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CUBIC AND QUARTIC PLANAR DIFFERENTIAL SYSTEMS WITH EXACT ALGEBRAIC LIMIT CYCLES

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Abstract. We construct cubic and quartic polynomial planar differential systems with exact limit cycles that are ovals of algebraic real curves of degree four. The result obtained for the cubic case generalizes a proposition of [9]. For the quartic case, we deduce for the first time a class of systems with four algebraic limit cycles and another for which nested configurations of limit cycles occur.

1. Introduction

In this article, we consider the autonomous planar polynomial system of ordinary differential equations

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = P(x,y), \\ \dot{y} &= \frac{dx}{dt} = Q(x,y), \end{aligned} \tag{1.1}$$

where P and Q are two polynomials of $\mathbb{R}[x,y]$ with no common factor, the derivatives are performed with respect to the time variable t. By definition, the degree of the system (1.1) is $n = \max(\deg(P), \deg(Q))$.

In the qualitative theory of planar dynamical systems see [5, 11], one of the most important topics is related to the second part of the unsolved Hilbert 16th problem: what is the maximum number $H(n)$ of limit cycles that system (1.1) can have for a given degree n and what are their disposition in the x, y? There is a huge literature about limit cycles, most of them deal essentially with their detection, their number and their stability and rare are papers concerned by giving them explicitly. In the last years however, some papers on planar systems with one or more exact non trivial limit cycles of higher degrees were written (see for instance [1, 2, 9] and references therein). We recall that in the phase plane, a limit cycle of system (1.1) is a closed periodic orbit in the set of all its periodic orbits. A limit cycle is stable if all other solutions approach the limit cycle either from its interior or from exterior asymptotically as $t \to +\infty$ and when the limit cycle is unique and stable it dominates the global behavior of the system.

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An algebraic limit cycle is a non-singular compact component (or an oval) of a real algebraic curve. Also, according to Harnack theorem, the maximum number of ovals that an algebraic real curve of degree n, can have is at most $\frac{1}{2}(n-1)(n-2)+1$ and when this bound is reached, the corresponding curve is called an M-curve. It is strongly expected [5, 7, 10] that a polynomial planar system of degree n has at most $\frac{1}{2}(n-1)(n-2)+1$ algebraic limit cycles and generally we look for them as non-singular compact components of invariant algebraic curves.

For $U \in \mathbb{R}[x, y]$, the algebraic curve $U = 0$ is called an invariant curve of the polynomial system (1.1), if for some polynomial $K \in \mathbb{R}[x, y]$ called the cofactor of the algebraic curve, we have

$$P\frac{\partial U}{dx} + Q\frac{\partial U}{dy} = KU. \tag{1.2}$$

Simple analysis of equation (1.2) shows that the degree of the cofactor is at most $n-1$ and that the curve $U = 0$ is formed by trajectories of the system (1.1). Also, if the curve $U = 0$ is non-singular, the equilibrium points of the system are contained either in its non-bounded components or are located on the curve $K = 0$.

This paper is concerned by systems with maximum number of algebraic limit cycles, we show that this is possible for $n = 3$ and $n = 4$ by giving their exact analytic expressions. More precisely, we prove that the cubic class admits two algebraic limit cycles of degree four, by the way a result obtained in citel1 becomes a particular case. Concerning the quartic case, we present a class of systems with four exact limit cycles and an other one with two nested limit cycles, this classes are new in the literature. This work is organized as follows:

In the second section, we construct a class of cubic systems admitting two algebraic limit cycles of degree four analytically given. This generalizes the example in [9].

Section three is devoted to an effective construction of classes of quartic systems with one, two and four exact algebraic limit cycles. To obtain the result, we use a theorem by Christoffer [4].

2. Cubic Systems

Consider the class of cubic systems

$$\begin{aligned}
\dot{x} &= P(x, y) \\
&= a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy \\
&\quad + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \\
\dot{y} &= Q(x, y) \\
&= b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy \\
&\quad + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3.
\end{aligned} \tag{2.1}$$

For a sub-class of (2.1), we prove the existence of two algebraic limit cycles, moreover these limit cycles are explicitly given.

We introduce the quartic curve $U = 0$ where

$$U(x, y) = y^2 + \delta(x^2 - \alpha^2)(x^2 - \beta^2), \tag{2.2}$$

and α, β, γ and δ are real constants such that $\delta > 0$, $0 < \alpha < \beta$. To formulate and prove the main result of this section, we need to establish some auxiliary lemmas.

