_n, v_n)\,dx\\
&\geq m_0 \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx + m_0 \int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\\
&\quad - c_1 \int_\Omega |u_n|^{\alpha_1}\,dx - c_1 \int_\Omega |v_n|^{\beta_1}\,dx - c_1|\Omega|\\
&\geq \frac{m_0}{p^+}\|u_n\|_p^{p^-} + \frac{m_0}{q^+}\|v_n\|_q^{q^-} - c_3\|u_n\|_p^{\alpha_1} - c_2\|v_n\|_q^{\beta_1} - c_1|\Omega|,
\end{aligned}
$$

where $|\Omega|$ denotes the measure of Ω. Without loss of generality, we may assume $\|u_n\|_p \geq \|v_n\|_q$. Hence,

$$
J(u_n, v_n) \geq \frac{m_0}{p^+}\|u_n\|_p^{p^-} - c_3\|u_n\|_p^{\alpha_1} - c_2\|u_n\|_p^{\beta_1} - c_1|\Omega|, \tag{3.1}
$$

By the definition of norm on W, we have $\|(u_n, v_n)\| = \|u_n\|_p \to +\infty$. In view of (3.1) and the assumptions on α_1 and β_1, we can easily see that $J(u_n, v_n) \to +\infty$ as $n \to +\infty$; i.e., J is a coercive functional. Since J also is weakly lower semi-continuous, J has a minimum point (u, v) in W, and (u, v) is a weak solution pair which may be trivial of (1.1). The proof is completed. □

Lemma 3.2. *Let (u_n, v_n) be a Palais-Smale sequence for the Euler-Lagrange functional J. If* (H2), (H4), (H5) *are satisfied then (u_n, v_n) is bounded.*

Proof. Let (u_n, v_n) be a Palais-Smale sequence for the functional J. This means that $J(u_n, v_n)$ is bounded and $\|J'(u_n, v_n)\|_* \to 0$ as $n \to +\infty$. Then, there is a positive constant c_0 such that

$$
\begin{aligned}
c_0 &\geq J(u_n, v_n)\\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\Big) - \int_\Omega F(x, u_n, v_n)\,dx\\
&\geq (1-\mu)M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\\
&\quad - \int_\Omega \frac{u_n}{\theta_1}\frac{\partial F}{\partial u}(x, u_n, v_n)\,dx + (1-\mu)M\Big(\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\Big)\\
&\quad \times \int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx - \int_\Omega \frac{v_n}{\theta_2}\frac{\partial F}{\partial v}(x, u_n, v_n)\,dx - c_4,
\end{aligned}
$$

where c_4 is some positive constant. Then

$$
\begin{aligned}
c_0 &\geq J(u_n, v_n)\\
&\geq \big(\frac{1-\mu}{p^+} - \frac{1}{\theta_1}\big)M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)}\,dx + \frac{1}{\theta_1}D_1J(u_n, v_n)(u_n)\\
&\quad + \big(\frac{1-\mu}{q^+} - \frac{1}{\theta_2}\big)M\Big(\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\Big)\int_\Omega |\nabla v_n|^{q(x)}\,dx\\
&\quad + \frac{1}{\theta_2}D_2J(u_n, v_n)(v_n) - c_4
\end{aligned}
$$

$$
\begin{aligned}
&\geq \big(\frac{1-\mu}{p^+}-\frac{1}{\theta_1}\big)m_0\int_\Omega |\nabla u_n|^{p(x)}\,dx+\big(\frac{1-\mu}{q^+}-\frac{1}{\theta_2}\big)m_0\int_\Omega |\nabla v_n|^{q(x)}\,dx\\
&\quad -\frac{1}{\theta_1}\|D_1J(u_n,v_n)\|_{*,p}\|u_n\|_p-\frac{1}{\theta_2}\|D_2J(u_n,v_n)\|_{*,q}\|v_n\|_q-c_4.
\end{aligned}
$$

Now, suppose that the sequence (u_n,v_n) is not bounded. Without loss of generality, we may assume $\|u_n\|_p\geq\|v_n\|_q$.

