Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 151, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OSCILLATION CRITERIA FOR DAMPED QUASILINEAR SECOND-ORDER ELLIPTIC EQUATIONS

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Dedicated to my mother Meguem Ghomsi Mabou and all her Meguems

Abstract. In 2010, Yoshida [13] stated that oscillation criteria for the superlinear-sublinear elliptic equation equation

$$\nabla \cdot \big(A(x)\Phi(\nabla v)\big) + (\alpha+1)B(x)\cdot\Phi(\nabla v) + C(x)\phi_\beta(v) + D(x)\phi_\gamma(v) = f(x)$$

were not known. In this article, we provide some answers to this question using boundedness conditions on the coefficients of half-linear quasilinear elliptic equations. This is obtained by using some comparison methods and Picone-type formulas.

1. Introduction

In [13], for $A \in C^1(\mathbb{R}^n,\mathbb{R})$, $C, D, f \in C(\mathbb{R}^n,\mathbb{R})$ and $B \in C(\mathbb{R}^n,\mathbb{R}^n)$, the equation

$$\nabla \cdot \big(A(x)\Phi(\nabla v)\big) + (\alpha+1)B(x)\cdot\Phi(\nabla v) + C(x)\phi_\beta(v) + D(x)\phi_\gamma(v) = f(x) \quad (1.1)$$

was given. Here the central dot denotes the Euclidean scalar product between elements of $\mathbb{R}^n$. Let α be a positive fixed number. We define the following functions for $(t,\zeta) \in \mathbb{R}\times\mathbb{R}^n$ and $\nu > 0$,

$$\phi(t) := |t|^{\alpha-1}t; \quad \Phi(\zeta) := |\zeta|^{\alpha-1}\zeta, \quad \phi_\nu(t) := |t|^{\nu-1}t; \quad \Phi_\nu(\zeta) := |\zeta|^{\nu-1}\zeta.$$

Recall that for any $\alpha > 0$, the function $\phi = \phi_\alpha$ has the following properties:

$$\forall t, s \in \mathbb{R}, \quad \phi(t)\phi(s) = \phi(ts), \quad t\phi'(t) = \alpha\phi(t), \quad t\phi(t) = |t|^{\alpha+1};$$
$$\forall (s,\zeta) \in \mathbb{R}\times\mathbb{R}^n, \quad \phi(s)\Phi(\zeta) = \Phi(s\zeta); \quad \zeta\Phi(\zeta) = |\zeta|^{\alpha+1}.$$

The quest is to investigate oscillation criteria for equations similar to (1.1), following some different process but still based on Picone-type formulae.

2. One-dimensional and radially symmetric equations

First, we consider the simple equation

$$\{a(t)\phi(y')\}' + c(t)\phi(y) + h(t,y,y') = 0 \quad (2.1)$$

where $\{.\}'$ denotes the derivative with respect to the variable t. In the sequel, we assume

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2000 *Mathematics Subject Classification.* 34C10, 34K15, 35J70.
Key words and phrases. Picone; oscillation criteria for half-linear elliptic equations.

Submitted July 21, 2011. Published November 8, 2011.

(H0) a is a positive constant or $a \in C^1(\mathbb{R}, (0,\infty))$ and is non decreasing; the other coefficients are continuous in all their arguments.

Also we need some definitions:

A function u will be said to be a (regular) solution of (2.1) if there exists $T > 0$ such that u is locally piecewise C^2 and $u, \phi(u')$ are C^1 in $D_T := (T, \infty)$.

This indicates that our focus is on the behaviour of the solutions in exterior domains.

Definition 2.1. Let $u \in C(\mathbb{R}, \mathbb{R})$.

(1) A nodal set of u, is any bounded open and connected set $D = D(u) \neq \emptyset$ such that $u|_{\partial D} = 0$ and $u \neq 0$ in D.
(2) A function u is said to be (weakly) oscillatory (in $\mathbb{R}$) if it has a zero in any D_T and is strongly oscillatory if it has a nodal set in any D_T.
(3) An equation will be said to be oscillatory if any of its non-trivial solutions is oscillatory.
(4) An equation will be said to be homogeneous if whenever u is a solution so is also λu for all $\lambda \in \mathbb{R} \setminus \{0\}$. When this holds only for $\lambda = -1$ or 1 the equation is said to be odd.

Remark 2.2. When $h \equiv 0$, equation (2.1) is homogeneous and odd. From the definitions above, a function u would be non-oscillatory if it is eventually non zero; i.e., there exists $T > 0$ and $u(t) \neq 0$ for all $t \in D_T$. If such a non-oscillatory function happens to be a solution of an odd equation, we can freely chose it to be eventually positive or eventually negative.

