Electronic Journal of Differential Equations, Vol. 2011 (2011), No. 154, pp. 1–19.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE OF POSITIVE SOLUTIONS FOR NONLINEAR SYSTEMS OF SECOND-ORDER DIFFERENTIAL EQUATIONS WITH INTEGRAL BOUNDARY CONDITIONS ON AN INFINITE INTERVAL IN BANACH SPACES

XU CHEN, XINGQIU ZHANG

Abstract. The article shows the existence of positive solutions for systems of nonlinear singular differential equations with integral boundary conditions on an infinite interval in Banach spaces. Our main tool is the Mönch fixed point theorem combined with a monotone iterative technique. In addition, an explicit iterative approximation of the solution is provided.

1. Introduction

The theory of boundary-value problems with integral boundary conditions for ordinary differential equations arises in different areas of applied mathematics and physics. For example, heat conduction, chemical engineering,underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. There are many excellent results about the existence of positive solutions for integral boundary value problems in scalar case (see, for instance, [7, 9, 10, 13] and references therein). Very recently, by using Schauder fixed point theorem, Guo [2] obtained the existence of positive solutions for a class of nth-order nonlinear impulsive singular integro-differential equations in a Banach space.

Recently, Zhang et al [14], using the cone theory and monotone iterative technique, investigated the existence of minimal nonnegative solution of the following nonlocal boundary value problems for second-order nonlinear impulsive differential equations on an infinite interval with an infinite number of impulsive times

$$\begin{gathered}
-x''(t) = f(t, x(t), x'(t)), \quad t \in J, t \neq t_k, \\
\Delta x|_{t=t_k} = I_k(x(t_k)), \quad k = 1, 2, \dots, \\
\Delta x'|_{t=t_k} = \overline{I}_k(x(t_k)), \quad k = 1, 2, \dots, \\
x(0) = \int_0^\infty g(t)x(t)\,\mathrm{d}t, \quad x'(\infty) = 0,
\end{gathered}$$

2000 *Mathematics Subject Classification.* 34B15, 34B16, 34B40.
Key words and phrases. Singular differential equations; integral boundary; cone; ordering; positive solution; Mönch fixed point theorem.

Submitted August 22, 2011. Published November 10, 2011.

where $J = [0, +\infty)$, $f \in C(J \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty]$, $0 < t_1 < t_2 < \dots < t_k < \dots$, $t_k \to \infty$, $I_k \in C[\mathbb{R}^+, \mathbb{R}^+]$, $\overline{I}_k \in C[\mathbb{R}^+, \mathbb{R}^+]$, $g(t) \in C[\mathbb{R}^+, \mathbb{R}^+)$, with $\int_0^\infty g(t)\,\mathrm{d}t < 1$.

To the best of our knowledge, only a few authors have studied integral boundary value problems in Banach spaces and results for systems of second-order differential equation are rarely seen. Motivated by Zhang and Guo's work, in this paper, we consider the following singular integral boundary value problem on an infinite interval in a Banach space E:

$$\begin{gathered} x''(t) + f(t, x(t), x'(t), y(t), y'(t)) = 0, \\ y''(t) + g(t, x(t), x'(t), y(t), y'(t)) = 0, \quad t \in J_+, \\ x(0) = \int_0^\infty q(t)x(t)\,\mathrm{d}t, \quad x'(\infty) = x_\infty, \\ y(0) = \int_0^\infty h(t)y(t)\,\mathrm{d}t, \quad y'(\infty) = y_\infty, \end{gathered} \tag{1.1}$$

where $J = [0, \infty)$, $J_+ = (0, \infty)$, and the functions $q(t), h(t)$ are in $L[0, \infty)$ with $\int_0^\infty q(t)\,\mathrm{d}t < 1$, $\int_0^\infty h(t)\,\mathrm{d}t < 1$ and $\int_0^\infty tq(t)\,\mathrm{d}t < \infty$, $\int_0^\infty th(t)\,\mathrm{d}t < \infty$. $x'(\infty) = \lim_{t\to\infty} x'(t)$, $y'(\infty) = \lim_{t\to\infty} y'(t)$. The nonlinear terms $f(t, x_0, x_1, y_0, y_1)$ and $g(t, x_0, x_1, y_0, y_1)$ permit singularities at $t = 0$ and $x_i, y_i = \theta$ $(i = 0, 1)$, where θ denotes the zero element of Banach space E. By singularity, we mean that $\|f(t, x_0, x_1, y_0, y_1)\| \to \infty$ as $t \to 0^+$ or $x_i, y_i \to \theta$.

The main features of this article are as follows: Firstly, compared with [14], the systems of integral boundary value problem we discussed here is in Banach spaces and nonlinear term permits singularity not only at $t = 0$ but also at $x_i, y_i = \theta$ $(i = 0, 1)$. Secondly, compared with [2], the problem we discussed here is systems of integral boundary value problem, since the problem we discuss is the integral boundary value problems, the construction of bounded convex closed set is different from that in [2]. Furthermore, the relative compact conditions we used are weaker. Finally, an iterative sequence for the solution under some normal type conditions is established which makes it convenient in applications.

2. Preliminaries

Let

$$FC[J, E] = \{x \in C[J, E] : \sup_{t\in J} \frac{\|x(t)\|}{t+1} < \infty\},$$

$$DC^1[J, E] = \{x \in C^1[J, E] : \sup_{t\in J} \frac{\|x(t)\|}{t+1} < \infty \text{ and } \sup_{t\in J} \|x'(t)\| < \infty\}.$$

Evidently, $C^1[J, E] \subset C[J, E]$ and $DC^1[J, E] \subset FC[J, E]$. It is easy to see that $FC[J, E]$ is a Banach space with norm

$$\|x\|_F = \sup_{t\in J} \frac{\|x(t)\|}{t+1},$$

and $DC^1[J, E]$ is also a Banach space with norm

$$\|x\|_D = \max\{\|x\|_F, \|x'\|_1\},$$

where

$$\|x'\|_1 = \sup_{t\in J} \|x'(t)\|.$$

Let $X = DC^1[J,E] \times DC^1[J,E]$ with norm $\|(x,y)\|_X = \max\{\|x\|_D, \|y\|_D\}$, for $(x,y) \in X$. Then $(X, \|\cdot,\cdot\|_X)$ is also a Banach space. The basic space using in this paper is $(X, \|\cdot,\cdot\|_X)$.

Let P be a normal cone in E with normal constant N which defines a partial ordering in E by $x \le y$. If $x \le y$ and $x \ne y$, we write $x < y$. Let $P_+ = P\backslash\{\theta\}$. So, $x \in P_+$ if and only if $x > \theta$. For details on cone theory, see [4].

In what follows, we always assume that $x_\infty \ge x_0^*$, $y_\infty \ge y_0^*$, $x_0^*, y_0^* \in P_+$. Let $P_{0\lambda} = \{x \in P : x \ge \lambda x_0^*\}$, $P_{1\lambda} = \{y \in P : y \ge \lambda y_0^*\}(\lambda > 0)$. Obviously, $P_{0\lambda}, P_{1\lambda} \subset P_+$ for any $\lambda > 0$. When $\lambda = 1$, we write $P_0 = P_{01}, P_1 = P_{11}$; i.e., $P_0 = \{x \in P : x \ge x_0^*\}$, $P_1 = \{y \in P : y \ge y_0^*\}$. Let $P(F) = \{x \in FC[J,E] : x(t) \ge \theta, \forall\ t \in J\}$, and $P(D) = \{x \in DC^1[J,E] : x(t) \ge \theta, x'(t) \ge \theta, \forall\ t \in J\}$. It is clear, $P(F)$ and $P(D)$ are cones in $FC[J,E]$ and $DC^1[J,E]$, respectively. A map $(x,y) \in DC^1[J,E] \cap C^2[J_+,E]$ is called a positive solution of (1.1) if $(x,y) \in P(D) \times P(D)$ and $(x(t), y(t))$ satisfies (1.1).

Let $\alpha, \alpha_F, \alpha_D, \alpha_X$ denote the Kuratowski measure of non-compactness in the sets $E, FC[J,E], DC^1[J,E]$ and X, respectively. For details on the definition and properties of the measure of non-compactness, the reader is referred to references [1, 3, 4, 6]. Let

$$D_0 = \Big(1 + \frac{\int_0^\infty q(t)\,\mathrm{d}t}{1 - \int_0^\infty q(t)\,\mathrm{d}t}\Big), \quad D_1 = \Big(1 + \frac{\int_0^\infty h(t)\,\mathrm{d}t}{1 - \int_0^\infty h(t)\,\mathrm{d}t}\Big), \tag{2.1}$$

$D^* = \max\{D_0, D_1\}$. Denote

$$\lambda_0^* = \min\{\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1 - \int_0^\infty q(t)\,\mathrm{d}t}, 1\}, \quad \lambda_1^* = \min\{\frac{\int_0^\infty th(t)\,\mathrm{d}t}{1 - \int_0^\infty h(t)\,\mathrm{d}t}, 1\}$$

Let us list some conditions for convenience.