Therefore, for n large enough, we have

$$
c_5\geq \big(\frac{1-\mu}{p^+}-\frac{1}{\theta_1}\big)m_0\|u_n\|_p^{p^-}-\Big(\frac{1}{\theta_1}\|D_1J(u_n,v_n)\|_{*,p}+\frac{1}{\theta_2}\|D_2J(u_n,v_n)\|_{*,q}\Big)\|u_n\|_p.
$$

But, this cannot hold true since $p^->1$. Hence, $\{\|(u_n,v_n)\|\}$ is bounded. $\square$

In the following lemma, we show every bounded Palais-Smale sequence for the functional J contains a convergence subsequence.

Lemma 3.3. *Let (u_n,v_n) be a bounded Palais-Smale sequence for the Euler-Lagrange functional J. If* (H1), (H4) *are satisfied, then (u_n,v_n) contains a convergent subsequence.*

Proof. Let (u_n,v_n) be a bounded Palais-Smale sequence for the functional J. Then there is a subsequence still denoted by (u_n,v_n) which converges weakly in W. Without loss of generality, we assume that $(u_n,v_n)\rightharpoonup(u,v)$, then $J'(u_n,v_n)(u_n-u,v_n-v)\to 0$. Thus, we have

$$
\begin{aligned}
&J'(u_n,v_n)(u_n-u,v_n-v)\\
&=M\Big(\int_\Omega\frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega|\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\\
&\quad+M\Big(\int_\Omega\frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\Big)\int_\Omega|\nabla v_n|^{q(x)-2}\nabla v_n(\nabla v_n-\nabla v)\,dx\\
&\quad-\int_\Omega\frac{\partial F}{\partial u}(x,u_n,v_n)(u_n-u)\,dx-\int_\Omega\frac{\partial F}{\partial v}(x,u_n,v_n)(v_n-v)\,dx\to 0.
\end{aligned}
$$

On the other hand, let $\widetilde{\alpha}$, $\widetilde{\beta}$ be two continuous and positive functions on $\overline{\Omega}$ such that

$$
\frac{2\alpha(x)+\widetilde{\alpha}(x)}{p^*(x)}+\frac{2\beta(x)+\widetilde{\beta}(x)}{q^*(x)}=1,\quad\forall x\in\overline{\Omega}.
$$

Using the Young inequality, we obtain

$$
|s|^{\alpha(x)}|t|^{\beta(x)}\leq |s|^{\frac{\alpha(x)p^*(x)}{2\alpha(x)+\widetilde{\alpha}(x)}}+|t|^{\frac{\beta(x)q^*(x)}{2\beta(x)+\widetilde{\beta}(x)}}=|s|^{p_2(x)}+|t|^{q_2(x)},
$$

where $p_2(x):=\frac{\alpha(x)p^*(x)}{2\alpha(x)+\widetilde{\alpha}(x)}<p^*(x)$, $q_2(x):=\frac{\beta(x)q^*(x)}{2\beta(x)+\widetilde{\beta}(x)}<q^*(x)$. From (H1), we can obtain that there exist $p_3(x),q_3(x)\in C_+(\overline{\Omega})$ with $p_3(x)<p^*(x)$, $q_3(x)<q^*(x)$ in $\overline{\Omega}$ such that

$$
|F(x,s,t)|\leq c_6\big(1+|s|^{p_3(x)}+|t|^{q_3(x)}\big).
$$

From this inequality, Propositions 2.4 and 2.5, we can easily obtain

$$
\int_\Omega\frac{\partial F}{\partial u}(x,u_n,v_n)(u_n-u)\,dx\to 0
$$

and

$$
\int_\Omega\frac{\partial F}{\partial v}(x,u_n,v_n)(v_n-v)\,dx\to 0. \tag{3.2}
$$

Therefore, we have

$$
\begin{aligned}
&M\Big(\int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}\,dx\Big)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\to 0,\\
&M\Big(\int_\Omega \frac{1}{q(x)}|\nabla v_n|^{q(x)}\,dx\Big)\int_\Omega |\nabla v_n|^{q(x)-2}\nabla v_n(\nabla v_n-\nabla v)\,dx\to 0.
\end{aligned}
$$