The main strategy in this work is to use some comparison methods via Picone-type formulae to obtain oscillation criteria of some general equations. Of course for some of the simpler equations, the oscillation criteria will be obtained through direct investigations as in [1]. As examples of simple strongly oscillatory equations, for $\alpha > 0$, we have

$$\{\phi_\alpha(u')\}' + \alpha\phi_\alpha(u) = 0 \tag{2.2}$$

whose solutions are the generalized sine functions $S := S_\alpha$ [1, 3] with the following properties:

$$\begin{gathered} |S_\alpha(t)|^{\alpha+1} + |S'_\alpha(t)|^{\alpha+1} = 1, \quad S_\alpha(t + \pi_\alpha) = -S_\alpha(t), \\ \text{where } \pi_\alpha = \frac{2\pi}{(\alpha+1)\sin\{\frac{\pi}{\alpha+1}\}}. \end{gathered} \tag{2.3}$$

When $\alpha = 1$ the above functions are the usual trigonometric functions.

Easy calculations show that for $k \in \mathbb{R}\backslash\{0\}$, the function $W(t) := S_\alpha(e^{kt})$ satisfies

$$\{e^{-k\alpha t}\phi_\alpha(W')\}' + |k|^{\alpha+1}\alpha e^{kt}\phi_\alpha(W) = 0, \tag{2.4}$$

and the function $Y(t) := S_\alpha(t^k)$ with $t \geq 0$ satisfies

$$\{t^{(1-k)\alpha}\phi_\alpha(Y')\}' + |k|^{\alpha+1}t^{k-1}\alpha\phi_\alpha(Y) = 0. \tag{2.5}$$

A one-dimensional equation associated to (1.1), for some $\beta, \gamma > 0$, is

$$\{a(t)\phi(y')\}' + c(t)\phi_\beta(z) + d(t)\phi_\gamma(z) = f(t) \tag{2.6}$$

where the coefficients satisfy (H0). Also assume that

(H1) there exists $T > 0$ such that $c, d > 0$ and $f \leq 0$ on D_T.

Lemma 2.3. *Assume that* (H1) *holds and there is a bounded non-trivial solution* z *of* (2.6).

(1) If the coefficient a is a positive constant and z is eventually positive, then the derivative z' is also eventually positive and decreases to 0 *at* ∞.

(2) If $a' > 0$ and decreases to 0 at ∞, and c is unbounded, then the conclusion in still (1) holds if $f \not\equiv 0$ in some D_T. However, if $f(t) \equiv 0$ in some D_T, then conclusion in (1) holds provided that the solution z is eventually greater than a positive constant.

Proof. (1) If $a \equiv 1$, from (2.6) for a large T and $t > s > T$,

$$\phi(z'(t)) - \phi(z'(\tau)) = -\int_\tau^t \{c(s)\phi_\beta(z) + d(s)\phi_\gamma(z) - f(s)\}ds$$

whose second member is strictly negative. So z' is eventually decreasing and tends to 0 since z is bounded.

(2) Also from (2.6),

$$\begin{aligned} a'(t)\phi(z') + a(t)\frac{z''}{z'}z'\phi'(z') &= a'(t)\phi(z') + a(t)\alpha z''\frac{\phi(z')}{z'} \\ &= -\{c(t)\phi_\beta(z) + d(t)\phi_\gamma(z) - f(t)\}. \end{aligned}$$

As a' decays to 0, the last member is eventually negative while the one before last has the same sign as z'' eventually, if $\phi(z) > m > 0$ eventually. We then have the same conclusion as in (1). $\square$

Theorem 2.4. *Let z be a bounded non-trivial solution of* (2.6).

Under the hypotheses of (1), and (H1) of Lemma 2.3, z is oscillatory in $\mathbb{R}$.

Under the hypotheses of (2), and (H1) of lemma 2.3, z is oscillatory if $f \not\equiv 0$ in any D_T; otherwise it will be oscillatory unless

$$\lim \inf_{t \nearrow \infty} |z(t)| = 0\,. \tag{2.7}$$

It is easy to verify that under the conditions that $|f(t)|$ is eventually bounded and the functions c and d are eventually positive and unbounded, the conclusions of the theorem still hold.

Proof of Theorem 2.4. Assume that there is such a non-oscillatory solution z; i.e., $z > 0$ in some D_T. Then the non-negative function

$$H(t) := \frac{a(t)\phi(z')}{\phi(z)} = a(t)\phi(\frac{z'}{z})$$

satisfies, eventually,

$$H'(t) = -\{c(t)|z|^{\beta-\alpha} + d(t)|z|^{\gamma-\alpha}\} + \frac{f(t)}{\phi(z)} - \frac{\alpha a(t)}{\phi(z)}|z'|^{\alpha+1} \le \frac{f(t)}{\phi(z)}. \tag{2.8}$$

Therefore, $H(t) \le H(T) + \int_T^t \frac{f(s)}{\phi(z)}ds$ which is invalid for large $T > 0$ as the right hand side is eventually negative. Such a solution cannot be non-oscillatory unless (2.7) holds for the case 2. $\square$