(H1) $f, g \in C[J_+ \times P_{0\lambda} \times P_{0\lambda} \times P_{1\lambda} \times P_{1\lambda}, P]$ for any $\lambda > 0$ and there exist $a_i, b_i, c_i \in L[J_+, J]$ and $z_i \in C[J_+ \times J_+ \times J_+ \times J_+, J]$ $(i = 0,1)$ such that

$$\|f(t, x_0, x_1, y_0, y_1)\| \le a_0(t) + b_0(t) z_0(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$

for all $t \in J_+, x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*}$;

$$\|g(t, x_0, x_1, y_0, y_1)\| \le a_1(t) + b_1(t) z_1(\|x_0\|, \|x_1\|, \|y_0\|, \|y_1\|),$$

for all $t \in J_+, x_i \in P_{0\lambda_0^*}, y_i \in P_{1\lambda_1^*}$; and

$$\frac{\|f(t, x_0, x_1, y_0, y_1)\|}{c_0(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \to 0,$$
$$\frac{\|g(t, x_0, x_1, y_0, y_1)\|}{c_1(t)(\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\|)} \to 0,$$

as $x_i \in P_{0\lambda_0^*}$, $y_i \in P_{1\lambda_1^*}$ $(i = 0,1)$, $\|x_0\| + \|x_1\| + \|y_0\| + \|y_1\| \to \infty$, uniformly for $t \in J_+$; and for $i = 0,1$:

$$\int_0^\infty a_i(t)\mathrm{d}t = a_i^* < \infty, \quad \int_0^\infty b_i(t)\mathrm{d}t = b_i^* < \infty, \quad \int_0^\infty c_i(t)(1+t)\mathrm{d}t = c_i^* < \infty.$$

(H2) For any $t \in J_+$ and countable bounded set $V_i \subset DC^1[J, P_{0\lambda_0^*}]$, $W_i \subset DC^1[J, P_{1\lambda_1^*}]$ $(i = 0,1)$, there exist $L_i(t), K_i(t) \in L[J,J]$ $(i = 0,1)$ such that

$$\alpha(f(t,V_0(t),V_1(t),W_0(t),W_1(t))) \le \sum_{i=0}^{1} L_{0i}(t)\alpha(V_i(t)) + K_{0i}(t)\alpha(W_i(t)),$$
$$\alpha(g(t,V_0(t),V_1(t),W_0(t),W_1(t))) \le \sum_{i=0}^{1} L_{1i}(t)\alpha(V_i(t)) + K_{1i}(t)\alpha(W_i(t)),$$

with

$$G_i^* = \int_0^{+\infty} [(L_{i0}(s)+K_{i0}(s))(1+s) + L_{i1}(s)+K_{i1}(s)]\mathrm{d}s < \infty \quad (i=0,1),$$
$$G^* = \max\{G_0^*, G_1^*\}.$$

(H3) $t \in J_+$, $\lambda_0^* x_0^* \le x_i \le \overline{x}_i$, $\lambda_1^* y_0^* \le y_i \le \overline{y}_i$ $(i=0,1)$ imply

$$f(t,x_0,x_1,y_0,y_1) \le f(t,\overline{x}_0,\overline{x}_1,\overline{y}_0,\overline{y}_1), \quad g(t,x_0,x_1,y_0,y_1) \le g(t,\overline{x}_0,\overline{x}_1,\overline{y}_0,\overline{y}_1).$$

In what follows, we write

$$Q_1 = \{x \in DC^1[J,P] : x^{(i)}(t) \ge \lambda_0^* x_0^*, \forall\ t \in J, i=0,1\},$$
$$Q_2 = \{y \in DC^1[J,P] : y^{(i)}(t) \ge \lambda_1^* y_0^*, \forall\ t \in J, i=0,1\},$$

and $Q = Q_1 \times Q_2$. Evidently, Q_1, Q_2 and Q are closed convex set in $DC^1[J,E]$ and X, respectively.

We shall reduce BVP (1.1) to a system of integral equations in E. To this end, we first consider operator A defined by

$$A(x,y)(t) = (A_1(x,y)(t), A_2(x,y)(t)), \tag{2.2}$$

where

$$\begin{aligned} A_1(x,y)(t) = \frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{ & x_\infty \int_0^\infty tq(t)\,\mathrm{d}t \\ & + \int_0^\infty \int_0^\infty q(t)G(t,s)f(s,x(s),x'(s),y(s),y'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\} \\ & + tx_\infty + \int_0^\infty G(t,s)f(s,x(s),x'(s),y(s),y'(s))\,\mathrm{d}s, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} A_2(x,y)(t) = \frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{ & y_\infty \int_0^\infty th(t)\,\mathrm{d}t \\ & + \int_0^\infty \int_0^\infty h(t)G(t,s)g(s,x(s),x'(s),y(s),y'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\} \\ & + ty_\infty + \int_0^\infty G(t,s)g(s,x(s),x'(s),y(s),y'(s))\,\mathrm{d}s, \end{aligned} \tag{2.4}$$

where

$$G(t,s) = \begin{cases} t, & 0 \le t \le s < +\infty, \\ s, & 0 \le s \le t < +\infty. \end{cases}$$

Lemma 2.1. *If* (H1) *is satisfied, then the operator A defined by* (2.2) *is a continuous operator from Q to Q.*

Proof. Let

$$\varepsilon_0=\min\Big\{\frac{1}{8c_0^*\big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\big)},\frac{1}{8c_1^*\big(1+\frac{\int_0^\infty h(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\big)}\Big\},\tag{2.5}$$

$$r=\min\Big\{\frac{\lambda_0^*\|x_0^*\|}{N},\frac{\lambda_1^*\|y_0^*\|}{N}\Big\}>0.\tag{2.6}$$

By (H1), there exists a $R>r$ such that

$$\|f(t,x_0,x_1,y_0,y_1)\|\le\varepsilon_0c_0(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|),$$

for all $t\in J_+$, $x_i\in P_{0\lambda_0^*}$, $y_i\in P_{1\lambda_1^*}$ $(i=0,1)$, $\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|>R$; and

$$\|f(t,x_0,x_1,y_0,y_1)\|\le a_0(t)+M_0b_0(t),$$

for all $t\in J_+$, $x_i\in P_{0\lambda_0^*}$, $y_i\in P_{1\lambda_1^*}$ $(i=0,1)$, $\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|\le R$, where

$$M_0=\max\{z_0(u_0,u_1,v_0,v_1):r\le u_i,v_i\le R\ (i=0,1)\}.$$

Hence

$$\|f(t,x_0,x_1,y_0,y_1)\|\le\varepsilon_0c_0(t)(\|x_0\|+\|x_1\|+\|y_0\|+\|y_1\|)+a_0(t)+M_0b_0(t),\tag{2.7}$$

for all $t\in J_+$, $x_i\in P_{0\lambda_0^*}$, $y_i\in P_{1\lambda_1^*}$ $(i=0,1)$. Let $(x,y)\in Q$. By (2.7) we have

$$\begin{aligned}
&\|f(t,x(t),x'(t),y(t),y'(t))\|\\
&\le\varepsilon_0c_0(t)(1+t)\Big(\frac{\|x(t)\|}{t+1}+\frac{\|x'(t)\|}{t+1}+\frac{\|y(t)\|}{t+1}+\frac{\|y'(t)\|}{t+1}\Big)+a_0(t)+M_0b_0(t)\\
&\le\varepsilon_0c_0(t)(1+t)(\|x\|_F+\|x'\|_1+\|y\|_F+\|y'\|_1)+a_0(t)+M_0b_0(t)\\
&\le2\varepsilon_0c_0(t)(1+t)(\|x\|_D+\|y\|_D)+a_0(t)+M_0b_0(t),\\
&\le4\varepsilon_0c_0(t)(1+t)\|(x,y)\|_X+a_0(t)+M_0b_0(t),\quad\forall\ t\in J_+,
\end{aligned}\tag{2.8}$$

which together with (H1) implies the convergence of the infinite integral

$$\int_0^\infty\|f(s,x(s),x'(s),y(s),y'(s))\|\mathrm{d}s,\tag{2.9}$$

which together with (2.3) and (H1) implies

$$\begin{aligned}
&\|A_1(x,y)(t)\|\\
&\le\frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{\int_0^\infty\int_0^\infty q(t)G(t,s)\|f(s,x(s),x'(s),y(s),y'(s))\|\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+\|x_\infty\|\int_0^\infty tq(t)\,\mathrm{d}t\Big\}+t\|x_\infty\|+\int_0^\infty G(t,s)\|f(s,x(s),x'(s),y(s),y'(s))\|\,\mathrm{d}s.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\|A_1(x,y)(t)\|}{1+t} \\
&\leq \frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{\int_0^\infty\int_0^\infty q(t)\|f(s,x(s),x'(s),y(s),y'(s))\|\,\mathrm{d}s\,\mathrm{d}t \\
&\quad +\|x_\infty\|\int_0^\infty tq(t)\,\mathrm{d}t\Big\}+\|x_\infty\|+\int_0^\infty \|f(s,x(s),x'(s),y(s),y'(s))\|\,\mathrm{d}s \\
&\leq \Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)[4\varepsilon_0 c_0^*\|(x,y)\|_X+a_0^*+M_0b_0^*] \\
&\quad +\Big(1+\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\| \\
&\leq \frac{1}{2}\|(x,y)\|_X+\Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)(a_0^*+M_0b_0^*) \\
&\quad +\Big(1+\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\|.
\end{aligned} \tag{2.10}$$

