Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 146, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OUTPUT-FEEDBACK STABILIZATION AND CONTROL OPTIMIZATION FOR PARABOLIC EQUATIONS WITH NEUMANN BOUNDARY CONTROL

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Abstract. Both of feedback stabilization and optimal control problems are analyzed for a parabolic partial differential equation with Neumann boundary control. This PDE serves as a model of heat exchangers in a conducting rod. First, we explicitly construct an output-feedback operator which exponentially stabilizes the abstract control system representing the model. Second, we derive a controller which, simultaneously, stabilizes the associated output an minimizes a suitable cost functional.

1. Introduction

In this article, we study the parabolic equation

$$\begin{gathered} z_t(t,x) = [\varepsilon(x) z_x(t,x)]_x + b(x) z_x(t,x) + a(x) z(t,x), \quad \text{in } (0,\infty)\times(0,1), \\ z_x(t,0) = \rho z(t,0), \quad z_x(t,1) = u(t), \quad \text{in } (0,\infty), \\ z(0,x) = z^0(x), \quad \text{in } (0,1), \end{gathered} \tag{1.1}$$

with a control $u(t)$ placed at the extremity $x = 1$, via Neumann boundary condition, where the parameters ε, a, b, ρ, satisfy the assumptions

$$-\infty < \rho \leq +\infty, \quad a \in C^1[0,1], \quad b, \varepsilon \in C^2[0,1], \quad \inf_{x\in[0,1]} \varepsilon(x) > 0. \tag{1.2}$$

Equation (1.1) can be interpreted, in thermodynamic point of view, as a model of heat conducting rod in which not only the heat is being diffused and bifurcated ($(\varepsilon z_x)_x + b z_x$) but also a destabilizing heat is generating (az). System (1.1) also represents very well a linearized model of chemical tubular reactor [3] and it can further approximates a linearized model of unstable burning in solid propellant rockets [4].

The stabilization problem of parabolic systems is treated by several authors with different approaches. Stability by boundary control in the optimal control setting is discussed by Bensoussan et al. [2]. In [11, 15], the open-loop system is separated into an infinite-dimensional stable part and a finite-dimensional unstable part. A boundary control stabilizing the unstable part and leaving the stable part stable is derived. In [16, 17], the stabilizability problem for parabolic systems is

2000 *Mathematics Subject Classification.* 34K35.
Key words and phrases. C_0-semigroup; feedback theory for regular linear systems.

Submitted September 7, 2010. Published November 2, 2011.

approached using the feedback theory for (autonomous) regular linear systems. In time depend setting, the stabilizability and the controllability for non-autonomous parabolic systems are discussed in [14] by developing the so called non-autonomous regular linear systems. The finite-dimensional backstepping is applied in [1] to the discritized version of (1.1), and shown to be convergent in L^∞. The backstepping method with continuous kernel is investigated in [7, 10, 12] to construct boundary feedback laws making the closed-loop systems exponentially stable. The backstepping idea is to convert the parabolic system into a well known one using an integral transformation with a kernel satisfying an adequate PDE.

In this paper, we combine the feedback theory for regular linear system [16] and the backstepping method to design an output-feedback which exponentially stabilizes the abstract control system representing system (1.1). To be more precise, system (1.1) is written in a suitable state space as an abstract control system; $z_t(t) = Az(t) + Bu(t),\ t > 0,\ z(0) = 0$, where A represents the evolution of the open-loop system and B is an appropriate control operator. For any $\lambda > 0$, we explicitly construct an admissible observation operator C^λ which exponentially stabilizes (A, B) at the desired rate of λ. The stabilizing observation operator is given in term of the solution of an adequate kernel PDE which depends on λ. On the other hand, we erect a controller which solves, simultaneously, both the stabilization and the control optimization problems associated with (1.1). In particular, we design a controller which not only stabilizes the output of the concerned control system but also minimizes an adapted cost functional.

The paper is organized as follows: In Section 2, we present the stabilizability concept associated with regular linear systems. The abstract control system representing (1.1) is derived in Section 3. In Section 4, an explicit construction of the observation operator stabilizing (1.1) is given. Section 5 is devoted to study the λ-exponential stability of the closed-loop system. Finally, the optimal control problem of system(1.1) is treated in Section 6.