3. Some Picone-type formulae and results in one-dimensional equations

We consider the equations

$$\begin{gathered}\{a(t)\phi(y')\}' + c(t)\phi(y) = 0,\\ \{a_1(t)\phi(z')\}' + c_1(t)\phi(z) + h(t,z,z') = 0\end{gathered} \tag{3.1}$$

and define the two-form ζ on $C^1(\mathbb{R},\mathbb{R})$ for $\gamma > 0$ and $u, v \in C^1(\mathbb{R})$, by

$$\zeta_\gamma(u,v) := |u'|^{\gamma+1} - (\gamma+1)\phi_\gamma(\frac{u}{v}v')u' + \gamma|\frac{u'}{v}v'|^{\gamma+1} \tag{3.2}$$

which is non negative and null only if there exists $k \in \mathbb{R}$ such that $u = kv$. (see e.g. [3]). Easy verifications show that if y and z are solutions of (3.1), then wherever $z \neq 0$,

$$\begin{aligned}\{ya(t)\phi(y') - y\phi(\frac{y}{z})a_1(t)\phi(z')\}' &= a_1(t)\zeta_\alpha(y,z) + [a(t) - a_1(t)]|y'|^{\alpha+1}\\ &\quad + [c_1(t) - c(t)]|y|^{\alpha+1} + |y|^{\alpha+1}\frac{h(t,z,z')}{\phi(z)}\end{aligned} \tag{3.3}$$

Given the importance of the half-linear equations (when $h \equiv 0$ in (3.1)) in our investigation, we have the following result.

Theorem 3.1. *Assume that $a \in C^1(\mathbb{R},(0,\infty))$ is non-decreasing and $c \in C(\mathbb{R},\mathbb{R})$ is strictly positive in some D_T. Then for any $\alpha > 0$ the half-linear equation*

$$\{a(t)\phi_\alpha(z')\}' + c(t)\phi_\alpha(z) = 0, \quad t > 0 \tag{3.4}$$

is strongly oscillatory. Moreover, if M is a positive constant, then

$$\{a(t)\phi_\alpha(u')\}' + c(t)\phi_\alpha(u) + a(t)M = 0, \quad t > 0 \tag{3.5}$$

is strongly oscillatory.

In case where $M < 0$ but large enough, we have the same conclusion unless

$$\liminf_{t\nearrow\infty} |u(t)| = 0.$$

Proof of Theorem 3.1. Assume that there is a solution z of (3.4) which is strictly positive in D_T. From (2.4), $\{a_0(t)\phi_\alpha(y')\}' + c_0(t)\phi_\alpha(y) = 0$ is strongly oscillatory where for some $k_0 \le 0$, $a(t) \le \exp\{-k_0\alpha t\} := a_0(t)$ and $c(t) \ge |k_0|^{\alpha+1}\alpha\exp\{k_0 t\} := c_0(t)$ in some D_T.

substituting a and c in (3.3) (where $h \equiv 0$), we obtain, in D_T,

$$\begin{aligned}&\{ya_0(t)\phi(y') - y\phi(\ \frac{y}{z}\ )a(t)\phi(z')\}'\\ &= a(t)\zeta_\alpha(y,z) + [a_0(t) - a(t)]|y'|^{\alpha+1} + [c(t) - c_0(t)]|y|^{\alpha+1} > 0.\end{aligned}$$

The integration over any nodal set $D(y) \subset D_T$ of the above equation leads to an absurdity as the right-hand side will be strictly positive. The solution z cannot be eventually positive.

Let u be a bounded and non-trivial solution of (3.5). Wherever it is non-null, (3.3) applied to (3.4) and (3.5) gives

$$\{az\phi(z') - z\phi(\frac{z}{u})a\phi(u')\}' = a(t)\zeta_\alpha(z,u) + |z|^{\alpha+1}a(t)\frac{M}{\phi(u)}$$

and the conclusion follows as before if $M \geq 0$. If M is a negative but large enough and $u > \mu > 0$ in D_T for some $\mu > 0$ then in some D_T, $a(t)\{\zeta_\alpha(z,u)+|z|^{\alpha+1}\frac{M}{\phi(u)}\} < 0$ and we reach the same conclusion. □

Remark 3.2. The result of Theorem 3.1 includes the case where $a(t) \equiv c(t)$ is an increasing function and positive in some D_T.

Theorem 3.1 shows that besides some well known oscillation criteria for half-linear elliptic equations [13, 5, 10], (H0) and (H1) provide some other important criteria.