Differentiating (2.3), we obtain

$$A_1'(x,y)(t)=\int_t^\infty f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s+x_\infty. \tag{2.11}$$

Hence,

$$\begin{aligned}
\|A_1'(x,y)(t)\| &\leq \int_0^{+\infty}\|f(s,x(s),x'(s),y(s),y'(s))\|\mathrm{d}s+\|x_\infty\| \\
&\leq 4\varepsilon_0 c_0^*\|(x,y)\|_X+a_0^*+M_0b_0^*+\|x_\infty\| \\
&\leq \frac{1}{2}\|(x,y)\|_X+a_0^*+M_0b_0^*+\|x_\infty\|, \quad \forall t\in J.
\end{aligned} \tag{2.12}$$

It follows from (2.10) and (2.12) that

$$\begin{aligned}
\|A_1(x,y)\|_D \leq \frac{1}{2}\|(x,y)\|_X &+\Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)(a_0^*+M_0b_0^*) \\
&+\Big(1+\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\|.
\end{aligned} \tag{2.13}$$

So, $A_1(x,y)\in DC^1[J,E]$. On the other hand, it can be easily seen that

$$\begin{gathered}
A_1(x,y)(t)\geq \Big(\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)x_\infty \geq \lambda_0^* x_\infty\geq \lambda_0^* x_0^*, \quad \forall\ t\in J, \\
A_1'(x,y)(t)\geq x_\infty\geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*, \quad \forall\ t\in J.
\end{gathered}$$

so, $A_1(x,y)\in Q_1$. In the same way, we can easily obtain

$$\begin{aligned}
\|A_2(x,y)\|_D \leq \frac{1}{2}\|(x,y)\|_X &+\Big(1+\frac{\int_0^\infty h(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big)(a_0^*+M_0b_0^*) \\
&+\Big(1+\frac{\int_0^\infty th(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big)\|y_\infty\|,
\end{aligned} \tag{2.14}$$

and

$$A_2(x,y)(t) \geq \Big(\frac{\int_0^\infty h(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big) y_\infty \geq \lambda_1^* y_\infty \geq \lambda_1^* y_0^*, \quad \forall\, t \in J,$$
$$A_2'(x,y)(t) \geq y_\infty \geq \lambda_1^* y_\infty \geq \lambda_1^* y_0^*, \quad \forall t \in J.$$

where $M_1 = \max\{z_1(u_0,u_1,v_0,v_1) : r \leq u_i, v_i \leq R\ (i=0,1)\}$. Thus, we have proved that A maps Q to Q and we have

$$\|A(x,y)\|_X \leq \frac{1}{2}\|(x,y)\|_X + \gamma, \tag{2.15}$$

where

$$\begin{gathered}\gamma = \max\Big\{\Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)(a_0^* + M_0 b_0^*) + \Big(1+\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\|, \\ \Big(\Big(1+\frac{\int_0^\infty h(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big)(a_0^* + M_0 b_0^*) + \Big(1+\frac{\int_0^\infty th(t)\,\mathrm{d}t}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big)\|y_\infty\|\Big\}.\end{gathered} \tag{2.16}$$

Finally, we show that A is continuous. Let $(x_m,y_m), (\overline{x},\overline{y}) \in Q$, $\|(x_m,y_m) - (\overline{x},\overline{y})\|_X \to 0$ $(m \to \infty)$. Then $\{(x_m,y_m)\}$ is a bounded subset of Q. Thus, there exists $r > 0$ such that $\sup_m \|(x_m,y_m)\|_X < r$ for $m \geq 1$ and $\|(\overline{x},\overline{y})\|_X \leq r+1$. Similar to (2.10) and (2.12), it is easy to show that

$$\begin{aligned}&\|A_1(x_m,y_m) - A_1(\overline{x},\overline{y})\|_X \\ &\leq \int_0^\infty \|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) - f(s,\overline{x}(s),\overline{x'}(s),\overline{y}(s),\overline{y'}(s))\|\mathrm{d}s \\ &\quad + \frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\int_0^\infty \|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) \\ &\quad - f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\|\mathrm{d}s.\end{aligned} \tag{2.17}$$

It is clear that

$$f(t,x_m(t),x_m'(t),y_m(t),y_m'(t)) \to f(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t)) \tag{2.18}$$

as $m \to \infty$, for all $t \in J_+$. By (2.8), we obtain

$$\begin{aligned}&\|f(t,x_m(t),x_m'(t),y_m(t),y_m'(t)) - f(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t))\| \\ &\leq 8\epsilon_0 c_0(t)(1+t)r + 2a_0(t) + 2M_0 b_0(t) \\ &= \sigma_0(t), \quad \sigma_0(t) \in L[J,J], \quad m = 1,2,3,\dots,\ \forall t \in J_+.\end{aligned} \tag{2.19}$$

It follows from (2.18), (2.19), and the dominated convergence theorem that

$$\lim_{m\to\infty}\int_0^\infty \|f(s,x_m(s),x_m'(s),y_m(s),y_m'(s)) - f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\|\mathrm{d}s = 0.$$

It follows from the above inequality and (2.17) that $\|A_1(x_m,y_m) - A_1(\overline{x},\overline{y})\|_D \to 0$ as $m \to \infty$. By the same method, we have $\|A_2(x_m,y_m) - A_2(\overline{x},\overline{y})\|_D \to 0$ as $m \to \infty$. Therefore, the continuity of A is proved. $\square$

Lemma 2.2. *Under assumption* (H1), $(x,y) \in Q \cap (C^2[J_+,E] \times C^2[J_+,E])$ *is a solution of* (1.1) *if and only if* $(x,y) \in Q$ *is a fixed point of* A.

Proof. Suppose that $(x,y) \in Q \cap (C^2[J_+,E] \times C^2[J_+,E])$ is a solution of (1.1). For $t \in J$, integrating (1.1) from 0 to t, we have

$$\begin{aligned} -x'(t) + x'(0) &= \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s, \\ -y'(t) + y'(0) &= \int_0^t g(s,x(s),x'(s))\,\mathrm{d}s. \end{aligned} \tag{2.20}$$

Taking the limit as $t \to \infty$, we obtain

$$\begin{aligned} -x_\infty + x'(0) &= \int_0^\infty f(s,x(s),x'(s))\,\mathrm{d}s, \\ -y_\infty + y'(0) &= \int_0^\infty g(s,x(s),x'(s))\,\mathrm{d}s, \end{aligned} \tag{2.21}$$

Thus,

$$\begin{aligned} x'(0) &= x_\infty + \int_0^\infty f(s,x(s),x'(s))\,\mathrm{d}s, \\ y'(0) &= y_\infty + \int_0^\infty g(s,x(s),x'(s))\,\mathrm{d}s. \end{aligned} \tag{2.22}$$

We obtain

$$\begin{aligned} x'(t) &= x_\infty + \int_0^\infty f(s,x(s),x'(s))\,\mathrm{d}s - \int_0^t f(s,x(s),x'(s))\,\mathrm{d}s, \\ y'(t) &= y_\infty + \int_0^\infty g(s,x(s),x'(s))\,\mathrm{d}s - \int_0^t g(s,x(s),x'(s))\,\mathrm{d}s; \end{aligned} \tag{2.23}$$

$$x'(t) = x_\infty + \int_t^\infty f(s,x(s),x'(s))\,\mathrm{d}s, \tag{2.24}$$

$$y'(t) = y_\infty + \int_t^\infty g(s,x(s),x'(s))\,\mathrm{d}s. \tag{2.25}$$

Integrating (2.24) and (2.25) from 0 to t, we obtain

$$x(t) = x(0) + tx_\infty + \int_0^\infty G(t,s) f(s,x(s),x'(s))\,\mathrm{d}s, \tag{2.26}$$

$$y(t) = y(0) + ty_\infty + \int_0^\infty G(t,s) g(s,x(s),x'(s))\,\mathrm{d}s, \tag{2.27}$$

which together with the boundary-value condition implies that

$$\begin{aligned} x(0) &= \int_0^\infty q(t)x(t)\,\mathrm{d}t = x(0)\int_0^\infty q(t)\,\mathrm{d}t + x_\infty \int_0^\infty tq(t)\,\mathrm{d}t \\ &\quad + \int_0^\infty \int_0^\infty q(t)G(t,s) f(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t, \\ y(0) &= \int_0^\infty h(t)x(t)\,\mathrm{d}t = y(0)\int_0^\infty h(t)\,\mathrm{d}t + y_\infty \int_0^\infty th(t)\,\mathrm{d}t \\ &\quad + \int_0^\infty \int_0^\infty h(t)G(t,s) g(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t. \end{aligned} \tag{2.28}$$