In view of (H4), we have

$$
\begin{aligned}
&\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n(\nabla u_n-\nabla u)\,dx\to 0,\\
&\int_\Omega |\nabla v_n|^{q(x)-2}\nabla v_n(\nabla v_n-\nabla v)\,dx\to 0.
\end{aligned}
$$

Using Proposition 2.6, we have $u_n\to u$ in $W_0^{1,p(x)}(\Omega)$ and $v_n\to v$ in $W_0^{1,q(x)}(\Omega)$, which implies that $(u_n,v_n)\to(u,v)$ in W. This completes the proof. $\square$

Theorem 3.4. *If hypotheses* (H1)–(H5) *hold, then* (1.1) *has at least one weak solution.*

Proof. Let us show that J satisfies the conditions of Mountain Pass Theorem (see [36, Theorem 2.10]). By Lemmas 3.2 and 3.3, J satisfies Palais-Smale condition in W.

For $\|(u,v)\|<1$, using the Young's inequality, the fact $\frac{2\alpha(x)}{p^*(x)}+\frac{2\beta(x)}{q^*(x)}<1$ in $\overline{\Omega}$, Propositions 2.2 and 2.4, we obtain

$$
\int_\Omega |u|^{\alpha(x)}|v|^{\beta(x)}\,dx\le \frac12\int_\Omega |u|^{2\alpha(x)}\,dx+\frac12\int_\Omega |v|^{2\beta(x)}\,dx\le c_7(\|u\|_p^{2\alpha^-}+\|v\|_q^{2\beta^-}).
$$

On the other hand, assuming (H1), $W_0^{1,p(x)}(\Omega)\hookrightarrow L^{p^+}(\Omega)$, and $W_0^{1,q(x)}(\Omega)\hookrightarrow L^{q^+}(\Omega)$. Then there exists $c_8,c_9>0$ such that

$$
\begin{aligned}
&|u|_{p^+}\le c_8\|u\|_p \quad \text{for } u\in W_0^{1,p(x)}(\Omega)\\
&|v|_{q^+}\le c_9\|v\|_q \quad \text{for } v\in W_0^{1,q(x)}(\Omega),
\end{aligned}
$$

where $|\cdot|_r$ denote the norm on $L^{r(x)}(\Omega)$ with $r\in C_+(\overline{\Omega})$. Let $\varepsilon>0$ be small enough such that $\varepsilon c_8^{p^+}\le\frac{m_0}{2p^+}$ and $\varepsilon c_9^{q^+}\le\frac{m_0}{2q^+}$. By the assumptions (H1) and (H3), we have

$$
|F(x,s,t)|\le\varepsilon\big(|s|^{p^+}+|t|^{q^+}\big)+c(\varepsilon)(|s|^{p_1(x)}+|t|^{q_1(x)}+|s|^{\alpha(x)}|t|^{\beta(x)})
$$

for all $(x,s,t)\in\Omega\times\mathbb{R}^2$. In view of (H4) and and the above inequality, for $\|(u,v)\|$ sufficiently small, noting Proposition 2.2, we have