2. Preliminaries

Throughout this paper, U, X, Y, are Hilbert spaces. $A : D(A) \subset X \to X$ is the generator of a C_0-semigroup T. We denote by X_1 the Hilbert space $D(A)$ endowed with the graph norm; $\|x\|_1 = \|x\| + \|Ax\|$.

We further set $R(\lambda, A) = (\lambda - A)^{-1}$ for λ in the resolvent set $\varrho(A)$. The Hilbert space X_{-1} is the completion of X with respect to the norm $\|x\|_{-1} := \|R(\lambda, A)x\|$ for some $\lambda \in \varrho(A)$. Then, T is extended to a C_0-semigroup T_{-1} on X_{-1}. The generator of T_{-1} is denoted by A_{-1} which is an extension of A to X. For more detail on extrapolation theory we refer to [8].

Let $B \in \mathcal{L}(U, X_{-1})$, $C \in \mathcal{L}(X_1, Y)$ and on X_{-1} consider the abstract linear system

$$z_t(t) = A_{-1}z(t) + Bu(t), \quad z(0) = z^0, \tag{2.1}$$

$$y(t) = Cz(t), \quad t > 0, \tag{2.2}$$

where $u \in L^2_{loc}([0, \infty), U)$.

The well-posedness of system (2.1)–(2.2) requires a certain regularity of the triplet (A, B, C), due to [16, 17]. Moreover, if one relates the output y to the input u by an adequate (feedback) operator K; $u = Ky$, $K \in \mathcal{L}(Y, U)$, we obtain a new system called the closed-loop system. From [16], the well-posedness of the closed-loop system requires that the feedback operator should be admissible for the transfer function $H(\cdot) := CR(\cdot, A_{-1})B$; i.e., the operator $I_Y - H(\cdot)K$ is uniformly invertible in some half plan $\mathbb{C}_s := \{\lambda \in \mathbb{C} : \Re\lambda > s\}$. If it is the case, then due to Weiss [16], the operator representing the closed-loop system.

$$A^I := A_{-1} + BC_L \quad \text{with} \quad D(A^I) := \{x \in X : (A_{-1} + BC_L)x \in X\}, \tag{2.3}$$

generates a C_0 semigroup T^I.

In practice, many control systems are unstable. However, if one feeds back the output of an unstable system to the input by an appropriate feedback law $u = Ky$, it is possible to obtain a stable closed-loop system. This is called the *feedback stabilizability* of the open-loop system. An extensive survey on the stabilizability concept of linear systems can be found in [13]. Here, we are concerned with the concept of exponential stabilizability as presented in [17].

Definition 2.1 ([17])**.** Consider an abstract control system with open-loop generator A and control operator $B \in \mathcal{L}(U, X_{-1})$. We say that $C \in \mathcal{L}(X, U)$ stabilizes (A, B) if

(a) (A, B, C) is a regular triple,
(b) I_U is an admissible feedback operator for $H(\cdot) = CR(\cdot, A_{-1})B$,
(c) the operator A^I, defined in (2.3), generates an exponentially stable semigroup.

3. The abstract control system associated with (1.1)

Without loss of generality we set in what follows $b \equiv 0$ since it can be eliminated from equation (1.1) using the transformation

$$\tilde{z}(t, x) := \exp\Big(\int_0^x \frac{b(s)}{2\varepsilon(s)} ds\Big) z(t, x) \tag{3.1}$$

with the compatible changes of parameters

$$\begin{gathered} \tilde{\varepsilon}(x) := \varepsilon(x), \quad \tilde{a}(x) := a(x) - \frac{b'(x)}{2} - \frac{b^2(x)}{4\varepsilon(x)}, \\ \widetilde{\rho} := \rho + \frac{b(0)}{2\varepsilon(0)}, \quad \tilde{u}(t) := \exp\Big(\int_0^1 \frac{b(s)}{2\varepsilon(s)} ds\Big) u(t), \end{gathered} \tag{3.2}$$

In fact, one can easily see that

$$\widetilde{z}_t - (\widetilde{\varepsilon}\widetilde{z}_x)_x - \widetilde{a}\widetilde{z} = \{z_t - (\varepsilon z_x)_x - bz_x - az\} \exp\Big(\int_0^x \frac{b(s)}{2\varepsilon(s)} ds\Big).$$

Then, z satisfies (1.1) if and only if $\widetilde{z}$ satisfies (1.1) with the parameters $\widetilde{\varepsilon}, 0, \widetilde{a}, \widetilde{\rho}, \widetilde{u}$, instead of $\varepsilon, b, a, \rho, u$. Moreover, provided that $b \in C^2$, the parameters $\widetilde{\varepsilon}, 0, \widetilde{a}, \widetilde{\rho}$, satisfy (1.2).