Thus,

$$
\begin{aligned}
x(0)=&\frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{x_\infty\int_0^\infty tq(t)\,\mathrm{d}t\\
&+\int_0^\infty\int_0^\infty q(t)G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\},
\end{aligned}
\tag{2.29}
$$

$$
\begin{aligned}
y(0)=&\frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{y_\infty\int_0^\infty th(t)\,\mathrm{d}t\\
&+\int_0^\infty\int_0^\infty h(t)G(t,s)g(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\}.
\end{aligned}
\tag{2.30}
$$

Substituting (2.29) and (2.30) in (2.26) and (2.27),

$$
\begin{aligned}
x(t)=&\frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{x_\infty\int_0^\infty tq(t)\,\mathrm{d}t\\
&+\int_0^\infty\int_0^\infty q(t)G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\}\\
&+tx_\infty+\int_0^\infty G(t,s)f(s,x(s),x'(s))\,\mathrm{d}s,\\
y(t)=&\frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{y_\infty\int_0^\infty th(t)\,\mathrm{d}t\\
&+\int_0^\infty\int_0^\infty h(t)G(t,s)g(s,x(s),x'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\}\\
&+ty_\infty+\int_0^\infty G(t,s)g(s,x(s),x'(s))\,\mathrm{d}s.
\end{aligned}
\tag{2.31}
$$

Obviously, the integral $\int_0^t\int_s^\infty f(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s$ and the integral $\int_0^t\int_s^\infty g(\tau,x(\tau),x'(\tau),y(\tau),y'(\tau))\mathrm{d}\tau\mathrm{d}s$ are convergent. Therefore, (x,y) is a fixed point of operator A. Conversely, if (x,y) is fixed point of operator A, then direct differentiation gives the proof. $\square$

Lemma 2.3. *Let* (H1) *be satisfied, and V be a bounded set of Q. Then $\frac{(A_iV)(t)}{1+t}$ and $(A_i'V)(t)$ are equicontinuous on any finite subinterval of J; and for any $\epsilon>0$, there exists $N=\max\{N_1,N_2\}>0$ such that*

$$
\|\frac{A_i(x,y)(t_1)}{1+t_1}-\frac{A_i(x,y)(t_2)}{1+t_2}\|<\epsilon,\quad \|A_i'(x,y)(t_1)-A_i'(x,y)(t_2)\|<\epsilon
$$

uniformly with respect to $(x,y)\in V$ as $t_1,t_2\geq N$.

Proof. We only give the proof for operator A_1. For $(x,y)\in V$, $t_2>t_1$, we have

$$
\begin{aligned}
&\|\frac{A_1(x,y)(t_1)}{1+t_1}-\frac{A_1(x,y)(t_2)}{1+t_2}\|\\
&\leq \big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\big|\Big(\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\|\\
&\quad+\big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\big|\|x_\infty\|+\Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\\
&\quad\times\Big\{\|\frac{t_1}{1+t_1}\int_{t_1}^\infty f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\\
&\quad-\frac{t_2}{1+t_2}\int_{t_2}^\infty f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\\
&\quad+\|\int_0^{t_1}\frac{s}{1+t_1}f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\\
&\quad-\int_0^{t_2}\frac{s}{1+t_2}f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\Big\}\\
&\leq \big|t_1-t_2\big|\Big(1+\frac{\int_0^\infty tq(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\|x_\infty\|+\Big(1+\frac{\int_0^\infty q(t)\,\mathrm{d}t}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big)\\
&\quad\times\Big\{\big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\big|\,\|\int_0^\infty f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\\
&\quad+\|\int_{t_1}^{t_2}sf(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\\
&\quad+\frac{t_2}{1+t_2}\|\int_{t_1}^{t_2}f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\\
&\quad+\big|\frac{1}{1+t_1}-\frac{1}{1+t_2}\big|\,\|\int_0^{t_1}sf(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\\
&\quad+\big|\frac{t_1}{1+t_1}-\frac{t_2}{1+t_2}\big|\,\|\int_0^{t_1}f(s,x(s),x'(s),y(s),y'(s)\mathrm{d}s\|\Big\}.
\end{aligned}
\tag{2.32}
$$

Then, it is easy to see that by the above inequality and (H1), $\{\frac{A_1V(t)}{1+t}\}$ is equicontinuous on any finite subinterval of J.

Since $V\subset Q$ is bounded, there exists $r>0$ such that for any (x,y) in V, $\|(x,y)\|_X\leq r$. By (2.11), we obtain

$$
\begin{aligned}
&\|A_1'(x,y)(t_1)-A_1'(x,y)(t_2)\|\\
&=\|\int_{t_1}^{t_2}f(s,x(s),x'(s),y(s),y'(s))\mathrm{d}s\|\\
&\leq\int_{t_1}^{t_2}[4\epsilon_0rc_0(s)(1+s)+a_0(s)+M_0b_0(s)]\mathrm{d}s.
\end{aligned}
\tag{2.33}
$$

It follows from (2.33), (H1), and the absolute continuity of Lebesgue integral that $\{A_1'V(t)\}$ is equicontinuous on any finite subinterval of J.

We are in position to show that for any $\epsilon>0$, there exists $N_1>0$ such that

$$
\|\frac{A_1(x,y)(t_1)}{1+t_1}-\frac{A_1(x,y)(t_2)}{1+t_2}\|<\epsilon,\quad \|A_1'(x,y)(t_1)-A_1'(x,y)(t_2)\|<\epsilon,
$$

uniformly with respect to $x \in V$ as $t_1, t_2 \geq N_1$. Combining this with (2.32), we need only to show that for any $\epsilon > 0$, there exists sufficiently large $N_1 > 0$ such that

$$\begin{aligned} &\| \int_0^{t_1} \frac{s}{1+t_1} f(s, x(s), x'(s), y(s), y'(s)) \mathrm{d}s \\ &- \int_0^{t_2} \frac{s}{1+t_2} f(s, x(s), x'(s), y(s), y'(s)) \mathrm{d}s \| < \epsilon, \end{aligned}$$

for all $x \in V$ as $t_1, t_2 \geq N_1$. The rest part of the proof is very similar to [8, Lemma 2.3], we omit the details.

The proof for operator A_2 can be given in a similar way. Then the proof is complete. □

Lemma 2.4. *Let* (H1) *be satisfied,* V *be a bounded set in* $DC^1[J, E] \times DC^1[J, E]$. *Then*

$$\alpha_D(A_i V) = \max \big\{ \sup_{t \in J} \alpha\Big(\frac{(A_i V)(t)}{1+t}\Big), \sup_{t \in J} \alpha((A_i V)'(t)) \big\} \quad (i = 0, 1).$$

The proof of the above lemma is similar to that of [8, Lemma 2.4], we omit it.

Lemma 2.5 (Mönch Fixed-Point Theorem [1, 3])**.** *Let* Q *be a closed convex set of* E *and* $u \in Q$. *Assume that the continuous operator* $F : Q \to Q$ *has the following property:* $V \subset Q$ *countable,* $V \subset \overline{\mathrm{co}}(\{u\} \cup F(V)) \Rightarrow V$ *is relatively compact. Then* F *has a fixed point in* Q.

Lemma 2.6. *If* (H3) *is satisfied, then for* $x, y \in Q$, $x^{(i)} \leq y^{(i)}$, $t \in J$ $(i = 0, 1)$ *imply* $(Ax)^{(i)} \leq (Ay)^{(i)}$, $t \in J$ $(i = 0, 1)$.

It is easy to see that the above lemma follows from (2.3) (2.4) (2.11) and condition (H3).

Lemma 2.7 ([5])**.** *Let* D *and* F *be bounded sets in* E, *then*

$$\widetilde{\alpha}(D \times F) = \max\{\alpha(D), \alpha(F)\},$$

where $\widetilde{\alpha}$ *and* α *denote the Kuratowski measure of non-compactness in* $E \times E$ *and* E, *respectively.*

Lemma 2.8 ([5]). *Let* P *be normal (fully regular) in* E, $\widetilde{P} = P \times P$, *then* $\widetilde{P}$ *is normal (fully regular) in* $E \times E$.