Lemma 2.1. *The curve $U = 0$ is a non-singular quartic formed by two ovals.*

Proof. Let $\delta > 0, \alpha, \beta$ such that $\alpha < \beta$. We observe that the curve $U = 0$ exits only when $x \in [-\beta, -\alpha] \cup [\alpha, \beta]$ and is formed by the union of the two arcs (C_1) an (C_2), where

$$\begin{gathered}(C_1) : \{(x, y) : x \in [-\beta, -\alpha] \cup [\alpha, \beta] \wedge y = y_1(x)\}, \\ (C_2) : \{(x, y) : x \in [-\beta, -\alpha] \cup [\alpha, \beta] \wedge y = y_2(x)\}, \\ y_1(x) = -\sqrt{\delta}\sqrt{(x^2 - \alpha^2)(x^2 - \beta^2)}, \quad y_2(x) = -y_1(x).\end{gathered}$$

Since $y_2(x) - y_1(x) = 2\sqrt{\delta}\sqrt{(x^2 - \alpha^2)(x^2 - \beta^2)} \geq 0$, the second arc is always above the first one with the common points $(-\beta, 0), (-\alpha, 0), (\alpha, 0), (\beta, 0)$ as points of meeting, hence $U = 0$ is composed of the two closed curves

$$\begin{gathered}\{(x, y) : x \in [-\beta, -\alpha] \wedge y = y_2(x)\} \cup \{(x, y) : x \in [-\beta, -\alpha] \wedge y = y_1(x)\}, \\ \{(x, y) : x \in [\alpha, \beta] \wedge y = y_1(x)\} \cup \{(x, y) : x \in [\alpha, \beta] \wedge y = y_2(x)\}.\end{gathered}$$

By construction of the curve $U = 0$, the problem of smoothness can occur only at the points of meeting $(-\beta, 0), (-\alpha, 0), (\alpha, 0), (\beta, 0)$ of (C_1) and (C_2), but this is not the case since at this points the tangent to the curve is just parallel to the y-axis according to the simple fact that

$$\frac{dy}{dx} = \pm x \frac{\sqrt{\delta}}{\sqrt{(\alpha^2 - x^2)(\beta^2 - x^2)}} (-2x^2 + \alpha^2 + \beta^2).$$

□

Lemma 2.2. *The most general cubic planar polynomial system admitting the curve $U = 0$ as invariant curve is the system:*

$$\begin{aligned}\dot{x} &= -\frac{1}{2}((\alpha^2 + \beta^2)a_{21} + \frac{1}{\delta}b_{21})y + a_{11}xy + a_{21}x^2y + a_{12}xy^2, \\ \dot{y} &= 2\delta\alpha^2\beta^2a_{11} - \frac{1}{2}(\delta(\alpha^2 - \beta^2)^2a_{21} + (\alpha^2 + \beta^2)b_{21})x \\ &\quad + 2\delta\alpha^2\beta^2a_{12}y - \delta(\alpha^2 + \beta^2)a_{11}x^2 + 2a_{11}y^2 \\ &\quad + b_{21}x^3 - \delta(\alpha^2 + \beta^2)a_{12}x^2y + 2a_{21}xy^2 + 2a_{12}y^3.\end{aligned} \tag{2.3}$$

Proof. Considering system (2.1), we perform an Euclidean division of the polynomial $P(x, y)\frac{\partial U}{dx}(x, y) + Q(x, y)\frac{\partial U}{dy}(x, y)$ over the polynomial $U(x, y)$ with respect to the y variable. The curve $U = 0$ is invariant for this system if and only if the remainder vanishes identically. We are lead to a linear system of sixteen equations of the twenty unknowns a_{ij} and b_{ij}, $i + j = 0, 1, 2, 3$. Using Maple, we obtain

$$\begin{gathered}a_{00} = a_{10} = a_{20} = a_{02} = a_{30} = a_{03} = b_{11} = 0; \\ a_{01} = -\frac{1}{2}((\alpha^2 + \beta^2)a_{21} + \frac{1}{\delta}b_{21}); \quad b_{00} = 2\delta\alpha^2\beta^2a_{11}; \\ b_{10} = -\frac{1}{2}(\delta(\alpha^2 - \beta^2)^2a_{21} + (\alpha^2 + \beta^2)b_{21}); \quad b_{01} = 2\delta\alpha^2\beta^2a_{12}; \\ b_{20} = -\delta(\alpha^2 + \beta^2)a_{11}; \quad b_{02} = 2a_{11}; \quad b_{12} = b_{30}; \quad b_{21} = b_{03} = 2a_{12}.\end{gathered}$$