Now we consider the equation

$$\{a(t)\phi(z')\}' + c(t)\phi(z) + q(t)\phi(z') = f(t); \quad t > 0. \tag{3.6}$$

Theorem 3.3. *Assume that*

- (i) $a \in C^1(\mathbb{R},(0,\infty))$ *is non decreasing with decaying* a' *and* $c \in C(\mathbb{R},\mathbb{R}))$ *is strictly positive in some* D_T*;*
- (ii) $q \in C(\mathbb{R})$ *is bounded and* $f \in C(\mathbb{R},\mathbb{R})$ *is non positive.*

(a) If q *is eventually positive then any non-trivial and bounded solution* z *of* (3.6) *is oscillatory .*
(b) But if q *is not eventually positive,* z *is oscillatory unless*

$$\liminf_{t\nearrow\infty} |z(t)| = 0. \tag{3.7}$$

Proof. As before, from the hypotheses, eventually from (3.6),

$$a'(t)\phi(z') + \alpha a(t)z''\frac{\phi(z')}{z'} = -\{c(t)\phi(z) + q(t)\phi(z') - f(t)\} < 0$$

with $|a'(t)\phi(z')|$ decaying to 0. As for Lemma 2.3, z and z' are both positive with z' decreasing to 0. Let M be a very large positive number and y an oscillatory solution of

$$\{a(t)\phi_\alpha(y')\}' + c(t)\phi_\alpha(y) + a(t)M = 0.$$

Assume that there is such a solution z of (3.6) which is eventually positive. Then as in (3.3), wherever $z > 0$,

$$\begin{aligned} &\{ya(t)\phi(y') - y\phi(\frac{y}{z})a(t)\phi(z')\}' \\ &= a(t)\zeta_\alpha(y,z) + a(t)y\{M + q(t)\phi(\frac{y}{z}z')\} - |y|^{\alpha+1}\frac{f(t)}{\phi(z)}. \end{aligned} \tag{3.8}$$

If we integrate over a nodal set $D(y) \subset D_T$, where elements are positive, the above equation yields

$$0 = \int_{D(y)} \{a(t)\zeta_\alpha(y,z) + a(t)y\{M + q(t)\phi(\frac{y}{z}z')\} - |y|^{\alpha+1}\frac{f(t)}{\phi(z)}\}dt. \tag{3.9}$$

(a) As in the proof of Lemma 2.3, if a is a positive constant then if z' is eventually positive, so is z' and (even without the help of M) the right-hand side of (3.8) is strictly positive which is a contradiction.

(b) In this case, if $M > 0$ is large enough, we obtain the same conclusion provided that eventually $z > \mu > 0$ for some $\mu > 0$; in fact $M+q(t)\phi(\frac{y}{z}z')$ needs to be positive for a fixed large M. □

For $a \in C^1(\mathbb{R}^n, (0,\infty))$ and $c \in C(\mathbb{R},\mathbb{R})$ such that for some $T > 0$, c, $a' > 0$ in D_T and a large M, we consider a strongly oscillatory solution y of

$$\{a(t)\phi(y')\}' + c(t)\phi(y) - a(t)M = 0. \tag{3.10}$$

Consider the equation

$$\{a(t)\phi(z')\}' + c(t)\phi(z) + q(t)\phi(z') = 0 \tag{3.11}$$

where there exists $Q \in C^1(\mathbb{R},\mathbb{R})$ and $k \in C(\mathbb{R},\mathbb{R})$; $\quad Q'(t) = q(t) + k(t)$.

For a solution z of (3.11) and y that of (3.10), wherever $z \neq 0$,

$$\begin{aligned} &\big\{a(t)y\phi(y') - y\phi(\frac{y}{z})a(t)\phi(z') - y\phi(\frac{y}{z})a(t)Q(t)\phi(z')\big\}' \\ &= a(t)\zeta_\alpha(y,z) + a(t)y\{M - k(t)\phi(\frac{y}{z}z')\} - Q(t)\Big(ya(t)\phi(\frac{y}{z}z')\Big)'. \end{aligned} \tag{3.12}$$

We then have the following result.

Theorem 3.4. *Assume that there are*

(i) $Q \in C^1(\mathbb{R},\mathbb{R}); q,\ k \in C(\mathbb{R},\mathbb{R})$ *such that* $Q'(t) = q(t) + k(t)$*;*
(ii) $a \in C^1(\mathbb{R},(0,\infty))$ *and* $c \in C(\mathbb{R},\mathbb{R})$ *such that* $c, a' > 0$ *in some* D_T.

Then any non-trivial and bounded solution z *of*

$$\{a(t)\phi(z')\}' + c(t)\phi(z) + q(t)\phi(z') = 0 \tag{3.13}$$

(a) is oscillatory if $k \equiv 0$*;*
(b) is oscillatory if $k \not\equiv 0$ *and bounded in* D_T*, unless* $\liminf_{t\nearrow\infty} |z(t)| = 0$.