3. Main results

Theorem 3.1. *Assume* (H1), (H2) *and that* $2D^* \cdot G^* < 1$. *Then* (1.1) *has a positive solution* $(\overline{x}, \overline{y}) \in (DC^1[J, E] \cap C^2[J_+, E]) \times (DC^1[J, E] \cap C^2[J_+, E])$ *satisfying* $(\overline{x})^{(i)}(t) \geq \lambda_0^* x_0^*, (\overline{y})^{(i)}(t) \geq \lambda_1^* y_0^*$ *for* $t \in J$ $(i = 0, 1)$.

Proof. By Lemma 2.1, the operator A defined by (2.2) is a continuous operator from Q to Q. By Lemma 2.2, we need only to show that A has a fixed point $(\overline{x}, \overline{y})$ in Q. Choose $R > 2\gamma$ and let $Q^* = \{(x, y) \in Q : \|(x, y)\|_X \leq R\}$. Obviously, Q^* is a bounded closed convex set in space $DC^1[J, E] \times DC^1[J, E]$. It is easy to see that Q^* is not empty since $((1+t)x_\infty, (1+t)y_\infty) \in Q^*$. It follows from (2.15) (2.16) that $(x, y) \in Q^*$ implies that $A(x, y) \in Q^*$; i.e., A maps Q^* to Q^*. Let

$V=\{(x_m,y_m):m=1,2,\dots\}\subset Q^*$ satisfying $V\subset\overline{co}\{\{(u_0,v_0)\}\cup AV\}$ for some $(u_0,v_0)\in Q^*$. Then $\|(x_m,y_m)\|_X\le R$. By (2.3) and (2.11), we have

$$\begin{aligned}
&A_1(x_m,y_m)(t)\\
&=\frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{\int_0^\infty\int_0^\infty q(t)G(t,s)f(s,x_m(s),x_m'(s),y_m(s),y_m'(s))\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+x_\infty\int_0^\infty tq(t)\,\mathrm{d}t\Big\}+tx_\infty+\int_0^\infty G(t,s)f(s,x_m(s),x_m'(s),y_m(s),y_m'(s))\,\mathrm{d}s,
\end{aligned}\tag{3.1}$$

and

$$A_1'(x_m,y_m)(t)=\int_t^\infty f(s,x_m(s),x_m'(s),y_m(s),y_m'(s))\mathrm{d}s+x_\infty.\tag{3.2}$$

By Lemma 2.4, we have

$$\alpha_D(A_1V)=\max\Big\{\sup_{t\in J}\alpha((A_1V)'(t)),\ \sup_{t\in J}\alpha\Big(\frac{(A_1V)(t)}{1+t}\Big)\Big\},\tag{3.3}$$

where $(A_1V)(t)=\{A_1(x_m,y_m)(t):m=1,2,\dots\}$, $(A_1V)'(t)=\{A_1'(x_m,y_m)(t):m=1,2,\dots\}$.

By (2.9), we know that the infinite integral $\int_0^\infty\|f(t,x(t),x'(t),y(t),y'(t))\|\mathrm{d}t$ is convergent uniformly for for $m=1,2,3,\dots$. So, for any $\epsilon>0$, we can choose a sufficiently large $T>\xi_i$ $(i=1,2,\dots,m-2)>0$ such that

$$\int_T^\infty\|f(t,x(t),x'(t),y(t),y'(t))\|\mathrm{d}t<\epsilon.\tag{3.4}$$

Then, by Guo et al. [9, Theorem 1.2.3], (3.1), (3.2), (3.4), (H2), and Lemma 2.7, we obtain

$$\begin{aligned}
&\alpha\Big(\frac{(A_1V)(t)}{1+t}\Big)\\
&\le\frac{D_0}{1+t}\Big\{2\int_0^T\alpha(f(t,x_m(t),x_m'(t),y_m(t),y_m'(t))\,\mathrm{d}t+2\varepsilon\Big\}\\
&\le 2D_0\int_0^\infty\alpha(f(t,x_m(t),x_m'(t),y_m(t),y_m'(t))\,\mathrm{d}t+2\varepsilon\\
&\le 2D_0\alpha_X(V)\int_0^\infty(L_{00}(s)+K_{00}(s))(1+s)+(L_{01}(s)+K_{01}(s))\,\mathrm{d}t+2\varepsilon.\\
&\le 2D_0G_0^*\alpha_X(V)+2\varepsilon,
\end{aligned}\tag{3.5}$$

and

$$\alpha((A_1V)'(t))\le 2\int_0^\infty\alpha(f(t,x_m(t),x_m'(t),y_m(t),y_m'(t))\,\mathrm{d}s+2\varepsilon\le 2G_0^*\alpha_X(V)+2\varepsilon.$$

From this inequality, (3.3) and(3.5), it follows that

$$\alpha_D(A_1V)\le 2D_0\alpha_X(V)G_0^*.\tag{3.6}$$

In the same way, we obtain

$$\alpha_D(A_2V)\le 2D_1\alpha_X(V)G_1^*.\tag{3.7}$$

On the other hand, $\alpha_X(V)\le\alpha_X\{\overline{\mathrm{co}}(\{u\}\cup(AV))\}=\alpha_X(AV)$. Then, (3.6), (3.7), (H2), and Lemma 2.7 imply $\alpha_X(V)=0$; i.e., V is relatively compact in

$DC^1[J,E] \times DC^1[J,E]$. Hence, the Mönch fixed point theorem guarantees that A has a fixed point $(\overline{x}, \overline{y})$ in Q_1. □

Theorem 3.2. *Let cone P be normal and conditions* (H1)–(H3) *be satisfied. Then* (1.1) *has a positive solution $(\overline{x}, \overline{y}) \in Q \cap (C^2[J_+,E] \times C^2[J_+,E])$ which is minimal in the sense that $u^{(i)}(t) \geq \overline{x}^{(i)}(t), v^{(i)}(t) \geq \overline{y}^{(i)}(t)$, $t \in J$ $(i=0,1)$ for any positive solution $(u,v) \in Q \cap (C^2[J_+,E] \times C^2[J_+,E])$ of* (1.1). *Moreover, $\|(\overline{x}, \overline{y})\|_X \leq 2\gamma + \|(u_0, v_0)\|_X$, and there exists a monotone iterative sequence $\{(u_m(t), v_m(t))\}$ such that $u_m^{(i)}(t) \to \overline{x}^{(i)}(t), v_m^{(i)}(t) \to \overline{y}^{(i)}(t)$ as $m \to \infty (i=0,1)$ uniformly on J and $u_m''(t) \to \overline{x}''(t), v_m''(t) \to \overline{y}''(t)$ as $m \to \infty$ for any $t \in J_+$, where*

$$\begin{aligned} &u_0(t) \\ &= \frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{x_\infty \int_0^\infty tq(t)\,\mathrm{d}t + \int_0^\infty \int_0^\infty q(t)G(t,s)f(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \\ &\quad \lambda_1^* y_0^*)\,\mathrm{d}s\,\mathrm{d}t\Big\} + tx_\infty + \int_0^\infty G(t,s)f(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\,\mathrm{d}s, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &v_0(t) \\ &= \frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{y_\infty \int_0^\infty th(t)\,\mathrm{d}t + \int_0^\infty \int_0^\infty h(t)G(t,s)g(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \\ &\quad \lambda_1^* y_0^*)\,\mathrm{d}s\,\mathrm{d}t\Big\} + ty_\infty + \int_0^\infty G(t,s)g(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_1^* y_0^*, \lambda_1^* y_0^*)\,\mathrm{d}s, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &u_m(t) \\ &= \frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{x_\infty \int_0^\infty tq(t)\,\mathrm{d}t + \int_0^\infty \int_0^\infty q(t)G(t,s)f(s, u_{m-1}(s), u'_{m-1}(s), \\ &\quad v_{m-1}(s), v'_{m-1}(s))\,\mathrm{d}s\,\mathrm{d}t\Big\} + tx_\infty + \int_0^\infty G(t,s)f(s, u_{m-1}(s), u'_{m-1}(s), v_{m-1}(s), \\ &\quad v'_{m-1}(s))\,\mathrm{d}s, \quad \forall t \in J (m=1,2,3,\dots), \end{aligned} \tag{3.10}$$

$$\begin{aligned} &v_m(t) \\ &= \frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{y_\infty \int_0^\infty th(t)\,\mathrm{d}t + \int_0^\infty \int_0^\infty h(t)G(t,s)g(s, u_{m-1}(s), u'_{m-1}(s), \\ &\quad v_{m-1}(s), v'_{m-1}(s))\,\mathrm{d}s\,\mathrm{d}t\Big\} + ty_\infty + \int_0^\infty G(t,s)g(s, u_{m-1}(s), u'_{m-1}(s), v_{m-1}(s), \\ &\quad v'_{m-1}(s))\,\mathrm{d}s, \quad \forall t \in J (m=1,2,3,\dots). \end{aligned} \tag{3.11}$$