To present system (1.1) as an abstract control system, we define on the state space $X = L^2(0, 1)$ the operators

$$\begin{gathered} Af := (\varepsilon f_x)_x + af, \quad D(A) := \{f \in H^2(0, 1) : f_x(0) = \rho f(0),\ f_x(1) = 0\}, \\ Bu := -uA_{-1}\psi, \quad B \in \mathcal{L}(\mathbb{C}, X_{-1}). \end{gathered} \tag{3.3}$$

where ψ is the unique H^2-solution of the ordinary differential equation

$$\begin{gathered} (\varepsilon\psi_x)_x + a\psi = 0, \quad 0 \le x \le 1, \\ \psi_x(0) = \rho\psi(0), \quad \psi_x(1) = 1. \end{gathered} \tag{3.4}$$

The smoothness of the solution of (3.4) is shown as in [5, VIII.4]. We first confirm the well-posedness of the evolution equation corresponding to A and the admissibility of the control operator B (for A).

Lemma 3.1. (i) *A generates an analytic semigroup T on X;*
(ii) *B is an admissible control operator for T.*
Further, there exist constants $\theta, \alpha_0 > 0$ such that

$$\|R(s, A_{-1})B\|_{\mathcal{L}(\mathbb{C},X)} \leq \frac{\theta}{\sqrt{\Re s}} \tag{3.5}$$

for $\Re s > \alpha_0$.

Proof. (i) Observe that A is self-adjoint. Then A generates an analytic semigroup T on X; see e.g. [8]. (ii) Since T is analytic on the Hilbert space X, then due to De Simon [6],

$$\int_0^{t_0} u(t_0 - \sigma)T(\sigma)f d\sigma \in D(A),$$

for a.e. $t_0 > 0$, all $f \in X$, and $u \in L^2([0, t_0], \mathbb{C})$. Hence,

$$\Phi(t_0)u := \int_0^{t_0} T_{-1}(t_0 - \sigma)Bu(\sigma)d\sigma = -A\int_0^{t_0} u(t_0 - \sigma)T(\sigma)\psi d\sigma \in X$$

for some $t_0 > 0$. Therefore, B is an admissible control operator for T. Finally, the estimate (3.5) is a consequence of the admissibility of B for an analytic semigroup, see [18]. □

4. The observation operator

The idea of constructing the observation operator is to convert (1.1) into a well known equation by using the following transformation.

Lemma 4.1 ([12])**.** *Let $k \in H^2(\Delta)$, $\Delta := \{(x, y) : 0 \leq y \leq x \leq 1\}$, and define the linear bounded operator $\mathcal{T}_k : H^i(0, 1) \to H^i(0, 1)$, by*

$$(\mathcal{T}_k v)(x) := v(x) + \int_0^x k(x, y)v(y)dy.$$

Then, $\mathcal{T}_k$ has a linear bounded inverse $\mathcal{T}_k^{-1} : H^i(0, 1) \to H^i(0, 1)$, $i = 0, 1, 2$.

Next, assume that $z(t)$ satisfies (1.1) and for $t \geq 0$, $x \in [0, 1]$, set

$$w(t, x) := (\mathcal{T}_k z(t))(x) = z(t, x) + \int_0^x k(x, y)z(t, y)dy.$$

Then,

$$\begin{aligned} w_t(t, x) &= z_t(t, x) + \int_0^x k(x, y)z_t(t, y)dy \\ &= z_t(t, x) + \int_0^x k(x, y)\big[[\varepsilon(y)z_y(t, y)]_y + a(y)z(t, y)\big]dy \end{aligned}$$