After the substitution of these solution into system (2.1) and rewriting it in the standard form, we obtain the system (2.3). Then straightforward computations show that

$$P(x,y)\frac{\partial U}{dx}(x,y)+Q(x,y)\frac{\partial U}{dy}(x,y)=(4a_{12}y^2+(4a_{11}+4xa_{21})y)U(x,y),$$

which implies that the curve $U=0$ is an invariant curve and the associated cofactor is $K(x,y)=4a_{12}y^2+4(a_{11}+a_{21}x)y$. $\square$

Since we are interested by limit cycles rather than by invariant curves, we are constrained to impose additional conditions in order that system (2.3) admits the ovals of the curve $U=0$ as periodic solutions. If we put $a_{21}=a_{11}=0$, this system reduces to

$$\begin{aligned}\dot{x}&=-\frac{1}{2\delta}b_{21}y+a_{12}xy^2,\\ \dot{y}&=-\frac{1}{2}(\alpha^2+\beta^2)b_{21}x+2\delta\alpha^2\beta^2a_{12}y\\ &\quad+b_{21}x^3-\delta(\alpha^2+\beta^2)a_{12}x^2y+2a_{12}y^3.\end{aligned}\tag{2.4}$$

Lemma 2.3. *Suppose that $a_{12}\neq 0, b_{21}\neq 0$ and*

$$\Big(\frac{b_{21}}{a_{12}}\Big)^2>\tfrac{8}{27}\delta^3((\alpha^2+\beta^2)(\alpha^4-\tfrac{5}{2}\alpha^2\beta^2+\beta^4)-(\alpha^4-\beta^2\alpha^2+\beta^4)^{3/2}).\tag{2.5}$$

Then the ovals of the curve $U=0$ are periodic solutions of (2.4).

Proof. The singular points at finite distance of system (2.4) are the origin $(0,0)$, the points $(\pm\frac{\sqrt{2}}{2}\sqrt{\alpha^2+\beta^2},0)$, and for $y\neq 0$, the others that can probably lie on the curve $U=0$ are the points $(x_0,\frac{1}{2\delta}\frac{b_{21}}{a_{12}x_0})$ where x_0 is solution of the equation $4\delta^3a_{12}^2(x^2-\beta^2)(x^2-\alpha^2)x^2+b_{21}^2=0$. Thanks to condition (2.5), this equation has no real solutions. Consequently, these ovals are periodic solutions, one oval surrounds the point $(-\frac{\sqrt{2}}{2}\sqrt{\alpha^2+\beta^2},0)$, the other surrounds the point $(\frac{\sqrt{2}}{2}\sqrt{\alpha^2+\beta^2},0)$. $\square$

We recall that our goal is to show that the ovals of the curve $U=0$ are if fact limit cycles. The main result of this section is as follows.

Theorem 2.4. *If the constants α,β,γ and δ are such that $\delta>0$, $0<\alpha<\beta$ and the constants $a_{12}\neq 0$ and b_{21} satisfy the inequality* (2.5)*, the system* (2.4) *admits as limit cycles the two ovals of the algebraic curve $U=0$.*

Proof. From the preceding lemmas, the cofactor is $K(x,y)=4a_{12}y^2$, the two ovals of the algebraic curve $U=0$ are periodic orbits of (2.4).

If we denote by T_i, $i=1,2$, their corresponding periods, then since $a_{12}\neq 0$,

$$\int_0^{T_i}K(x(t),y(t))dt=4a_{12}\int_0^{T_i}y^2(t)dt\neq 0.$$

By [6, Theorem 1] we conclude. The hyperbolicity of the two limit cycles depends on the sign of a_{12}. $\square$

2.1. **Nature of the singular points.** It is useful to discuss the nature of the singular points of the system (2.4) in order to draw its phase portrait near this points. We just outline the situation for the interesting cases:

- For $b_{21}\neq\frac{1}{3}$, $4(\frac{b_{21}}{a_{12}})^2(3b_{21}-1)<\frac{\delta^3(\alpha^2-\beta^2)^4}{(\alpha^2+\beta^2)}$ and $a_{12}<0$, the points $(\pm\frac{\sqrt{2}}{2}\sqrt{\alpha^2+\beta^2},0)$ are stable foci;

- For $b_{21} \neq \frac{1}{3}$, $4(\frac{b_{21}}{a_{12}})^2(3b_{21}-1) < \frac{\delta^3(\alpha^2-\beta^2)^4}{(\alpha^2+\beta^2)}$ and $a_{12} > 0$, the points $(\pm\frac{\sqrt{2}}{2}\sqrt{\alpha^2+\beta^2}, 0)$ are unstable foci.

Finally, the origin is always a saddle and as the line $y = 0$ is an invariant curve, no other orbit can cross the y-axis.