$$
\begin{aligned}
J(u,v)&\ge\frac{m_0}{p^+}\int_\Omega|\nabla u|^{p(x)}\,dx+\frac{m_0}{q^+}\int_\Omega|\nabla v|^{q(x)}\,dx-\varepsilon\int_\Omega|u|^{p^+}\,dx-\varepsilon\int_\Omega|v|^{q^+}\,dx\\
&\quad -c(\varepsilon)\int_\Omega\Big(|u|^{p_1(x)}+|v|^{q_1(x)}+|u|^{\alpha(x)}|v|^{\beta(x)}\Big)\,dx\\
&\ge\frac{m_0}{p^+}\|u\|_p^{p^+}-\varepsilon c_8^{p^+}\|u\|_p^{p^+}+\frac{m_0}{q^+}\|v\|_q^{q^+}-\varepsilon c_9^{q^+}\|v\|_q^{q^+}\\
&\quad -c(\varepsilon)\Big(\|u\|_p^{p_1^-}+\|v\|_q^{q_1^-}+c_7\|u\|_p^{2\alpha^-}+c_7\|v\|_q^{2\beta^-}\Big)\\
&\ge\frac{m_0}{2p^+}\|u\|_p^{p^+}+\frac{m_0}{2q^+}\|v\|_q^{q^+}-c(\varepsilon)\Big(\|u\|_p^{p_1^-}+\|v\|_q^{q_1^-}+c_7\|u\|_p^{2\alpha^-}+c_7\|v\|_q^{2\beta^-}\Big).
\end{aligned}
$$

Since $p_1^-, 2\alpha^- > p^+$ and $q_1^-, 2\beta^- > q^+$, there exist $r > 0$, $\delta > 0$ such that $J(u) \geq \delta > 0$ for every $\|(u,v)\| = r$.

On the other hand, we have known that the assumption (H2) implies the following assertion: for every $x \in \overline{\Omega}$, $s, t \in \mathbb{R}$, the inequality

$$F(x,s,t) \geq c_{10}(|s|^{\theta_1} + |t|^{\theta_2} - 1) \tag{3.3}$$

holds; see [28]. When $t > t_0$, from (H5) we can easily obtain that

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{1/(1-\mu)}} t^{1/(1-\mu)} := c_{11} t^{1/(1-\mu)},$$

where t_0 is an arbitrarily positive constant. For $(\widetilde{u}, \widetilde{v}) \in W \setminus \{(0,0)\}$ and $t > 1$, we have

$$\begin{aligned}
J(t\widetilde{u}, t\widetilde{v}) &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |t\nabla \widetilde{u}|^{p(x)}\,dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)} |t\nabla \widetilde{v}|^{q(x)}\,dx\Big) \\
&\quad - \int_\Omega F(x, t\widetilde{u}, t\widetilde{v})\,dx \\
&\leq c_{12}\big(\int_\Omega |t\nabla \widetilde{u}|^{p(x)}\,dx\big)^{1/(1-\mu)} - c_{10} t^{\theta_1} \int_\Omega |\widetilde{u}|^{\theta_1}\,dx \\
&\quad + c_{13}\Big(\int_\Omega |t\nabla \widetilde{v}|^{q(x)}\,dx\Big)^{1/(1-\mu)} - c_{10} t^{\theta_2} \int_\Omega |\widetilde{v}|^{\theta_2}\,dx - c_{14} \\
&\leq c_{12} t^{\frac{p^+}{1-\mu}} \Big(\int_\Omega |\nabla \widetilde{u}|^{p(x)}\,dx\Big)^{1/(1-\mu)} - c_{10} t^{\theta_1} \int_\Omega |\widetilde{u}|^{\theta_1}\,dx \\
&\quad + c_{13} t^{\frac{q^+}{1-\mu}} \Big(\int_\Omega |\nabla \widetilde{v}|^{q(x)}\,dx\Big)^{1/(1-\mu)} - c_{10} t^{\theta_2} \int_\Omega |\widetilde{v}|^{\theta_2}\,dx - c_{14} \\
&\to -\infty, \quad \text{as } t \to +\infty,
\end{aligned}$$

due to $\theta_1 > \frac{p^+}{1-\mu}$ and $\theta_2 > \frac{q^+}{1-\mu}$. Since $J(0,0) = 0$, considering Lemmas 3.2 and 3.3, we see that J satisfies the conditions of Mountain Pass Theorem. So J admits at least one nontrivial critical point. □

Next we will prove under some symmetry condition on the function F that (1.1) possesses infinitely many nontrivial weak solutions.