By integrating by parts from 0 to x, for $t > 0$ and $\lambda > 0$, we obtain

$$\begin{aligned} & w_t - [\varepsilon w_x]_x + \lambda w \\ & = \big[(\lambda + a(x)) - 2\varepsilon(x)\frac{d}{dx}(k(x,x)) - \varepsilon'(x)k(x,x)\big] z(t,x) \\ & \quad + \int_0^x \big[(\lambda + a(y))k(x,y) + \big([\varepsilon(y)k_y(x,y)]_y - [\varepsilon(x)k_x(x,y)]_x\big)\big] z(t,y)dy \\ & \quad + [k_y(x,0) - \rho k(x,0)]\varepsilon(0)z(t,0). \end{aligned} \tag{4.1}$$

Then $w_t - [\varepsilon(x)w_x]_x + \lambda w = 0$, in $(0,\infty) \times (0,1)$, if and only if the kernel k satisfies the PDE

$$\begin{gathered} {}_x - [\varepsilon(y)k_y(x,y)]_y = a_\lambda(y)k(x,y), \quad 0 \le y \le x \le 1, \\ k_y(x,0) = \rho k(x,0), \quad 0 \le x \le 1, \\ k(x,x) = \frac{1}{2\sqrt{\varepsilon(x)}} \int_0^x \frac{a_\lambda(s)}{\sqrt{\varepsilon(s)}} ds =: g(x), \quad 0 \le x \le 1, \end{gathered} \tag{4.2}$$

where $a_\lambda(x) := a(x) + \lambda$. We note that the third (boundary) equation of (4.2) is obtained by solving the first order differential equation

$$2\varepsilon(x)\frac{d}{dx}(k(x,x)) + \varepsilon'(x)k(x,x) = a_\lambda(x)$$

with the initial condition $k(0,0) = 0$. The following well-posedness result of the kernel PDE (4.2) is proved in [7] which generalizes the one obtained in [10] for ε constant.

Lemma 4.2. *Assume that* (1.2) *holds. Then the kernel equation* (4.2) *has a unique solution* $k \in H^2(\Delta)$.

Now, let k^λ be the solution of the PDE (4.2) associated with some $\lambda > 0$. From (4.1), we obtain

$$w_t = [\varepsilon(x)w_x]_x - \lambda w \quad \text{in } (0,\infty) \times (0,1).$$

Moreover, it follows from the boundary conditions of (1.1) that

$$w_x(t,0) = \rho w(t,0), \quad w_x(t,1) = u(t) + k_0(1)z(t,1) + \langle k_1^\lambda, z(t)\rangle,$$

where $\langle \cdot,\cdot \rangle$ denotes the inner product on X and $k_0^\lambda(y) = k^\lambda(1,y)$, $k_1^\lambda(y) = k_x^\lambda(1,y)$. Thus, $w_x(t,1) = 0$ if and only if u satisfies the control law

$$u(t) = -k_0^\lambda(1)z(t,1) - \langle k_1^\lambda, z(t)\rangle. \tag{4.3}$$

This means that $\mathcal{T}_k$ converts the closed-loop system (1.1),(4.3), into

$$\begin{gathered} w_t(t,x) = [\varepsilon(x)w_x(t,x)]_x - \lambda w(t,x), \quad \text{in } (0,\infty) \times (0,1), \\ w_x(t,0) = \rho w_x(t,0), \quad w(t,1) = 0, \quad \text{in } (0,\infty), \\ w(0,x) = w^0(x), \quad \text{in } (0,1), \end{gathered} \tag{4.4}$$

where $w^0(x) := z^0(x) + \int_0^x k(x,y)z^0(y)dy$.

The following theorem states the well-posedness of the closed-loop system (1.1), (4.3) and also gives an estimation of the solution.

Theorem 4.3. *For any* $z^0 \in L^2(0,1)$*, the closed-loop system* (1.1),(4.3) *has a unique solution* $z(t,x) \in C^{1,2} := C^1\big((0,\infty) \times C^2[0,1]\big)$ *such that*

$$\|z(t)\| \le Me^{-\lambda t}\|z^0\|, \tag{4.5}$$

where M *is a positive constant independent of* z^0.