Proof. From (3.8),(3.9) one sees that $(u_0, v_0) \in C[J,E] \times C[J,E]$ and

$$u_0'(t) = \int_t^{+\infty} f(s, \lambda_0^* x_0^*, \lambda_0^* x_0^*, \lambda_0^* y_0^*, \lambda_0^* y_0^*)\mathrm{d}s + x_\infty. \tag{3.12}$$

By (3.8) and (3.12), we have $u_0^{(i)} \geq \lambda_0^* x_\infty \geq \lambda_0^* x_0^*$ $(i = 0, 1)$ and

$$
\begin{aligned}
&\frac{\|u_0(t)\|}{1+t}\\
&\leq \frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{\int_0^\infty\int_0^\infty q(t)G(t,s)\|f(s,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\|\,\mathrm{d}s\,\mathrm{d}t\\
&\quad+\|x_\infty\|\int_0^\infty tq(t)\,\mathrm{d}t\Big\}+t\|x_\infty\|+\int_0^\infty G(t,s)\|f(s,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\|\,\mathrm{d}s,\\
&\leq \Big(1+\frac{\int_0^\infty g(t)\,\mathrm{d}t}{1-\int_0^\infty g(t)\,\mathrm{d}t}\Big)\int_0^\infty a_0(s)+b_0(s)z_0(\|\lambda_0^*x_0^*\|,\|\lambda_0^*x_0^*\|,\|\lambda_0^*y_0^*\|,\|\lambda_0^*y_0^*\|\mathrm{d}s\\
&\quad+\Big(1+\frac{\int_0^\infty tg(t)\,\mathrm{d}t}{1-\int_0^\infty g(t)\,\mathrm{d}t}\Big)\|x_\infty\|,
\end{aligned}
$$

$$
\begin{aligned}
\|u_0'(t)\| &\leq \int_t^\infty \|f(s,\lambda_0^*x_0^*,\lambda_0^*x_0^*,\lambda_1^*y_0^*,\lambda_1^*y_0^*)\|\mathrm{d}\tau+\|x_\infty\|\\
&\leq \int_0^\infty a_0(s)+b_0(s)h_0(\|\lambda_0^*x_0^*\|,\|\lambda_0^*x_0^*\|,\|\lambda_0^*y_0^*\|,\|\lambda_0^*y_0^*\|)\mathrm{d}s+\|x_\infty\|,
\end{aligned}
$$

which implies $\|u_0\|_D < \infty$. Similarly, we have $\|v_0\|_D < \infty$. Thus, $(u_0, v_0) \in DC^1[J,E] \times DC^1[J,E]$. It follows from (2.3) and (3.10) that

$$
(u_m, v_m)(t) = A(u_{m-1}, v_{m-1})(t), \quad \forall t \in J,\ m = 1, 2, 3, \dots. \tag{3.13}
$$

By Lemma 2.1, we obtain $(u_m, v_m) \in Q$ and

$$
\|(u_m, v_m)\|_X = \|A(u_{m-1}, v_{m_1})\|_X \leq \frac{1}{2}\|(u_{m-1}, v_{m-1})\|_X + \gamma. \tag{3.14}
$$

By (H3) and (3.13), we have

$$
u_1(t) = A_1(u_0(t), v_0(t)) \geq A_1(\lambda_0^*x_0^*, \lambda_1^*y_0^*) = u_0(t), \quad \forall\ t \in J, \tag{3.15}
$$

and

$$
v_1(t) = A_2(u_0(t), v_0(t)) \geq A_2(\lambda_0^*x_0^*, \lambda_1^*y_0^*) = v_0(t), \quad \forall t \in J. \tag{3.16}
$$

By induction, we obtain

$$
(\lambda_0^*x_0^*, \lambda_1^*y_0^*) \leq (u_0(t), v_0(t)) \leq (u_1(t), v_1(t)) \leq \dots \leq (u_m(t), v_m(t)) \leq \dots, \tag{3.17}
$$

for all $t \in J$. By induction and Lemma 2.6 and (3.13), we have

$$
(\lambda_0^*x_0^*, \lambda_1^*y_0^*) \leq (u_0^{(i)}(t), v_0^{(i)}(t)) \leq (u_1^{(i)}(t), v_1^{(i)}(t)) \leq \dots \leq (u_m^{(i)}(t), v_m^{(i)}(t)) \leq \dots, \tag{3.18}
$$

for all $t \in J$. It follows from (3.14), by induction, that

$$
\begin{aligned}
\|(u_m, v_m)\|_X &\leq \gamma + \frac{1}{2}\gamma + \dots + \big(\frac{1}{2}\big)^{m-1}\gamma + \big(\frac{1}{2}\big)^m\|(u_0, v_0)\|_X\\
&\leq \frac{\gamma(1-(\frac{1}{2})^m)}{1-\frac{1}{2}} + \|(u_0, v_0)\|_X\\
&\leq 2\gamma + \|(u_0, v_0)\|_X \quad (m = 1, 2, 3, \dots).
\end{aligned} \tag{3.19}
$$

Let $K = \{(x, y) \in Q : \|(x, y)\|_X \leq 2\gamma + \|(u_0, v_0)\|_X\}$. Then K is a bounded closed convex set in space $DC^1[J,E] \times DC^1[J,E]$ and operator A maps K into K. Clearly, K is not empty since $(u_0, v_0) \in K$. Let $W = \{(u_m, v_m) : m = 0, 1, 2, \dots\}$, $AW = \{A(u_m, v_m) : m = 0, 1, 2, \dots\}$. Obviously, $W \subset K$ and $W = \{(u_0, v_0)\} \cup A(W)$. As in the proof of Theorem 3.1, we obtain $\alpha_X(AW)=0$; i.e., W is relatively compact in $DC^1[J,E]\times DC^1[J,E]$. So, there exists a $(\overline{x},\overline{y})\in DC^1[J,E]\times DC^1[J,E]$ and a subsequence $\{(u_{m_j},v_{m_j}):j=1,2,3,\dots\}\subset W$ such that $\{(u_{m_j},v_{m_j})(t):j=1,2,3,\dots\}$ converges to $(\overline{x}^{(i)}(t),\overline{y}^{(i)}(t))$ uniformly on J $(i=0,1)$. Since that P is normal and $\{(u_m^{(i)}(t),v_m^{(i)}(t)):m=1,2,3,\dots\}$ is nondecreasing, by Lemma 2.8 it is easy to see that the entire sequence $\{(u_m^{(i)}(t),v_m^{(i)}(t)):m=1,2,3,\dots\}$ converges to $(\overline{x}^{(i)}(t),\overline{y}^{(i)}(t))$ uniformly on J $(i=0,1)$. Since $(u_m,v_m)\in K$ and K is a closed convex set in space $DC^1[J,E]\times DC^1[J,E]$, we have $(\overline{x},\overline{y})\in K$. It is clear,

$$f(s,u_m(s),u_m'(s),v_m(s),v_m'(s))\to f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s)),\tag{3.20}$$

as $m\to\infty$, for all $s\in J_+$. By (H1) and (3.19), we have

$$\begin{aligned}&\|f(s,u_m(s),u_m'(s),v_m(s),v_m'(s))-f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\|\\&\le 8\epsilon_0c_0(s)(1+s)\|(u_m,v_m)\|_X+2a_0(s)+2M_0b_0(s)\\&\le 8\epsilon_0c_0(s)(1+s)(2\gamma+\|(u_0,v_0)\|_X)+2a_0(s)+2M_0b_0(s)\end{aligned}\quad .\tag{3.21}$$

Noticing (3.20) and (3.21) and taking limit as $m\to\infty$ in (3.10), we obtain

$$\begin{aligned}\overline{x}(t)&=\frac{1}{1-\int_0^\infty q(t)\,\mathrm{d}t}\Big\{x_\infty\int_0^\infty tq(t)\,\mathrm{d}t\\&\quad+\int_0^\infty\int_0^\infty q(t)G(t,s)f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\}\\&\quad+tx_\infty+\int_0^\infty G(t,s)f(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\,\mathrm{d}s,\end{aligned}\tag{3.22}$$

In the same way, taking limit as $m\to\infty$ in (3.11), we obtain

$$\begin{aligned}\overline{y}(t)&=\frac{1}{1-\int_0^\infty h(t)\,\mathrm{d}t}\Big\{y_\infty\int_0^\infty th(t)\,\mathrm{d}t\\&\quad+\int_0^\infty\int_0^\infty h(t)G(t,s)g(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\,\mathrm{d}s\,\mathrm{d}t\Big\}\\&\quad+ty_\infty+\int_0^\infty G(t,s)g(s,\overline{x}(s),\overline{x}'(s),\overline{y}(s),\overline{y}'(s))\,\mathrm{d}s,\end{aligned}\tag{3.23}$$

which together with (3.22) and Lemma 2.2 implies that $(\overline{x},\overline{y})\in K\cap C^2[J_+,E]\times C^2[J_+,E]$ and $(\overline{x}(t),\overline{y}(t))$ is a positive solution of (1.1). Differentiating (3.10) twice, we obtain

$$u_m''(t)=-f(t,u_{m-1}(t),u_{m-1}'(t),v_{m-1}(t),v_{m-1}'(t)),\quad\forall t\in J_+',\ m=1,2,3,\dots.$$