Theorem 3.5. *Assume* (H1), (H2), (H4), (H5), *and that $F(x,u,v)$ is even in u, v. Then* (1.1) *has a sequence of solutions $\{(\pm u_k, \pm v_k)\}_{k=1}^\infty$ such that $J(\pm u_k, \pm v_k) \to +\infty$ as $k \to +\infty$.*

Because $W_0^{1,p(x)}$ and $W_0^{1,q(x)}$ are a reflexive and separable Banach space, then W and W^* are too. There exist $\{e_j\} \subset W$ and $\{e_j^*\} \subset W^*$ such that

$$W = \overline{\operatorname{span}\{e_j : j = 1,2,\dots\}}, \quad W^* = \overline{\operatorname{span}\{e_j^* : j = 1,2,\dots\}},$$

and

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between W and W^*. For convenience, we write $X_j = \operatorname{span}\{e_j\}$, $Y_k = \oplus_{j=1}^k X_j$, $Z_k = \overline{\oplus_{j=k}^\infty X_j}$. We will use the following "Fountain theorem" to prove Theorem 3.5.

Lemma 3.6 ([36]). *Assume*

(A1) *X is a Banach space, $I \in C^1(X, \mathbb{R})$ is an even functional.*
(A2) *For each $k = 1, 2, \dots$, there exist $\rho_k > r_k > 0$ such that*
(A2) $\inf_{u \in Z_k, \|u\| = r_k} I(u) \to +\infty$ *as $k \to +\infty$.*
(A3) $\max_{u \in Y_k, \|u\| = \rho_k} I(u) \le 0$.
(A4) *I satisfies Palais-Smale condition for every $c > 0$.*

Then I has a sequence of critical values tending to $+\infty$.

For every $a > 1$, $u, v \in L^a(\Omega)$, we define

$$|(u, v)|_a := \max\{|u|_a, |v|_a\}.$$

Set

$$\begin{aligned} a &:= \max_{x \in \overline{\Omega}} \{2\alpha(x), 2\beta(x), p_1(x), q_1(x)\} > \min\{p^-, q^-\}, \\ b &:= \min_{x \in \overline{\Omega}} \{2\alpha(x), 2\beta(x), p_1(x), q_1(x)\} > 0. \end{aligned}$$

Then we have the following result.

Lemma 3.7 ([28]). *Denote*

$$\beta_k = \sup\{|(u, v)|_a : \|(u, v)\| = 1, (u, v) \in Z_k\}.$$

Then $\lim_{k \to +\infty} \beta_k = 0$.

Proof of Theorem 3.5. According to the assumptions on F, Lemmas 3.2 and 3.3, J is an even functional and satisfies Palais-Smale condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that (A_2) and (A_3) holding. Thus, the conclusion can be obtained from Fountain theorem.

(A2): For any $(u_k, v_k) \in Z_k$, $\|u_k\|_p \ge 1$, $\|v_k\|_q \ge 1$ and $\|(u_k, v_k)\| = r_k$ (r_k will be specified below), we have

$$\begin{aligned} &J(u_k, v_k) \\ &= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)} |\nabla u_k|^{p(x)}\,dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)} |\nabla v_k|^{q(x)}\,dx\Big) - \int_\Omega F(x, u_k, v_k)\,dx \\ &\ge m_0 \int_\Omega \frac{1}{p(x)} |\nabla u_k|^{p(x)}\,dx + m_0 \int_\Omega \frac{1}{q(x)} |\nabla v_k|^{q(x)}\,dx - \int_\Omega F(x, u_k, v_k)\,dx \\ &\ge \frac{m_0}{p^+} \int_\Omega |\nabla u_k|^{p(x)}\,dx + \frac{m_0}{q^+} \int_\Omega |\nabla v_k|^{q(x)}\,dx \\ &\quad - c \int_\Omega \Big(1 + |u_k|^{p_1(x)} + |v_k|^{q_1(x)} + |u_k|^{\alpha(x)} |v_k|^{\beta(x)}\Big)\,dx \\ &\ge \frac{m_0}{p^+} \|u_k\|_p^{p^-} + \frac{m_0}{q^+} \|v_k\|_q^{q^-} - c|u_k|_{p_1(x)}^{p_1(\xi_1^k)} - c|v_k|_{q_1(x)}^{q_1(\xi_2^k)} \\ &\quad - c_{15} |u_k|_{2\alpha(x)}^{2\alpha(\eta_1^k)} - c_{15} |v_k|_{2\beta(x)}^{2\beta(\eta_2^k)} - c|\Omega|, \end{aligned}$$