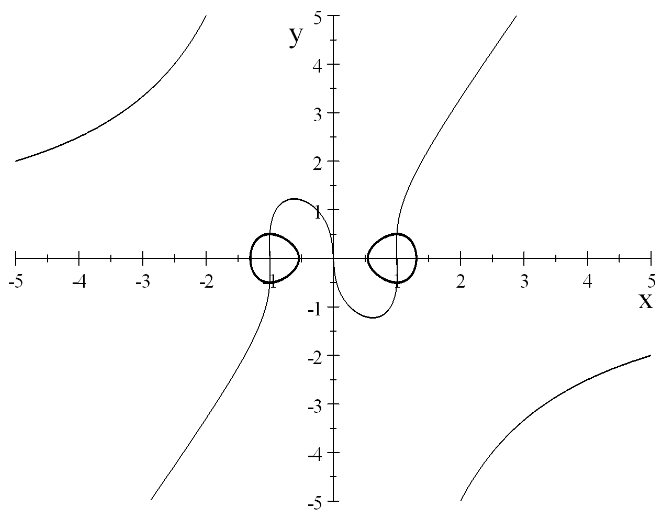

FIGURE 2.1. Limit cycles and the null-clines

2.2. **Example.** This example shows that [9, proposition 19] is a particular result of Theorem 2.4. In fact if we take $\delta = \frac{1}{2}$, $a_{12} = 2$, $b_{21} = -20$, $\alpha^2\beta^2 = \frac{1}{2}$ and $\alpha^2 + \beta^2 = 2$, the system (2.3) reads

$$\begin{aligned} \dot{x} &= 20y + 2xy^2, \\ \dot{y} &= 20x + y - 20x^3 - 2x^2y + 4y^3, \end{aligned} \tag{2.6}$$

which is nothing but system [8, (50)]. It is easy to see that Theorem 2.4 applies, and that the two ovals of the curve $y^2 + \frac{1}{2}x^4 - x^2 + \frac{1}{4} = 0$ are the corresponding algebraic limit cycles enclosing the unstable foci $(\pm 1, 0)$ (see the disposition of the limit cycles in figure 2.1, where the vertical isocline $\frac{dx}{dt} = 0$ is composed of the straight line $y = 0$ and the hyperbola $y = -10/x$, while the horizontal isocline $y = 0$ is the cubic $20x + y - 20x^3 - 2x^2y + 4y^3 = 0$). Inside the limit cycles, all solutions recede from the foci and spiral clockwise approaching the limit cycles asymptotically as $t \to +\infty$.

3. Quartic System

In this section, we give a feasible construction of quartic systems admitting exact hyperbolic algebraic limit cycles of degree four. In fact, since nested configurations of ovals can occur for an algebraic curve of degree equal or greater than 4, this allow us to present for the first time a class of quartic systems with nested configuration of algebraic limit cycles and also an other class with four exact algebraic limit cycles.

Moreover this limit cycles are explicitly given. For that, we consider the well known planar system

$$\begin{aligned}\dot{x} &= P(x,y) = af(x,y) - D(x,y)f_y(x,y),\\ \dot{y} &= Q(x,y) = bf(x,y) + D(x,y)f_x(x,y),\end{aligned} \tag{3.1}$$

where $f \in \mathbb{R}[x,y]$ is a polynomial of degree m and $D(x,y) = ux + vy + w$. This planar system belongs to a more general class of systems intervening in the inverse approach of dynamical systems. Christoffer [4] has proved that if the line $D(x,y) = 0$ lies outside all non-singular compact components (ovals) of the algebraic curve $f = 0$ and the constants a and b are chosen such that $au + bv \neq 0$, then this system admits all the bounded components of the curve $f = 0$ as hyperbolic limit cycles. Furthermore, the vector field (3.1) has no other limit cycles.

We introduce the real algebraic curve $f = 0$ given analytically for $\gamma \neq 0$ by

$$f(x,y) = y^4 + x^4 + \alpha y^2 + \beta x^2 + \gamma = 0. \tag{3.2}$$

The main result of this section is as follows.

Theorem 3.1. *Let the planar differential system*

$$\begin{aligned}\dot{x} &= a\gamma - 2w\alpha y + a\beta x^2 - 2u\alpha xy + (a-2v)\alpha y^2\\ &\quad - 4wy^3 + ax^4 - 4uxy^3 + (a-4v)y^4,\\ \dot{y} &= b\gamma + 2w\beta x + (b+2u)\beta x^2 + 2v\beta xy + b\alpha y^2\\ &\quad + 4wx^3 + (b+4u)x^4 + 4vx^3y + by^4,\end{aligned} \tag{3.3}$$

where the real constants $\alpha, \beta, \gamma \neq 0, a, b, u, v$ and w are such that $au + bv \neq 0$ and the line $ux + vy + w = 0$ do not intersect the algebraic curve (3.2). *If we assume additionally the conditions:*

$$(\alpha,\beta,\gamma) \in \mathbb{R}^3 \setminus \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^{*+} \tag{3.4}$$

$$\alpha^2 + \beta^2 - 4\gamma > 0; \tag{3.5}$$

$$\alpha^2 - 4\gamma \neq 0; \tag{3.6}$$

$$\beta^2 - 4\gamma \neq 0; \tag{3.7}$$

this system possesses exactly:

- *Four limit cycles each located strictly in one of the four quarter of the plane and having the four points $(\pm\frac{\sqrt{2}}{2}\sqrt{-\beta}, \pm\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ as centers if $\alpha < 0$, $\beta < 0$ and $\max\{\alpha^2, \beta^2\} < 4\gamma$;*
- *a nested configuration of two limit cycles with the origin as center for $\alpha < 0$, $\beta < 0$ and $0 < 4\gamma < \min\{\alpha^2, \beta^2\}$;*
- *two limit cycles symmetric with respect to the y-axis and centered at $(\pm\frac{\sqrt{2}}{2}\sqrt{-\beta}, 0)$ if $\alpha > 0$, $\beta < 0$ and $\alpha^2 < 4\gamma < \beta^2$;*
- *two limit cycles symmetric with respect to the x-axis and centered at $(0, \pm\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ for $\alpha < 0$, $\beta > 0$ and $\beta^2 < 4\gamma < \alpha^2$;*
- *one limit cycle centered at the origin if $\alpha \leq 0$, $\beta \leq 0$ and $\gamma < 0$.*
- *Moreover, these limit cycles are hyperbolic and analytically given as the ovals of the curve* (3.2).

To prove this theorem, we need some lemmas.

Lemma 3.2. *The curve* (3.2) *is composed only by ovals.*

Proof. We recall that the curve (3.2) is non-singular if and only if the following system has no solutions:

$$\begin{gathered}\frac{\partial f}{\partial x}(x,y)=0,\\ \frac{\partial f}{\partial y}(x,y)=0,\\ f(x,y)=0.\end{gathered}$$

Taking into account the symmetries of this curve, we are just lead to examine the following possible critical points of the function $f(x,y)$: $(0,0),(0,\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ for $\alpha\le 0$, $(\frac{1}{2}\sqrt{2}\sqrt{-\beta},0)$ for $\beta\le 0$ and the last one is $(\frac{\sqrt{2}}{2}\sqrt{-\beta},\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ for $\alpha\le 0$ and $\beta\le 0$. But from the assumptions of Theorem 3.1,

$$\begin{gathered}f(0,0)=-\gamma\neq 0,\\ f(0,\frac{\sqrt{2}}{2}\sqrt{-\alpha})=-\frac{1}{4}(\alpha^2-4\gamma)\neq 0,\\ f(\frac{\sqrt{2}}{2}\sqrt{-\beta},0)=-\frac{1}{4}(\beta^2-4\gamma)\neq 0,\\ f(\frac{\sqrt{2}}{2}\sqrt{-\beta},\frac{\sqrt{2}}{2}\sqrt{-\alpha})=-\frac{1}{4}(\alpha^2+\beta^2-4\gamma)<0,\end{gathered}$$

so this curve is non-singular.

Since $y=\pm\frac{\sqrt{2}}{2}\sqrt{-\alpha\pm\sqrt{-4x^4-4\beta x^2+\alpha^2-4\gamma}}$ cannot approach $\pm\infty$, the curve remains at finite distance from the origin, it is then bounded and consequently composed just by ovals. We can also deduce that this curve is bounded if we put (3.4) on the form:

$$(y^2+\frac{\alpha}{2})^2+(x^2+\frac{\beta}{2})^2=\frac{1}{4}(\alpha^2+\beta^2-4\gamma),$$

which shows by the same that conditions (3.4) and (3.5) are necessary and sufficient for the curve to be non-empty and not formed by a finite set of points. This complete the proof. □

The following lemma enumerates the number of ovals.

Lemma 3.3. *The curve* (3.2) *is composed of:*

- *Four ovals strictly located each strictly in one of the four quarters of the plane and having the four points* $(\pm\frac{\sqrt{2}}{2}\sqrt{-\beta},\pm\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ *as centers if* $\alpha<0$, $\beta<0$ *and* $\max\{\alpha^2,\beta^2\}<4\gamma$;
- *a nested configuration of two ovals centered at the origin for* $\alpha<0$, $\beta<0$ *and* $0<4\gamma<\min\{\alpha^2,\beta^2\}$;
- *two ovals symmetric with respect to the* y*-axis with the points* $(\pm\frac{\sqrt{2}}{2}\sqrt{-\beta},0)$ *as centers if* $\alpha>0$, $\beta<0$ *and* $\alpha^2<4\gamma<\beta^2$;
- *two ovals symmetric with respect to the* x*-axis with the points* $(0,\pm\frac{\sqrt{2}}{2}\sqrt{-\alpha})$ *as centers for* $\alpha<0$, $\beta>0$ *and* $\beta^2<4\gamma<\alpha^2$;
- *one oval centered at the origin if* $\alpha\le 0$, $\beta\le 0$ *and* $\gamma<0$.