Proof. It remains to show that the equivalent system (4.4) has a unique solution w satisfying

$$\|w(t)\| \le e^{-\lambda t}\|w^0\|. \tag{4.6}$$

In fact, consider on the state space X the operator

$$D(G) := \{f \in H^2(0,1): \ f_x(0) = \rho f(0), \ f_x(1) = 0\},$$
$$Gf := (\varepsilon f_x)_x - \lambda f, \quad \text{for } f \in D(G).$$

Observe that G is self adjoint. Moreover, by integrating by parts over $[0,1]$, we get

$$\langle Gf, f\rangle \ \le -\lambda\|f\|^2,$$

for every $f \in D(G)$. Then, see e.g. [2, p. 55], G generates a bounded analytic semigroup S such that

$$\|S(t)\| \le e^{-\lambda t}, \quad t \ge 0. \tag{4.7}$$

This means that for any $w^0 \in X$ system (4.4) has a unique solution $w = S(\cdot)w^0 \in C([0,\infty), X)$. Since S is analytic, $S(\cdot)w^0 \in C^1((0,\infty), D(G^\infty))$ for all $t > 0$, where $D(G^\infty) := \cap_{n=0}^{\infty} D(G^n)$; see e.g. [8, p. 93]. Now, the Sobolev embedding theorem leads us to conclude that $w \in C^{1,2}$. Moreover, (4.6) is an immediate consequence of (4.7).

System (1.1), (4.3) is well posed, since it can be transformed via the isomorphism $\mathcal{T}_k$ to the well posed system (4.4). Further, the fact that $\mathcal{T}_k^{-1}$ and $\mathcal{T}_k$ are bounded, then there exists a constant $\delta > 0$ such that

$$\|z(t)\| \le \delta\|w(t)\| \quad \text{and} \quad \|w^0\| \le \delta\|z^0\|, \tag{4.8}$$

for $t \ge 0$. Finally, (4.5), follows from (4.6) combined with (4.8). $\square$

Theorem 4.3 shows that the feedback law (4.3) forces the the open-loop system (1.1) to exhibit a behavior akin to $e^{-\lambda t}$ with L^2-norm (as $t \to \infty$). This leads us to choose as observation operator

$$C^\lambda f := -k_0^\lambda(1)f(1) - \langle k_1^\lambda, f\rangle, \quad C^\lambda \in \mathcal{L}(X, \mathbb{C}), \tag{4.9}$$

where k^λ is the solution of the kernel PDE (4.2) corresponding to some $\lambda > 0$. We will show in the following section that C^λ is an appropriate observation operator to create a stabilizing controller with respect to the open-loop system corresponding the aforesaid operators (A, B).

5. The closed-loop stability

We confirm in this section that C^λ is a suitable stabilizing output operator for the abstract control system represented by (A, B). The following theorem constitutes the first main result of this paper.

Theorem 5.1. *Consider (A,B) with representation (3.3) and define C^λ by (4.9). Then*

(i) *C^λ stabilizes (A, B),*
(ii) *the operator $A^I := A_{-1} + BC^\lambda$ with the domain $D(A^I) := \{f \in X : A_{-1}f + BC^\lambda f \in X\}$, generates a C_0-semigroup T^I such that*

$$\|T^I(t)z^0\| \le Me^{-\lambda t}\|z^0\|, \tag{5.1}$$

for $t \ge 0$ and any $z^0 \in X$, where M is a positive constant independent of z^0.

Proof. Since C^λ is a bounded perturbation of the Dirichlet trace, it follows that it is an admissible observation operator for the open-loop semigroup T and that its degree of unboundedness is 1/4, see e.g. [9]. Taking into account the analyticity of the open-loop semigroup T, the feedthrough operator is equal to zero and the control operator B also has the same degree of unboundedness 1/4. [9, Example 7.7.5] then shows that C^λ is an admissible state feedback operator. Thus due to [16], (A, B, C^λ) is a regular triple and the transfer function is given by

$$H(s) = C^\lambda R(s, A_{-1})B,$$

for a sufficiently large $\Re s$. On the other hand, due to Lemma 3.1, there exist α, $\theta > 0$ such that

$$\|H(s)\| = \|C^\lambda R(s, A_{-1})B\| \leq \frac{\theta \|C^\lambda\|_{\mathcal{L}(X,\mathbb{C})}}{\sqrt{\Re s}}, \quad \text{for } s \in \mathbb{C}_\alpha.$$