Hence, by (3.20), we obtain

$$\lim_{m\to\infty}u_m''(t)=-f(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t))=\overline{x}''(t),\quad\forall t\in J_+'.$$

Similarly, we have

$$\lim_{m\to\infty}v_m''(t)=-g(t,\overline{x}(t),\overline{x}'(t),\overline{y}(t),\overline{y}'(t))=\overline{y}''(t),\quad\forall\ t\in J_+'.$$

Let $(m(t),n(t))$ be any positive solution of (1.1). By Lemma 2.2, we have $(m,n)\in Q$ and $(m(t),n(t))=A(m,n)(t)$, for $t\in J$. It is clear that $m^{(i)}(t)\ge\lambda_0^*x_0^*>\theta$, $n^{(i)}(t)\ge\lambda_1^*y_0^*>\theta$ for any $t\in J$ $(i=0,1)$. So, by Lemma 2.6, we have $m^{(i)}(t)\ge$

$u_0^{(i)}(t)$, $n^{(i)}(t) \geq v_0^{(i)}(t)$ for any $t \in J$ $(i = 0, 1)$. Assume that $m^{(i)}(t) \geq u_{m-1}^{(i)}(t)$, $n^{(i)}(t) \geq v_{m-1}^{(i)}(t)$ for $t \in J$, $m \geq 1$ $(i = 0, 1)$. From Lemma 2.6 it follows that

$$(A_1^{(i)}(m,n)(t), A_2^{(i)}(m,n)(t)) \geq (A_1^{(i)}(u_{m-1}, v_{m-1}))(t), A_2^{(i)}(u_{m-1}, v_{m-1}))(t))$$

for $t \in J$ $(i = 0, 1)$; i.e., $(m^{(i)}(t), n^{(i)}(t)) \geq (u_m^{(i)}(t), v_m^{(i)}(t))$ for $t \in J$ $(i = 0, 1)$. Hence, by induction, we obtain

$$m^{(i)}(t) \geq \overline{x}_m^{(i)}(t), n^{(i)}(t) \geq \overline{y}_m^{(i)}(t) \quad \forall t \in J \ (i = 0, 1; m = 0, 1, 2, \dots). \tag{3.24}$$

Now, taking limits in (3.24), we obtain $m^{(i)}(t) \geq \overline{x}^{(i)}(t), n^{(i)}(t) \geq \overline{y}^{(i)}(t)$ for $t \in J$ $(i = 0, 1)$. The proof is complete. $\square$

Theorem 3.3. *Let cone P be fully regular and conditions* (H1) *and* (H3) *be satisfied. Then the conclusion of Theorem 3.2 holds.*

Proof. The proof is almost the same as that of Theorem 3.2. The only difference is that, instead of using condition (H2), the conclusion $\alpha_X(W) = 0$ is implied directly by (3.18) and (3.19), the full regularity of P and Lemma 2.8. $\square$

4. An example

Consider the infinite system of scalar singular second-order three-point boundary value problems:

$$\begin{aligned}
-x_n''(t) &= \frac{1}{9n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5 + y_n(t) + x_{2n}'(t) + y_{3n}'(t) \\
&\quad + \frac{2}{3n^2 x_n(t)} + \frac{4}{7n^5 x_{2n}'(t)}\Big)^{1/2} + \frac{1}{9\sqrt[6]{t}(1+3t)^2}\ln\big[(1+3t)x_n(t)\big], \\
-y_n''(t) &= \frac{1}{7n^4\sqrt[3]{e^{2t}}(4+5t)^8}\Big(6 + x_{3n}(t) + x_{4n}'(t) + \frac{1}{8n^3 y_{2n}(t)} \\
&\quad + \frac{5}{16n^4 y_{4n}'(t)}\Big)^{1/3} + \frac{1}{7\sqrt[6]{t}(3+4t)^3}\ln\big[(3+4t)y_{4n}'(t)\big], \\
x_n(0) &= \int_0^\infty e^{-t^2} x_n(t)\,\mathrm{d}t, \quad x_n'(\infty) = \frac{1}{n}, \\
y_n(0) &= \int_0^\infty \frac{1}{2} e^{\frac{-t^2}{2}} y_n(t)\,\mathrm{d}t, \quad y_n'(\infty) = \frac{1}{2n}, \quad (n = 1, 2, \dots).
\end{aligned} \tag{4.1}$$

Proposition 4.1. *System* (4.1) *has a minimal positive solution* $(x_n(t), y_n(t))$ *satisfying* $x_n(t), x_n'(t) \geq 1/n$, $y_n(t), y_n'(t) \geq 1/(2n)$ *for* $0 \leq t < +\infty$ $(n = 1, 2, 3, \dots)$.

Proof. Let $E = c_0 = \{x = (x_1, \dots, x_n, \dots) : x_n \to 0\}$ with the norm $\|x\| = \sup_n |x_n|$. Obviously, $(E, \|\cdot\|)$ is a real Banach space. Choose $P = \{x = (x_n) \in c_0 : x_n \geq 0, n = 1, 2, 3, \dots\}$. It is easy to verify that P is a normal cone in E with normal constant 1. Now we consider (4.1) which can be regarded as a boundary-value problem of form (1.1) in E with $x_\infty = (1, \frac{1}{2}, \frac{1}{3}, \dots)$, $y_\infty = (\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \dots)$. In this situation, $x = (x_1, \dots, x_n, \dots)$, $u = (u_1, \dots, u_n, \dots)$, $y = (y_1, \dots, y_n, \dots)$,

$v=(v_1,\dots,v_n,\dots)$, $f=(f_1,\dots,f_n,\dots)$, in which

$$\begin{aligned} &f_n(t,x,u,y,v)\\ &=\frac{1}{9n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5+y_n+u_{2n}+v_{3n}+\frac{2}{3n^2x_n}+\frac{4}{7n^5u_{2n}}\Big)^{1/2}\\ &\quad+\frac{1}{9\sqrt[6]{t}(1+3t)^2}\ln[(1+3t)x_n], \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &g_n(t,x,u,y,v))\\ &=\frac{1}{7n^4\sqrt[3]{e^{2t}}(4+5t)^8}\Big(6+x_{3n}+u_{4n}+\frac{1}{8n^3y_{2n}}+\frac{5}{16n^4v_{4n}}\Big)^{1/3}\\ &\quad+\frac{1}{7\sqrt[6]{t}(3+4t)^3}\ln[(3+4t)v_{4n}]. \end{aligned} \tag{4.3}$$

Let $x_0^*=x_\infty=(1,\frac12,\frac13,\dots)$, $y_0^*=y_\infty=(\frac12,\frac14,\frac16,\dots)$. Then $P_{0\lambda}=\{x=(x_1,x_2,\dots,x_n,\dots):x_n\geq\frac{\lambda}{n},\ n=1,2,3,\dots\}$, $P_{1\lambda}=\{y=(y_1,y_2,\dots,y_n,\dots):y_n\geq\frac{\lambda}{2n},\ n=1,2,3,\dots\}$, for $\lambda>0$. By a simple computation, we have $D_0=8.7912$, $D_1=2.6787$, $\int_0^\infty e^{-t^2}\,\mathrm{d}t\approx0.8863<1$, $\int_0^\infty te^{-t^2}\,\mathrm{d}t=0.5$, $\int_0^\infty\frac12e^{\frac{-t^2}{2}}\,\mathrm{d}t\approx0.6267<1$, $\int_0^\infty\frac12te^{\frac{-t^2}{2}}\,\mathrm{d}t=0.5$, $\lambda_0^*=\lambda_1^*=1$. It is clear, $f,g\in C[J_+\times P_{0\lambda}\times P_{0\lambda}\times P_{1\lambda}\times P_{1\lambda},P]$ for any $\lambda>0$. Note that $\sqrt[3]{e^{2t}}>\sqrt[6]{t}$ for $t>0$ for $t>0$, by (4.2) and (4.3), we obtain

$$\begin{aligned} \|f(t,x,u,y,v)\|\leq\frac{1}{9\sqrt[6]{t}(1+3t)^2}\Big\{&\Big(\frac{167}{21}+\|y\|+\|u\|+\|v\|\Big)^{1/2}\\ &+\ln\big[(1+3t)\|x\|\big]\Big\}, \end{aligned} \tag{4.4}$$

and

$$\|g(t,x,u,y,v)\|\leq\frac{1}{7\sqrt[6]{t}(3+4t)^2}\Big\{\big(9+\|x\|+\|u\|\big)^{1/3}+\ln\big[(3+4t)\|v\|\big]\Big\}, \tag{4.5}$$