Proof. The symmetries of the curve $f=0$ allow us to avoid a cumbersome proof. Let N_1 (resp. N_2) be the number of intersecting points of this curve with the x-axis (resp. the y-axis). To compute N_1 and N_2, we consider respectively the equations

$$x^4 + \beta x^2 + \gamma = 0, \tag{3.8}$$

$$y^4 + \alpha y^2 + \gamma = 0, \tag{3.9}$$

and let $\Delta(x) = -4x^4 - 4\beta x^2 + \alpha^2 - 4\gamma$.

Case 1: $\alpha^2 - 4\gamma < 0$ and $\beta^2 - 4\gamma < 0$. Equations (3.8) and (3.9) have no solutions so $N_1 = 0$ and $N_2 = 0$. Restricted to $x > 0$ and $y > 0$, the curve $f = 0$ is composed of an oval centered at the point $(\frac{\sqrt{2}}{2}\sqrt{-\beta}, \frac{\sqrt{2}}{2}\sqrt{-\alpha})$ and formed by the union of the arcs

$$x \mapsto y(x) = \frac{\sqrt{2}}{2}\sqrt{-\alpha + \sqrt{\Delta(x)}}, \quad x \mapsto y(x) = \frac{\sqrt{2}}{2}\sqrt{-\alpha - \sqrt{\Delta(x)}},$$

and $x \in [x_1, x_2]$, with

$$x_1 = \frac{\sqrt{2}}{2}\sqrt{-\beta - \sqrt{\alpha^2 + \beta^2 - 4\gamma}}, \quad x_2 = \frac{\sqrt{2}}{2}\sqrt{-\beta + \sqrt{\alpha^2 + \beta^2 - 4\gamma}}.$$

The same statement holds in the remaining quarters of the plan.

Case 2: These conditions $\alpha < 0$, $\beta < 0$ and $0 < 4\gamma < \min\{\alpha^2, \beta^2\}$ imply that the equations (3.8) and (3.9) admit four solutions each, so $N_1 = N_2 = 4$. Let us denote increasingly by $-x_2 < -x_1 < x_1 < x_2$ and $-y_2 < -y_1 < y_1 < y_2$ these solutions. In this case, the curve $f = 0$ is composed by two nested ovals centered both at the origin, the outer one passes through the points $(x_2, 0), (0, y_2), (-x_2, 0)$ and $(0, -y_2)$, and the inner one passes through the points $(x_1, 0), (0, y_1), (-x_1, 0)$ and $(0, -y_2)$. For $y > 0$, the union of the two arcs

$$x \mapsto y(x) = \frac{\sqrt{2}}{2}\sqrt{-\alpha + \sqrt{\Delta(x)}}, \quad x \mapsto y(x) = \frac{\sqrt{2}}{2}\sqrt{-\alpha - \sqrt{\Delta(x)}}$$

forms the two halves located above the x-axis of the two nested ovals, while the union of the arcs

$$x \mapsto y(x) = -\frac{\sqrt{2}}{2}\sqrt{-\alpha + \sqrt{\Delta(x)}}, \quad x \mapsto y(x) = -\frac{\sqrt{2}}{2}\sqrt{-\alpha - \sqrt{\Delta(x)}}$$

composes the remaining parts of these ovals located below the x-axis.

Case 3: $\alpha > 0, \beta < 0$ and $\alpha^2 < 4\gamma < \beta^2$. Equation (3.8) possesses four solutions $-x_2 < -x_1 < x_1 < x_2$ and (3.9) no solutions, so $N_1 = 4$ and $N_2 = 0$. The curve $f = 0$ is composed by two ovals symmetric with respect to the y-axis, the first one is centered at $(-\frac{\sqrt{2}}{2}\sqrt{-\beta}, 0)$ and passes through the points $(-x_2, 0)$ and $(-x_1, 0)$ and the second centered at $(\frac{\sqrt{2}}{2}\sqrt{-\beta}, 0)$ passes through the points $(x_1, 0)$ and $(x_2, 0)$.