Proof. If in (3.11) we replace Q by $Q + \mu$ with $\mu \in \mathbb{R}$, (3.12) remains valid with $Q+\mu$. If there is a solution z of (3.13) which is positive in some D_T, the integration of (3.12) over any nodal set $D(y^+) \subset D_T$ gives

$$\begin{aligned} 0 = &\int_{D(y^+)} a(t)\Big(\zeta_\alpha(y,z) + y\{M - k(t)\phi(\frac{y}{z}z')\}\Big)dt \\ &- \int_{D(y^+)} \big(Q(t)+\mu\big)[a(t)y\phi(\frac{y}{z}z')]'\,dt, \quad \forall \mu \in \mathbb{R}. \end{aligned} \tag{3.14}$$

The formula (3.14) can only hold if each integrand in the formula in null in D_T; in particular only if

$$a(t)[\zeta_\alpha(y,z) + y\{M - k(t)\phi(\frac{y}{z}z')\}] = 0$$

in any $D(y^+) \subset D_T$.

(a) If $k \equiv 0$ this is absurd for $M \geq 0$. Therefore, the assumption is false; z cannot be eventually positive.

(b) If $k \not\equiv 0$ but bounded with $z > \nu$ for some $\nu > 0$ in D_T, the same conclusion holds by choosing a large enough $M > 0$. □

4. Multidimensional case

If $w \in C^1(\mathbb{R}^n,\mathbb{R})$ is radially symmetric; i.e., $w(x) := W(r) := W(|x|)$ for some $W \in C^1(\mathbb{R})$ then easy but elaborate calculations show that

$$\nabla w(x) = W'(r)\frac{X}{|X|} \quad \text{and} \quad \nabla\cdot\{a(r)\Phi(\nabla w)\} = \frac{1}{r^{n-1}}\{r^{n-1}a(r)\phi(W')\}'$$

and for $B \in C(\mathbb{R}^n, \mathbb{R}^n)$, $B(x) \cdot \Phi(\nabla u) = B(x) \cdot \frac{X}{|X|}\phi(U')$, where $a \in C^1(\mathbb{R})$ say, $X =^t (x_1, x_2, \dots, x_n)$ denotes the position-vector and $r := |X| = \sqrt{\{\sum_{i=1}^n x_i^2\}}$ its module.

Consider the operators

$$P(u) := \nabla \cdot \{A(x)\Phi(\nabla u)\} + C(x)\phi(u) + B_1(x) \cdot \Phi(\nabla u); \tag{4.1}$$

$$R(u) := \nabla \cdot \{a(r)\Phi(\nabla u)\} + c(r)\phi(u) + B_2(x) \cdot \Phi(\nabla u) + F(x) \tag{4.2}$$

where the real functions a, A are positive and continuously differentiable, c, C, F are continuous in all their arguments and $B_i \in C(\mathbb{R}^n, \mathbb{R}^n)$. If a function u in (4.2) is radially symmetric; i.e.m $u(x) := U(|x|) = U(r)$, then, in terms of U, (4.2) reads

$$\begin{aligned} R_1(U) = \{&r^{n-1}\, a(r)\phi(U')\}' + r^{n-1}\, c(r)\phi(U) \\ &+ r^{n-1}\Big(B_2(x) \cdot \frac{X}{|X|}\phi(U') + F(x)\Big). \end{aligned} \tag{4.3}$$

If the regular functions u and v satisfy $Pu = Rv = 0$ in $\mathbb{R}^n$, then a Picone formula reads

$$\begin{aligned} &\nabla \cdot \{uA(x)\Phi(\nabla u) - u\phi(\frac{u}{v})a(r)\Phi(\nabla v)\} \\ &= a(r)Z_\alpha(u,v) + \big(A(x) - a(r)\big)|\nabla u|^{\alpha+1} + \big(c(r) - C(x)\big)|u|^{\alpha+1} \\ &\quad + |u|^{\alpha+1}[B_2(x) \cdot \Phi(\frac{\nabla v}{v}) - B_1(x) \cdot \Phi(\frac{\nabla u}{u})] + |u|^{\alpha+1}\frac{F(x)}{\phi(v)} \end{aligned} \tag{4.4}$$

where for all $\gamma > 0$ and all $u, v \in C^1(\mathbb{R}^n)$,

$$Z_\gamma(u,v) := |\nabla u|^{\gamma+1} - (\gamma+1)\Phi_\gamma(\frac{u}{v}\nabla v) \cdot \nabla u + \gamma|\frac{u}{v}\nabla v|^{\gamma+1}.$$

If the coefficients a and c were not radially symmetric, but $a_1(x)$ and $c_1(x)$ are, then (4.1) would be the same with $a_1(x)$ and $c_1(x)$ replacing them.

We recall that for all $\gamma > 0$ the two-form $Z_\gamma(u,v) \ge 0$ and is null only if either $uv = 0$ or there exist $k \in \mathbb{R}$ with $u = kv$. (see e.g. [5, 8, 10]).