where $\xi_1^k, \xi_2^k, \eta_1^k, \eta_2^k \in \Omega$. Therefore,

$$\begin{aligned} &J(u_k, v_k) \\ &\ge \frac{m_0}{\max\{p^+, q^+\}} \|(u_k, v_k)\|^{\min\{p^-, q^-\}} - c|u_k|_a^{p_1(\xi_1^k)} - c|v_k|_a^{q_1(\xi_2^k)} \\ &\quad - c|u_k|_a^{2\alpha(\eta_1^k)} - c|v_k|_a^{2\beta(\eta_2^k)} - c|\Omega| \\ &\ge \frac{m_0}{\max\{p^+, q^+\}} \|(u_k, v_k)\|^{\min\{p^-, q^-\}} - c(\beta_k \|(u_k, v_k)\|)^{p_1(\xi_1^k)} - c(\beta_k \|(u_k, v_k)\|)^{q_1(\xi_2^k)} \end{aligned}$$

$$
\begin{aligned}
&- c(\beta_k\|(u_k,v_k)\|)^{2\alpha(\eta_1^k)} - c(\beta_k\|(u_k,v_k)\|)^{2\beta(\eta_2^k)} - c|\Omega| \\
&\geq \frac{m_0}{\max\{p^+,q^+\}}\|(u_k,v_k)\|^{\min\{p^-,q^-\}} - c_{16}\beta_k^b\|(u_k,v_k)\|^a - c|\Omega|,
\end{aligned}
$$

where a, b are defined above. At this stage, we fix r_k as follows:

$$
r_k := \Big(\frac{m_0}{2c_{16}\max\{p^+,q^+\}\beta_k^b}\Big)^{1/(a-\min\{p^-,q^-\})} \to +\infty \quad \text{as } k \to +\infty.
$$

Consequently, if $\|(u_k,v_k)\| = r_k$ then

$$
J(u_k,v_k) \geq \frac{m_0}{2\max\{p^+,q^+\}}\|(u_k,v_k)\|^{\min\{p^-,q^-\}} - c|\Omega| \to +\infty \quad \text{as } k \to +\infty.
$$

(A3): From (H2), we have $F(x,u,v) \geq c_{10}(|u|^{\theta_1} + |v|^{\theta_2} - 1)$ for every $x \in \Omega$ and $u, v \in \mathbb{R}$. Therefore, for any $(u,v) \in Y_k$ with $\|(u,v)\| = 1$ and $1 < \rho_k = t_k$ with $t_k \to +\infty$, we have

$$
\begin{aligned}
&J(t_ku,t_kv) \\
&= \widehat{M}\Big(\int_\Omega \frac{1}{p(x)}|t_k\nabla u|^{p(x)}\,dx\Big) + \widehat{M}\Big(\int_\Omega \frac{1}{q(x)}|t_k\nabla v|^{q(x)}\,dx\Big) - \int_\Omega F(x,t_ku,t_kv)\,dx. \\
&\leq c_{17}\Big(\int_\Omega |t_k\nabla u|^{p(x)}\,dx\Big)^{1/(1-\mu)} + c_{18}\Big(\int_\Omega |t_k\nabla v|^{q(x)}\,dx\Big)^{1/(1-\mu)} \\
&\quad - c_{10}t_k^{\theta_1}\int_\Omega |u|^{\theta_1}\,dx - c_{10}t_k^{\theta_2}\int_\Omega |v|^{\theta_2}\,dx + c_{19}, \\
&\leq c_{17}t_k^{\frac{p^+}{1-\mu}}\Big(\int_\Omega |\nabla u|^{p(x)}\,dx\Big)^{1/(1-\mu)} - c_{10}t_k^{\theta_1}\int_\Omega |u|^{\theta_1}\,dx \\
&\quad + c_{18}t_k^{\frac{q^+}{1-\mu}}\Big(\int_\Omega |\nabla v|^{q(x)}\,dx\Big)^{1/(1-\mu)} - c_{10}t_k^{\theta_2}\int_\Omega |v|^{\theta_2}\,dx + c_{19}.
\end{aligned}
$$