Case 4: $\alpha < 0$, $\beta > 0$ and $\beta^2 < 4\gamma < \alpha^2$: The arguments are similar to those in the third case. The curve $f = 0$ is composed by two ovals where the first one passes through the points $(0, -y_2)$ and $(0, -y_1)$ and the second through the points $(0, y_1)$ and $(0, y_2)$.

Case 5: $\alpha \le 0$, $\beta \le 0$ and $\gamma < 0$. Equations (3.8) and (3.9) have two solutions each denoted by $-x_1, x_1$ and $-y_1, y_1$ so $N_1 = N_2 = 2$ and the curve $f = 0$ admits the origin as center and is formed by a single oval passing through the points $(x_1, 0), (0, y_1), (-x_1, 0)$ and $(0, -y_1)$. $\square$

We can now give the proof of the main result of this section.

Proof of Theorem 3.1. If we choose $f(x,y) = y^4 + x^4 + \alpha y^2 + \beta x^2 + \gamma$ in system (9), we obtain system (3.3) and of course the curve $f = 0$ is an invariant curve of the later, we can see that

$$K(x,y) = a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} = 4ax^3 + 2a\beta x + 4by^3 + 2b\alpha y$$

is the associated cofactor.

From Lemma 3.2, no equilibrium point of the system (3.3) can lie on the curve $f = 0$, so the ovals of this curve are periodic orbits. Since we have assumed that the line $ux + vy + w = 0$ do not intersect the algebraic curve $f = 0$, from the theorem by Christoffer [4] we deduce that these ovals are the only hyperbolic limit cycles of system (3.3). Their number is discussed in Lemma 3.3 and this completes the proof. □

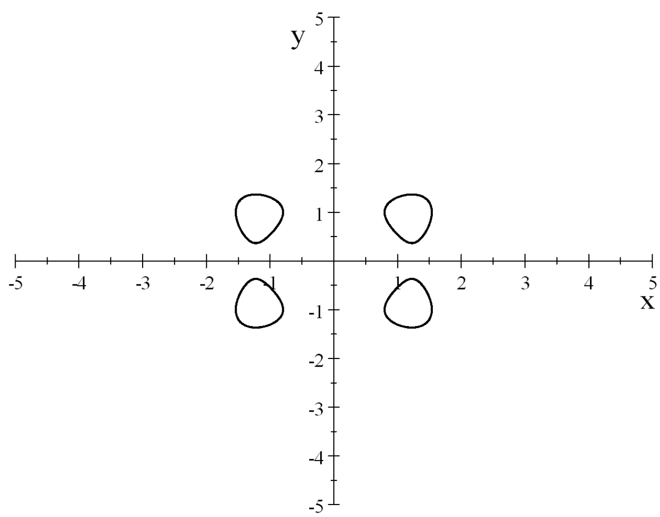

FIGURE 3.1. Four limit cycles for system (3.10)

Example 3.4. Let $a = b = 1$, $u = w = 0$, $v = 1$. The system

$$\begin{aligned} \dot{x} &= \frac{5}{2} - 3x^2 + 2y^2 + x^4 - 3y^4, \\ \dot{y} &= \frac{5}{2} - 3x^2 - 6xy - 2y^2 + x^4 + 4x^3y + y^4, \end{aligned} \tag{3.10}$$

admits exactly four hyperbolic limit cycles that are the ovals of the algebraic curve $y^4 + x^4 - 2y^2 - 3x^2 + \frac{5}{2} = 0$, each oval encloses one of the four equilibrium points approximately given by ($x = \pm 1.29$, $y = \pm 0.89$). See Figure 3.1.

Example 3.5. Let $a = b = 1$, $u = 0$, $v = 1$, $w = -3$: the system

$$\begin{aligned} \dot{x} &= 1 - 36y - 6x^2 + 6y^2 + 12y^3 + x^4 - 3y^4, \\ \dot{y} &= 1 + 36x - 6x^2 - 12xy - 6y^2 - 12x^3 + x^4 + 4x^3y + y^4, \end{aligned} \tag{3.11}$$