For easy writing we define for $h \in C(\mathbb{R}^n, \mathbb{R})$ and $H \in C(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned} h^+(r) &:= \max_{|x|=r} h(x), \quad H^+(r) := \max_{|x|=r} H(x) \cdot \frac{X}{|X|}, \\ h^-(r) &:= \min_{|x|=r} h(x), \quad H^-(r) := \min_{|x|=r} H(x) \cdot \frac{X}{|X|}. \end{aligned} \tag{4.5}$$

In [13], we have the equation

$$\nabla \cdot \big(\Phi_\alpha(\nabla v)\big) + \phi_\beta(v) + \phi_\gamma(v) = 0$$

where $0 < \gamma < \alpha < \beta$. Here we consider the more general equation

$$\nabla \cdot \big(\Phi_\alpha(\nabla v)\big) + \phi_\beta(v) + \phi_\gamma(v) + B(x) \cdot \Phi_\alpha(\nabla v) + F(x) = 0 \tag{4.6}$$

where $B \in C(\mathbb{R}^n, \mathbb{R}^n)$. If $v(x) := z(r)$ is a radially symmetric solution of (4.6), then

$$\Big(r^{n-1}\phi_\alpha(z')\Big)' + r^{n-1}\{\phi_\beta(z) + \phi_\gamma(z) + B(x) \cdot \frac{X}{|X|}\phi_\alpha(z') + F(x)\} = 0. \tag{4.7}$$

Let y be a strongly oscillatory solution of (see Remark 3.2 and Theorem 3.1)

$$\Big(r^{n-1}\phi_\alpha(y')\Big)' + r^{n-1}\big(\phi_\alpha(y) - M\big) = 0.$$

Then

$$
\begin{aligned}
&\{yr^{n-1}\phi_\alpha(y') - r^{n-1}y\phi_\alpha(\frac{y}{z})\phi_\alpha(z')\}' \\
&= r^{n-1}[\zeta_\alpha(y,z) + |y|^{\alpha+1}\Big(|z|^{\beta-\alpha} + |z|^{\gamma-\alpha} - 1\Big) \\
&\quad + y\{M + B(x)\cdot\frac{X}{|X|}\phi_\alpha(\frac{yz'}{z}) + F(x)\phi(\frac{y}{z})\}] \\
&= r^{n-1}\big[\zeta_\alpha(y,z) + |y|^{\alpha+1}\Big(|z|^{\beta-\alpha} + |z|^{\gamma-\alpha}\Big) \\
&\quad + y\{M - \phi_\alpha(y) + B(x)\cdot\frac{X}{|X|}\phi_\alpha(\frac{yz'}{z}) + F(x)\phi(\frac{y}{z})\}\big].
\end{aligned} \tag{4.8}
$$

For $R > 0$, define $\Omega_R := \{x \in \mathbb{R}^n : |x| > R\}$.

Theorem 4.1. *Assume that The functions $B(x)\cdot X/|X|$ and $F(x)$ are radially symmetric and bounded in some Ω_R. Then any non-trivial and bounded radially symmetric solution z of*

$$
\nabla\cdot\big(\Phi_\alpha(\nabla z)\big) + \phi_\beta(z) + \phi_\gamma(z) + B(x)\cdot\Phi_\alpha(\nabla z) + F(x) = 0 \tag{4.9}
$$

is oscillatory, unless

$$
\lim\inf_{r\nearrow\infty}|z(r)| = 0. \tag{4.10}
$$

Proof. It is sufficient to note that in (4.8), if $|z| > \mu > 0$ in Ω_R, as

$$
|\{B(x)\cdot\frac{X}{|X|}\phi_\alpha(\frac{yz'}{z}) + F(x)\phi(\frac{y}{z}) - \phi(y)\}|
$$

is uniformly bounded under the hypotheses, for $M > 0$ large enough,

$$
\{M - \phi(y) + B(x)\cdot\frac{X}{|X|}\phi_\alpha(\frac{yz'}{z}) + F(x)\phi(\frac{y}{z})\} > 0.
$$

So such a solution z cannot be eventually positive, unless (4.10) holds. □

5. Main results

We start with an important link between multi-dimensional and one-dimensional oscillation criterion for half-linear operators, and some oscillation criteria by means of the comparison method.

Theorem 5.1. *(1) For any regular functions $a, c \in C(\mathbb{R}^n, \mathbb{R})$, if the equation*

$$
\{r^{n-1}\, a^+(r)\phi(y')\}' + r^{n-1}c^-(r)\phi(y) = 0
$$

is oscillatory in $\mathbb{R}$, then so is $\nabla\cdot\{a(x)\Phi(\nabla u)\} + c(x)\phi(u) = 0$ in $\mathbb{R}^n$. (see [5, Theorem 3.1]*)*

(2) If $\nabla\cdot\{a(x)\Phi(\nabla u)\} + c(x)\phi(u) = 0$ is strongly oscillatory, then any bounded solution v of the equation

$$
\nabla\cdot\{a(x)\Phi(\nabla v)\} + c(x)\phi(v) + M = 0 \tag{5.1}
$$

is oscillatory: (i) for all $M \ge 0$; (ii) for all $M < 0$, provided that it is large enough, unless $\lim\inf_{|x|\nearrow\infty}|v(x)| = 0$.