Which implies that there exists $s_0 > \alpha$ such that $|H(s)| < 1$ for $s \in \mathbb{C}_{s_0}$. Consequently, $I_{\mathbb{C}}$ is an admissible feedback for H. According to Section 2, A^I generates a C_0-semigroup T^I. Which means that $T^I(\cdot)z^0$ is the unique classical solution of the evolution equation

$$\begin{gathered} z_t(t) = A_{-1}z(t) + BC^\lambda z(t), t > 0, \\ z(0) = z^0; \end{gathered}$$

i.e., $T^I(\cdot)z^0$ is the unique solution of the closed-loop system

$$\begin{gathered} z_t(t) = A_{-1}z(t) + Bv(t), z(0) = z^0, \\ y(t) = C^\lambda z(t), \\ v(t) = y(t), \quad t > 0. \end{gathered} \tag{5.2}$$

On the other hand, in view of Theorem 4.3, for a given $z^0 \in X$ the system (1.1), (4.3) has a unique solution $z = z(t, x, z^0) \in C^{1,2}$. Observe that, $z(t) - u(t)\psi \in D(A)$, for $t > 0$, and

$$z_t(t) = A(z(t) - u(t)\psi) = A_{-1}z(t) + Bu(t).$$

Moreover, the control law (4.3) means that

$$u(t) = C^\lambda z(t) = y(t).$$

This shows that z is also a solution of (5.2). Thus, $z(\cdot, z^0) = T^I(\cdot)z^0$. Finally, the estimate (5.1) is an immediate consequence of (4.5). □

Alternatively, instead of invoking [9, Example 7.7.5], one can use in the above proof, that the impulse response is in $L^1(0; 1)$ (which follows from analyticity of the semigroup and the degrees of unboundedness) and then use the reasoning involving the concept of well-posedness radius from [16] to show that C^λ is an admissible state feedback operator.

The scheme of Figure 1 makes understood the meaning of the stability result stated in Theorem 5.1, and shows how the controller (4.3) affects in a closed form the open-loop system (1.1),

In view of the scheme of Figure 1, in order to stabilize (1.1) in a closed form, for a given rate λ, one computes, for example by a numerical calculator, the quantity $q := -k_0^\lambda(1)z(t, 1) - \langle k_1^\lambda, z(t)\rangle$, and one injects, intermediary a dispositive described by the control operator B , the sum q at the extremity $x = 1$. The state of the resulting closed-loop system exhibits a behavior akin to $e^{-\lambda t}$ as $t \to \infty$.

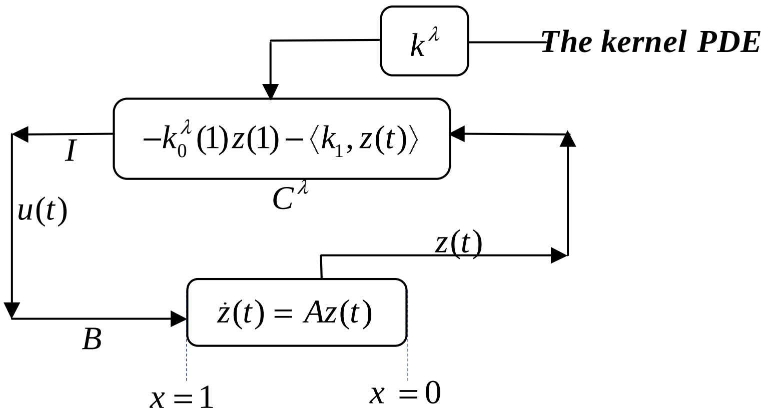

FIGURE 1. The closed-loop system of (1.1) associated with the control law (4.3)

Remark 5.2. Although of the results in the above sections are given for $b = 0$. However, if $b \neq 0$, one may consider, in view of (3.1)–(3.2), the observation operator

$$\widetilde{C}^\lambda f := \big(- \widetilde{k}_0^\lambda(1) f(1) - \langle \widetilde{k}_1^\lambda, f\rangle\big) e^{-\int_0^1 \frac{b(s)}{2\varepsilon(s)}}, \quad f \in X,$$

where $\widetilde{k}$ is the solution of the kernel PDE given for $\widetilde{\varepsilon}$, $\widetilde{a}$, $\widetilde{\rho}$ instead of ε, a, ρ.