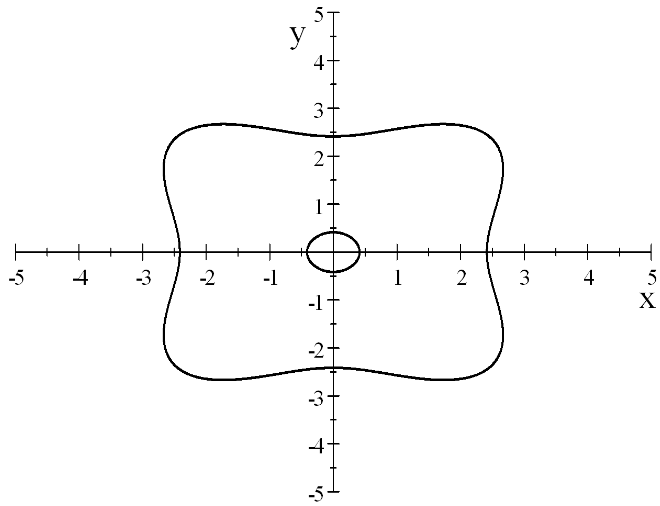

FIGURE 3.2. A nested configutation of two limit cycles for system (3.11)

admits two hyperbolic limit cycles that are the ovals of the algebraic curve $y^4+x^4-6y^2-6x^2+1=0$. The inner oval contains the equilibrium points ($x=-2.78\times10^{-2}$, $y=2.78\times10^{-2}$), and the set

$$(x=1.62, y=-0.21), \quad (x=-1.87, y=-1.56), \quad (x=-1.83, y=-0.21),$$
$$(x=-2.15, y=2.15), \quad (x=0.15, y=-1.65), \quad (x=1.56, y=-1.56)$$

of equilibrium points lie between the two ovals. See Figure 3.2.

Remark 3.6. The class of bounded non-singular quartic curves defined by (3.2) is the simplest one with all possible configurations of ovals and regular transforms of the plane or slight deformation directly operated on its equation do not alter its topology. We can illustrate this by the next example.

Example 3.7. The system

$$\begin{aligned}\dot{x} &= -\frac{3}{2}+40x-5y-66x^2+30xy-12y^2-88x^3-252x^2y\\ &\quad -66xy^2-21y^3+40x^4-24x^3y-144x^2y^2-68xy^3-13y^4,\\ \dot{y} &= -5-8x-62y-84x^2+108xy-30y^2+176x^3-72x^2y\\ &\quad +132xy^2-6y^3-64x^4+32x^3y-48x^2y^2+24xy^3,\end{aligned} \tag{3.12}$$

admits the nested asymmetric configuration of ovals composing the curve given by the full term equation:

$$\begin{aligned}&17y^4-24xy^3+48x^2y^2+24x^3y+17x^4+32y^3-48xy^2\\ &+24x^2y-4x^3-6y^2-24xy-24x^2-16y+8x-4=0\end{aligned}$$

as the only hyperbolic limit cycles. This result is derived from the preceding example when we perform the regular change of variables $x\to 2x+y$, $y\to x-2y-1$ on system (3.11). See Figure 3.3.

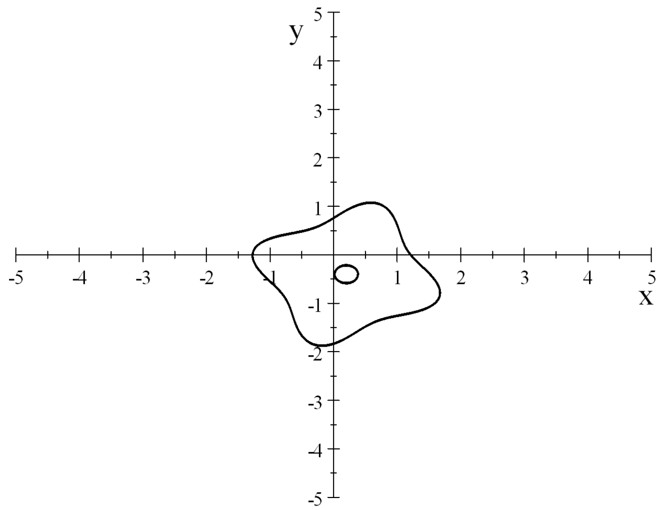

FIGURE 3.3. A nested configuration of limit cycles for system (3.12)

Conclusion and perspectives. The elementary method used in this paper seems to be fruitful to investigate more general planar dynamical systems in order to obtain explicitly some or all their limit cycles at least when it is question of the algebraic ones. In the spirit of the inverse approach to dynamical systems, we look for them as the ovals of suitably chosen invariant algebraic curves. The following questions can be raised:

- Does system (2.4) admits additional limit cycles? Can one transforms it into another cubic system with two algebraic limit cycles of degree greater than four?
- Are there quartic systems with three algebraic limit cycles?
- Is it right that the number $\frac{(n-2)(n-1)}{2} + 1$ is the upper bound of algebraic limit cycles for system (1.1) in general and for system for which this bound is reached (as for the cubic and quartic classes studied in this paper), do we have $H(n) - [\frac{(n-2)(n-1)}{2} + 1] = 0$?
- Finally, we know (see [3]) that nested configurations of algebraic limit cycles are not possible in quadratic systems, we claim that they are not possible in cubic systems to.

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