By $\theta_1 > \frac{p^+}{1-\mu}$, $\theta_2 > \frac{q^+}{1-\mu}$ and $\dim Y_k = k$, it is easy to see that $J(t_ku,t_kv) \to -\infty$ as $\|(t_ku,t_kv)\| \to +\infty$ for $(u,v) \in Y_k$.

The proof of Theorem 3.5 is completed by the Fountain theorem. $\square$

Acknowledgments. The author wishes to express his gratitude to the anonymous referee for reading the original manuscript carefully and making several corrections and remarks.

References

[1] E. Acerbi and G. Mingione; *Regularity results for stationary electro-rheologicaluids*, Arch. Ration. Mech. Anal., 164 (2002), 213–259.

[2] E. Acerbi, G. Mingione; *Gradient estimate for the $p(x)$-Laplacean system*, J. Reine Angew. Math., 584 (2005), 117–148.

[3] A. Arosio, S. Panizzi; *On the well-posedness of the Kirchhoff string*, Trans. Amer. Math. Soc. 348 (1996) 305–330.

[4] C. O. Alves, F. J. S. A. Corrêa, T. F. Ma; *Positive solutions for a quasilinear elliptic equation of Kirchhoff type*, Comput. Math. Appl. 49 (2005) 85–93.

[5] S. N. Antontsev, S. I. Shmarev; *A Model Porous Medium Equation with Variable Exponent of Nonlinearity: Existence, Uniqueness and Localization Properties of Solutions*, Nonlinear Anal. 60 (2005), 515–545.

[6] S. N. Antontsev , J. F. Rodrigues; *On Stationary Thermo-rheological Viscous Flows*, Ann. Univ. Ferrara, Sez. 7, Sci. Mat. 52 (2006), 19–36.

[7] S. Bernstein; *Sur une classe d'équations fonctionnelles aux dérivées partielles*, Bull. Acad. Sci. URSS. Ser. Math. [Izv. Akad. Nauk SSSR] 4 (1940) 17–26.