Proof. (1) As in (4.4),

$$\begin{aligned} &\nabla\cdot\{a^+(r)y\Phi(\nabla y)-y\phi(\frac{y}{u})a(x)\Phi(\nabla u)\}\\ &=a(x)Z(y,u)+(a^+-a)|\nabla y|^{\alpha+1}+(c-c^-)|y|^{\alpha+1} \end{aligned}\tag{5.2}$$

which for non-null and distinct u and y is strictly positive. If u is assumed to be eventually strictly positive in some Ω_T, then the integration of (5.2) over any nodal set $D(y)\subset\Omega_T$ would lead to a contradiction. Thus u cannot be eventually positive.

(2) In this case, with $\mu\in\{M,-M\}$, if we assume that $\nabla\cdot\{a(x)\Phi(\nabla v)\}+c(x)\phi(v)+\mu=0$ has a non-trivial and bounded solution v which is strictly positive in some Ω_T then in application of (4.4),

$$\nabla\cdot\{ua(x)\Phi(\nabla u)-u\phi(\frac{u}{v})a(x)\Phi(\nabla v)\}=a(x)Z(u,v)+\mu u\phi(\frac{u}{v}).$$

If $\mu\geq 0$ then the right-hand side of the above equation is strictly positive; but if $\mu<0$ but very large, $a(x)Z(u,v)+\mu u\phi(\frac{u}{v})<0$ provided that $v>\nu$ in Ω_T for some $\nu>0$. In both cases, integration over any nodal set $D(u)\subset\Omega_T$ leads to a contradiction. □

It is important to mention that for the above result, M can be replaced by $a(x)M$, a being that in the concerning equation $\nabla\cdot\{a(x)\Phi(\nabla u)\}+c(x)\phi(u)=0$ which is assumed bounded below in some Ω_T by a positive constant in the case where the result is based on "big M". In fact in this case the right hand side of the equation above reads $a(x)\{Z(u,v)+Mu\phi(\frac{u}{v})\}$.

Now we go back to the equation

$$\nabla\cdot\Big(A(x)\Phi(\nabla v)\Big)+A(x)B(x)\cdot\Phi(\nabla v)+C(x)\phi_\beta(v)+D(x)\phi_\gamma(v)+f(x)=0 \tag{5.3}$$

where as said before, $A\in C^1(\mathbb{R}^n,(0,\infty))$, $f,C,D\in C(\mathbb{R}^n,\mathbb{R})$; $B\in C(\mathbb{R}^n,\mathbb{R}^n)$ and $\beta,\gamma>0$. We suppose that there exists $b\in C^1(\mathbb{R}^n,\mathbb{R})$ such that

$$\nabla b(x)=B(x)+K(x),\tag{5.4}$$

where $K\in C(\mathbb{R}^n,\mathbb{R}^n)$ is bounded.

Let u be a strongly oscillatory solution of

$$\nabla\cdot\big(A(x)\Phi(\nabla u)\big)+C_1(x)\phi_\beta(u)-A(x)M=0\tag{5.5}$$

where $M>0$ and $\frac{C_1}{A}$ bounded.

Developments as those give for v in (5.3) (formally) lead to

$$\begin{aligned} &\nabla\cdot\{uA(x)\Phi(\nabla u)-u\phi(\frac{u}{v})A(x)\Phi(\nabla v)\}\\ &=A(x)Z_\alpha(u,v)+|u|^{\alpha+1}\Big(C(x)|v|^{\beta-\alpha}-C_1(x)|u|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\Big)\\ &\quad+u[A(x)\{B(x)\cdot\Phi(\frac{u}{v}\nabla v)+M\}+f(x)\phi(\frac{u}{v})]. \end{aligned}$$