6. Optimal control problem for (1.1)

In some applications, it is not benefic to stabilizes a system by a large cost. So, by stabilizing a system, a question should de asked. What is the cost of stabilizing the system? To this purpose, we devote this section to deal with the optimal control of system (1.1) coupled with the adequate output function

$$y(t) := 2\sqrt{1+\lambda}\langle k_0, z(t)\rangle, \tag{6.1}$$

where k is the solution of the kernel PDE (4.2) and $z(t) = z(t, u, z^0)$ is the solution of the system (1.1) corresponding to the initial condition z^0 and the control u. The optimal control problem that we address here, is to design a control u which, simultaneously, stabilizes the output function y and minimizes the cost functional

$$J(u) := \int_0^\infty y(t)^2 dt + \int_0^\infty \big\{\varepsilon_2 z_x(t,1) - Q(u)\big\}^2 dt \tag{6.2}$$

with

$$Q(u) := \varepsilon_1 z(t,1) - \langle p, z(t)\rangle,$$

where $\varepsilon_1 := \varepsilon(1)k_0'(1), \varepsilon_2 := \varepsilon(1)k_0(1)$ and $p(y) := \big[\varepsilon(x)k_x(x,y)\big]_{x|_{x=1}}$. We note here that J can be written as $\int_0^\infty \big(y(t)^2 + \|Ku(t)\|^2\big)dt$, where K is a linear operator chosen appropriately. Which shows that (6.2) has the usual form of a cost functional. The second main result of this paper is given by the following theorem.

Theorem 6.1. *The controller*

$$\varepsilon_2 u^{\rm opt}(t) = \varepsilon_1 z(t,1) + \langle 2k_0 - p, z(t)\rangle, \tag{6.3}$$

applied to (1.1)*, stabilizes the output function* y *and minimizes the cost* J*. Moreover, the optimal value for* J *is given by*

$$J^{\rm opt} = 2\langle k_0, z^0\rangle^2.$$

Proof. For $t \geq 0$, set

$$V(t) := \frac{1}{2}\langle k_0, z(t)\rangle^2.$$

By integrating by parts and using (4.2), we obtain

$$\begin{aligned}\dot V(t) &= \langle k_0, z(t)\rangle\big\{\varepsilon_2 z_x(t,1) - \varepsilon_1 z(t,1) + \langle p - \lambda k_0, z(t)\rangle\big\}\\ &= -\lambda\langle k_0, z(t)\rangle^2 + \langle k_0, z(t)\rangle\big[\varepsilon_2 z_x(t,1) - Q(u)\big],\end{aligned}$$

which can be written as

$$\begin{aligned}\dot V(t) =& \Big\{\langle k_0, z(t)\rangle + \frac{1}{2}\big[\varepsilon_2 z_x(t,1) - Q(u)\big]\Big\}^2\\ &- (1+\lambda)\langle k_0, z(t)\rangle^2 - \frac{1}{4}\big[\varepsilon_2 z_x(t,1) - Q(u)\big]^2.\end{aligned} \tag{6.4}$$

So,

$$\frac{1}{4}J(u) = V(0) - V(\infty) + \int_0^\infty \Big\{\langle k_0, z(t)\rangle + \frac{1}{2}\big[\varepsilon_2 z_x(t,1) - Q(u)\big]\Big\}^2 dt. \tag{6.5}$$

Choosing now the control $u^{\rm opt}$ as in (6.3), then the control law $z_x(t,1) = u^{\rm opt}(t)$ is equivalent to

$$\langle k_0, z(t)\rangle + \frac{1}{2}\big[\varepsilon_2 z_x(t,1) - Q(u)\big] = 0. \tag{6.6}$$

Substituting (6.6) in (6.4), we obtain $\dot V(t) \leq -2(1+\lambda)V(t)$, which implies

$$V(t) \leq e^{-2(1+\lambda)t}V(0) \quad \text{and} \quad y(t)^2 \leq e^{-2(1+\lambda)t}y(0)^2. \tag{6.7}$$

This proves that the control law $u^{\rm opt}(t) = z_x(t,1)$ stabilizes the output y.

On the other hand, from (6.7), one has $V(\infty) = 0$. Substituting (6.6) in (6.5), we obtain

$$J(u^{\rm opt}) = 4V(0) = J^{\rm opt}.$$

This completes the the proof. □

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