[8] M. M. Cavalcanti, V. N. Cavalcanti, J.A. Soriano; *Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation*, Adv. Differential Equations 6 (2001) 701–730.
[9] Y. Chen, S. Levine, M. Rao; *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl.Math. 66 (4) (2006), 1383–1406.
[10] F. J. S. A. Corrêa, G. M. Figueiredo; *On a elliptic equation of p-kirchhoff type via variational methods*, Bull. Austral. Math. Soc. Vol. 74 (2006) 263–277.
[11] F. J. S. A. Corrêa, R.G. Nascimento; *On a nonlocal elliptic system of p-Kirchhoff-type under Neumann boundary condition*, Mathematical and Computer Modelling 49 (2009) 598–604.
[12] G. Dai, R. Hao; *Existence of solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. 359 (2009) 275–284.
[13] L. Diening, P. Hästö, A. Nekvinda; *Open problems in variable exponent Lebesgue and Sobolev spaces*, in: P. Drábek, J. Rákosník, FSDONA04 Proceedings, Milovy, Czech Republic, 2004, pp.38–58.
[14] G. Dai, D. Liu; *Infinitely many positive solutions for a $p(x)$-Kirchhoff-type equation*, J. Math. Anal. Appl. 359 (2009) 704–710.
[15] G. Dai, J. Wei; *Infinitely many non-negative solutions for a $p(x)$-Kirchhoff-type problem with Dirichlet boundary condition*, Nonlinear Analysis 73 (2010) 3420–3430.
[16] P. D'Ancona, S. Spagnolo; *Global solvability for the degenerate Kirchhoff equation with real analytic data*, Invent. Math. 108 (1992) 247–262.
[17] D. G. De Figueiredo; *Semilinear elliptic systems: a survey of superlinear problems*, Resenhas 2 (1996) 373–391.
[18] M. Dreher; *The Kirchhoff equation for the p-Laplacian*, Rend. Semin. Mat. Univ. Politec. Torino 64 (2006) 217–238.
[19] M. Dreher; *The wave equation for the p-Laplacian*, Hokkaido Math. J. 36 (2007) 21–52.
[20] X. L. Fan; *On the sub-supersolution methods for $p(x)$-Laplacian equations*, J. Math. Anal. Appl. 330 (2007), 665–682.
[21] X. L. Fan and X. Y. Han; *Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathbb{R}^N$*, Nonlinear Anal. 59 (2004), 173–188.
[22] X. L. Fan, J. S. Shen, D. Zhao; *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl. 262 (2001), 749–760.
[23] X. L. Fan, Q. H. Zhang; *Existence of solutions for $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. 52 (2003), 1843–1852.
[24] X. L. Fan, Q. H. Zhang and D. Zhao; *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*, J. Math. Anal. Appl. 302 (2005), 306–317.
[25] X.L. Fan, D. Zhao; *On the Spaces $L^{p(x)}$ and $W^{m,p(x)}$*, J. Math. Anal. Appl. 263 (2001), 424–446.
[26] X. L. Fan, Y. Z. Zhao, Q. H. Zhang; *A strong maximum principle for $p(x)$-Laplace equations*, Chinese J. Contemp. Math. 24 (3) (2003) 277–282.
[27] X. L. Fan; *On nonlocal $p(x)$-Laplacian Dirichlet problems*, Nonlinear Anal. 72 (2010) 3314–3323.
[28] A. El Hamidi; *Existence results to elliptic systems with nonstandard growth conditions*, J. Math. Anal. Appl. 300 (2004) 30–42.
[29] P. Harjulehto, P. Hästö; *An overview of variable exponent Lebesgue and Sobolev spaces*, in Future Trends in Geometric Function Theory (D. Herron (ed.), RNC Work-shop), Jyväskylä, 2003, 85–93.
[30] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, 1883.
[31] J. L. Lions; *On some equations in boundary value problems of mathematical physics*, in: Contemporary Developments in Continuum Mechanics and Partial Differential Equations (Proc. Internat. Sympos., Inst. Mat. Univ. Fed. Rio de Janeiro, Rio de Janeiro, 1977), North-Holland Math. Stud., vol. 30, North-Holland, Amsterdam, 1978, pp. 284–346.
[32] T. F. Ma, J. E. Muñoz Rivera; *Positive solutions for a nonlinear nonlocal elliptic transmission problem*, Appl. Math. Lett. 16 (2003) 243–248.
[33] S. I. Pohožaev; *A certain class of quasilinear hyperbolic equations*, Mat. Sb. (N.S.) 96 (1975) 152–166, 168.
[34] M. Růžička; *Electrorheological Fluids: Modeling and Mathematical Theory*, Springer-Verlag, Berlin, 2000.

[35] S. Samko; *On a progress in the theory of Lebesgue spaces with variable exponent Maximal and singular operators*, Integral Transforms Spec. Funct. 16 (2005), 461–482.

[36] M. Willem; *Minimax Theorems*, Birkhäuser, Boston, 1996.

[37] V. V. Zhikov; *Averaging of functionals of the calculus of variations and elasticity theory*, Math. USSR. Izv. 9 (1987), 33–66.

[38] V. V. Zhikov, S. M. Kozlov, O. A. Oleinik; *Homogenization of Differential Operators and Integral Functionals*, Translated from the Russian by G.A. Yosifian, Springer-Verlag, Berlin, 1994.

[39] V. V. Zhikov; *On some variational problems*, Russian J. Math. Phys. 5 (1997), 105–116.

Guowei Dai

Department of Mathematics, Northwest Normal University, Lanzhou, 730070, China

E-mail address: daiguowei@nwnu.edu.cn