As

$$\begin{aligned} &\nabla\cdot\{u\phi(\frac{u}{v})b(x)A(x)\Phi(\nabla v)\}\\ &=b(x)\{A(x)\{|\nabla u|^{\alpha+1}-Z_\alpha(u,v)\}-u\Big(A(x)B(x)\cdot\Phi(\frac{u\nabla v}{v})+f(x)\phi(\frac{u}{v})\Big)\\ &\quad-|u|^{\alpha+1}[C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}]\}+uA(x)(B(x)+K(x))\cdot\Phi(\frac{u\nabla v}{v}), \end{aligned}$$

we have

$$\begin{aligned}
&\nabla\cdot\{uA(x)\Phi(\nabla u)-u\phi(\frac{u}{v})A(x)\Phi(\nabla v)-u\phi(\frac{u}{v})b(x)A(x)\Phi(\nabla v)\}\\
&=A(x)Z_\alpha(u,v)+|u|^{\alpha+1}\big[C(x)|v|^{\beta-\alpha}-C_1(x)|u|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\big]\\
&\quad+A(x)u\Big(M-K(x).\Phi(\frac{u\nabla v}{v})\Big)+uf(x)\phi(\frac{u}{v})\\
&+b(x)\{A(x)[Z_\alpha(u,v)-|\nabla u|^{\alpha+1}]+uA(x)B(x)\cdot\Phi(\frac{u\nabla v}{v})\\
&\quad+uf(x)\phi(\frac{u}{v})+|u|^{\alpha+1}\{C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\}\}
\end{aligned}$$

and

$$\begin{aligned}
&\nabla\cdot\{uA(x)\Phi(\nabla u)-u\phi(\frac{u}{v})A(x)\Phi(\nabla v)-u\phi(\frac{u}{v})b(x)A(x)\Phi(\nabla v)\}\\
&=A(x)Z_\alpha(u,v)+|u|^{\alpha+1}[C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}]\\
&\quad+A(x)u\Big(M-K(x)\cdot\Phi(\frac{u\nabla v}{v})-\frac{C_1(x)}{A(x)}\phi_\beta(u)\Big)+uf(x)\phi(\frac{u}{v})\\
&\quad+b(x)\{A(x)[Z_\alpha(u,v)-|\nabla u|^{\alpha+1}]+uA(x)B(x)\cdot\Phi(\frac{u\nabla v}{v})\\
&\quad+uf(x)\phi(\frac{u}{v})+|u|^{\alpha+1}\{C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\}\}.
\end{aligned}\tag{5.6}$$

Theorem 5.2. *Consider the equation* (5.3) *where*

- (i) $A\in C^1(\mathbb{R}^n,(0,\infty))$;
- (ii) $f,C,D\in C(\mathbb{R}^n,\mathbb{R})$ *are positive in some* Ω_R;
- (iii) *there exist* $b\in C^1(\mathbb{R}^n,\mathbb{R})$, $K\in C(\mathbb{R}^n,\mathbb{R}^n)$ *with* $b(x):=B(x)+K(x)$ *such that* K *is eventually bounded.*

If v is a bounded non-trivial solution of (5.3)*, then (a)* (5.3) *is strongly oscillatory if* $K\equiv 0$*; (b)* (5.3) *is strongly oscillatory if* $K\not\equiv 0$*, unless*

$$\lim\inf_{|x|\nearrow\infty}|v(x)|=0.$$

Proof. Assume that there is a solution v of (5.3) which is not oscillatory; e.g., There exists $\rho\ge R$ such that $v>0$ in Ω_ρ.

In (5.5) the function b can be replaced by a $b_1:=b+\mu$, for any constant $\mu\in\mathbb{R}$. So after such a replacement the integration of the resulting equation over any nodal set $D(u^+)\subset\Omega_\rho$ ($u>0$ in $D(u^+)$ and $u|_{\partial D(u^+)}=0$), we obtain that for all $\mu\in\mathbb{R}$,

$$\begin{aligned}
0=&\int_{D(u^+)}[A(x)Z_\alpha(u,v)+|u|^{\alpha+1}\Big(C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\Big)\\
&+A(x)u\Big(M-K(x)\cdot\Phi(\frac{u\nabla v}{v})-\frac{C_1(x)}{A(x)}\phi_\beta(u)\Big)+uf(x)\phi(\frac{u}{v})]dx\\
&+\int_{D(u^+)}\{b(x)+\mu\}\{A(x)[Z_\alpha(u,v)-|\nabla u|^{\alpha+1}]+uA(x)B(x)\cdot\Phi(\frac{u\nabla v}{v})\\
&+uf(x)\phi(\frac{u}{v})+|u|^{\alpha+1}\{C(x)|v|^{\beta-\alpha}+D(x)|v|^{\gamma-\alpha}\}\}dx
\end{aligned}\tag{5.7}$$

which could hold only if each integrand is null; in particular that in the first integral. In that integrand, all terms are non negative except for

$$\Big(M-K(x)\cdot\Phi(\frac{u\nabla v}{v})-\frac{C_1(x)}{A(x)}\phi_\beta(u)\Big).$$

But from the hypotheses, this formula is positive.

(a) if $K \equiv 0$ and $M > 0$ big enough. Then v cannot be eventually positive;

(b) if $K \not\equiv 0$ the same conclusion prevails unless, because of the term $\Phi(\frac{u\nabla v}{v})$, $\liminf_{|x|\nearrow\infty} |v(x)| = 0$. $\square$

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