which imply (H1) is satisfied for $a_0(t)=0$, $b_0(t)=c_0(t)=\frac{1}{9\sqrt[6]{t}(1+3t)^2}$, $a_1(t)=0$, $b_1(t)=c_1(t)=\frac{1}{7\sqrt[6]{t}(3+4t)^2}$ and

$$z_0(u_0,u_1,u_2,u_3)=\big(\frac{167}{21}+u_1+u_2+u_3\big)^{1/2}+\ln[(1+3t)u_0],$$

$$z_1(u_0,u_1,u_2,u_3)=\big(9+u_0+u_1\big)^{1/3}+\ln[(3+4t)u_3].$$

Let $f^1=\{f_1^1,f_2^1,\dots,f_n^1,\dots\}$, $f^2=\{f_1^2,f_2^2,\dots,f_n^2,\dots\}$, $g^1=\{g_1^1,g_2^1,\dots,g_n^1,\dots\}$, $g^2=\{g_1^2,g_2^2,\dots,g_n^2,\dots\}$, where

$$f_n^1(t,x,u,y,v)=\frac{1}{9n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(5+y_n+u_{2n}+v_{3n}+\frac{2}{3n^2x_n}+\frac{4}{7n^5u_{2n}}\Big)^{1/2}, \tag{4.6}$$

$$f_n^2(t,x,u,y,v)=\frac{1}{9\sqrt[6]{t}(1+3t)^2}\ln[(1+3t)x_n], \tag{4.7}$$

$$g_n^1(t,x,u,y,v)=\frac{1}{7n^4\sqrt[3]{e^{2t}}(4+5t)^8}\Big(6+x_{3n}+u_{4n}+\frac{1}{8n^3y_{2n}}+\frac{5}{16n^4v_{4n}}\Big)^{1/3}, \tag{4.8}$$

$$g_n^2(t,x,u,y,v) = \frac{1}{7\sqrt[6]{t}(3+4t)^3}\ln[(3+4t)v_{4n}]. \tag{4.9}$$

Let $t \in J_+$, and $R > 0$, and $\{z^{(m)}\}$ be any sequence in $f^1(t, P_{0R}^*, P_{0R}^*, P_{1R}^*, P_{1R}^*)$, where $z^{(m)} = (z_1^{(m)}, \dots, z_n^{(m)}, \dots)$. By (4.6), we have

$$0 \le z_n^{(m)} \le \frac{1}{9n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(\frac{167}{21} + 3R\Big)^{1/2} \quad (n, m = 1, 2, 3, \dots). \tag{4.10}$$

So, $\{z_n^{(m)}\}$ is bounded, and by the diagonal method we can choose a subsequence $\{m_i\} \subset \{m\}$ such that

$$\{z_n^{(m)}\} \to \overline{z}_n \quad \text{as } i \to \infty\ (n = 1, 2, 3, \dots), \tag{4.11}$$

which by (4.10) implies

$$0 \le \overline{z}_n \le \frac{1}{9n^3\sqrt[3]{e^{2t}}(2+5t)^9}\Big(\frac{167}{21} + 3R\Big)^{1/2} \quad (n = 1, 2, 3, \dots). \tag{4.12}$$

Hence $\overline{z} = (\overline{z}_1, \dots, \overline{z}_n, \dots) \in c_0$. It is easy to see from (4.10)-(4.12) that

$$\|z^{(m_i)} - \overline{z}\| = \sup_n |z_n^{(m_i)} - \overline{z}_n| \to 0 \quad \text{as } i \to \infty.$$

Thus, we have proved that $f^1(t, P_{0R}^*, P_{0R}^*, P_{1R}^*, P_{1R}^*)$ is relatively compact in c_0.

For any $t \in J_+, R > 0, x, y, \overline{x}, \overline{y} \in D \subset P_{0R}^*$, by (4.7) we have

$$\begin{aligned} |f_n^2(t,x,y,u,v) - f_n^2(t,\overline{x},\overline{y},\overline{u},\overline{v})| &= \frac{1}{9\sqrt[6]{t}(1+3t)^2}|\ln[(1+3t)x_n] - \ln[(1+3t)\overline{x}_n]| \\ &\le \frac{1}{9\sqrt[6]{t}(1+3t)}\frac{|x_n - \overline{x}_n|}{(1+3t)\xi_n}, \end{aligned} \tag{4.13}$$

where ξ_n is between x_n and $\overline{x}_n$. By (4.13), we obtain

$$\|f^2(t,x,y,u,v) - f^2(t,\overline{x},\overline{y},\overline{u},\overline{v})\| \le \frac{1}{9\sqrt[6]{t}(1+3t)}\|x - \overline{x}\|, \quad x, y, \overline{x}, \overline{y} \in D. \tag{4.14}$$

In the same way, we can prove that $g^1(t, P_{0R}^*, P_{0R}^*, P_{1R}^*, P_{1R}^*)$ is relatively compact in c_0. Also we can obtain

$$\|g^2(t,x,u,y,v) - g^2(t,\overline{x},\overline{u},\overline{y},\overline{v})\| \le \frac{1}{7\sqrt[6]{t}(3+4t)^2}\|v - \overline{v}\|, \quad x, y, \overline{x}, \overline{y} \in D.$$

From this inequality and (4.14), it is easy to see that (H2) holds for $L_{00}(t) = 1/(9\sqrt[6]{t}(1+3t))$, $L_{10}(t) = 1/(7\sqrt[6]{t}(3+4t)^2)$. By a simple computation, we have $G_0^* \approx 0.044$, $G_1^* \approx 0.013$, $2G^* \cdot D^* \approx 0.7736 < 1$. Our conclusion follows from Theorem 3.1. □

Acknowledgements. Supported by grants 20110491154 from the China Postdoctoral Science Foundation, BS2010SF004 from the Foundation for Outstanding Middle-Aged and Young Scientists of Shandong Province, and J10LA53 from a Project of Shandong Province Higher Educational Science and Technology Program. The authors want to thank the anonymous referee for his/her valuable comments.

References

[1] K. Demling; *Ordinary Differential Equations in Banach Spaces*, Springer-Verlag, Berlin, 1977.

[2] D. J. Guo; *Existence of positive solutions for nth-order nonlinear impulsive singular integro-differential equations in Banach spaces*, Nonlinear Anal. 68 (2008) 2727-2740.

[3] D. J. Guo, V. Lakshmikantham, X. Z. Liu; *Nonlinear Integral Equation in Abstract Spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

[4] D. J. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press, San Diego, 1988.

[5] D. J. Guo, V. Lakshmikantham; *Coupled fixed points of nonlinear operators with applications*, Nonlinear Anal. 11 (1987) 623-632.

[6] V. Lakshmikantham, S. Leela; *Nonlinear Differential Equation in Abstract Spaces*, Pergamon, Oxford, 1981.

[7] Y. H. Li, F. Y. Li; *Sign-changing solutions to second-order integral boundary value problems*, Nonlinear Anal. 69 (2008) 1179-1187

[8] Y. S. Liu; *Boundary value problems for second order differential equations on unbounded domains in a Banach space*, Appl. Math. Comput. 135 (2003) 569-583.

[9] Z. L. Yang; *Existence and nonexistence results for positive solutions of an integral boundary value problem*, Nonlinear Anal. 65 (2006) 1489-1511

[10] Z. L. Yang; *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl. 321 (2006) 751-765

[11] Z. Yang; *Positive solutions to a system of second-order nonlocal boundary value problems*, Nonlinear Anal. 62 (2005) 1251-1265.

[12] Z. Yang; *Positive solutions of a second-order integral boundary value problem*, J. Math. Anal. Appl. 321 (2006) 751-765.

[13] Y. Zhao, G. X. Song, X. Y. Sun; *Integral boundary value problems with causal operators*, Computers and Mathematics with Applications. 59 (2010) 2768-2775

[14] X. M. Zhang, X. Z. Yang, M. Q. Feng; *Minimal Nonnegative Solution of Nonlinear Impulsive Differential Equations on Infinite Interval*, Boundary Value Problems, Volume 2011, Article ID 684542, 15 pages.

[15] X. Q. Zhang; *Existence of positive solution for second-order nonlinear impulsive singular differential equations of mixed type in Banach spaces*, Nonlinear Anal. 70 (2009) 1620-1628

[16] X. Q. Zhang; *Successive iteration and positive solutions for a second-order multi-point boundary value problem on a half-line*, Computers and Mathematics with Applications. 58 (2009) 528-535

Xu Chen

School of Mathematics, Liaocheng University, Liaocheng, Shandong, China

E-mail address: `woshchxu@163.com`

Xingqiu Zhang

School of Mathematics, Liaocheng University, Liaocheng, Shandong, China

E-mail address: `zhxq197508@163.com`