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EXISTENCE OF POSITIVE SOLUTIONS FOR SEMILINEAR ELLIPTIC SYSTEMS WITH INDEFINITE WEIGHT

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ABSTRACT. This article concerns the existence of positive solutions of semilinear elliptic system

$$\begin{aligned} -\Delta u &= \lambda a(x) f(v), \quad \text{in } \Omega, \\ -\Delta v &= \lambda b(x) g(u), \quad \text{in } \Omega, \\ u &= 0 = v, \quad \text{on } \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary $\partial\Omega$ and λ is a positive parameter. $a, b : \Omega \to \mathbb{R}$ are sign-changing functions. $f, g : [0, \infty) \to \mathbb{R}$ are continuous with $f(0) > 0$, $g(0) > 0$. By applying Leray-Schauder fixed point theorem, we establish the existence of positive solutions for λ sufficiently small.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ ($N \geq 1$) be a bounded domain with a smooth boundary $\partial\Omega$ and $\lambda > 0$ a parameter. Let $a, b : \Omega \to \mathbb{R}$ be sign-changing functions. We are concerned with the existence of positive solutions of the semilinear elliptic system

$$\begin{aligned} -\Delta u &= \lambda a(x) f(v), \quad \text{in } \Omega, \\ -\Delta v &= \lambda b(x) g(u), \quad \text{in } \Omega, \\ u &= 0 = v, \quad \text{on } \partial\Omega. \end{aligned} \tag{1.1}$$

In the past few years, the existence of positive solutions of the nonlinear eigenvalue problem

$$-\Delta u = \lambda f(u) \tag{1.2}$$

has been studied extensively by many authors. It is well-known that many problems in mathematical physics may lead to problem (1.2). See, for example, fluid dynamics [1], combustion theory [2, 10], nonlinear field equations [3], wave phenomena [15], etc. Lions [14] studied the existence of positive solutions of Dirichlet problem

$$\begin{aligned} -\Delta u &= \lambda a(x) f(u), \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned} \tag{1.3}$$

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with the weight function and nonlinearity satisfy $a \geq 0$, $f \geq 0$, respectively. Problem (1.3) with indefinite weight $a(\cdot)$ is more interesting, and which has been studied by Brown [4, 5], Cac [6], Hai [11] and the references therein.

In recent years, a good amount of research is established for reaction-diffusion systems. Reaction-diffusion systems model many phenomena in Biology, Ecology, combustion theory, chemical reactors, population dynamics etc. And the elliptic system

$$\begin{gathered} -\Delta u = \lambda f(v), \quad \text{in } \Omega, \\ -\Delta v = \lambda g(u), \quad \text{in } \Omega, \\ u = 0 = v, \quad \text{on } \partial\Omega \end{gathered} \tag{1.4}$$

has been considered as a typical example of these models. The existence of positive solutions of (1.4) is established by de Figueiredo [9] et al, by an Orlicz space setting for $N \geq 3$. Hulshof et al [13] established the existence of positive solutions for (1.4) by variational technique for $N \geq 1$. Dalmasso [7] proved the existence of positive solutions of (1.4) by Schauder's fixed point theorem. Hai and Shivaji [12] established the existence of positive solution of (1.4) for λ large, by using the method of sub and supersolutions and Schauder's fixed point theorem.

Recently, Tyagi [16] studied the existence of positive solutions of (1.1) by the method of monotone iteration and Schauder's fixed point theorem. He assumed that $a, b \in L^\infty(\Omega)$ and

(H1) $f, g : [0, \infty) \to [0, \infty)$ which are continuous and nondecreasing on $[0, \infty)$;

(H2) There exists $\mu_1 > 0$ such that
$$\int_\Omega G(x,y)a^+(y)dy \geq (1+\mu_1)\int_\Omega G(x,y)a^-(y)dy, \quad \forall x \in \Omega;$$

(H3) There exists $\mu_2 > 0$ such that
$$\int_\Omega G(x,y)b^+(y)dy \geq (1+\mu_2)\int_\Omega G(x,y)b^-(y)dy, \quad \forall x \in \Omega,$$
where $G(x,y)$ is the Green's function of $-\Delta$ associated with Dirichlet boundary condition.

Here a^+, b^+ are positive parts of a and b; while a^- and b^- are the negative parts. The main result of Tyagi [16] reads as follows.

Theorem 1.1. *Assume $f(0) > 0$, $g(0) > 0$, f and g both are nondecreasing, and continuous functions. Also assume* (H2), (H3). *Then there exists $\lambda^* > 0$ depending on $f, g, a, b, \mu_i, i = 1, 2$ such that* (1.1) *has a nonnegative solution for $0 \leq \lambda \leq \lambda^*$.*

Motivated by the above references, the purpose of the present article is to study the existence of positive solutions of (1.1) by using the Leray-Schauder fixed point theorem:

Lemma 1.2 ([8])**.** *Let X be a Banach space and $T : X \to X$ a completely continuous operator. Suppose that there exists a constant $M > 0$, such that each solution $(x, \sigma) \in X \times [0, 1]$ of*
$$x = \sigma T x, \quad \sigma \in [0,1], \ x \in X$$
satisfies $\|x\|_X \leq M$. Then T has a fixed point.

Next, we state the main result of this article, under the assumption

(H1') $f, g : [0, \infty) \to \mathbb{R}$ are continuous with $f(0) > 0, g(0) > 0$.

Theorem 1.3. *Let a, b be nonzero continuous functions on $\overline{\Omega}$. Assume that* (H1'), (H2), (H3) *hold. Then there exists a positive number λ^* such that* (1.1) *has a positive solution for $0 < \lambda < \lambda^*$.*

Remark 1.4. Assumption (H1') implies that the nonlinearities f and g can change their signs, but can not be monotone; thus (H1') is much weaker than the assumption (H1) used in Tyagi [16]. We obtain a similar result as Theorem 1.1 under the weaker condition (H1'). It is worth remarking that in proving the Theorem 1.3, we extend the results in Hai [11].

As a consequence of Theorem 1.3, we have the following result.

Corollary 1.5. *Assume that (H1') holds. Let a, b be nonzero integrable functions on $[0, 1]$. Suppose that there exist two positive constants $k_1 > 1$ and $k_2 > 1$ such that*

$$
\begin{gathered}
\int_0^t s^{N-1}a^+(s)ds \geq k_1 \int_0^t s^{N-1}a^-(s)ds, \quad \forall t \in [0,1], \\
\int_0^t s^{N-1}b^+(s)ds \geq k_2 \int_0^t s^{N-1}b^-(s)ds, \quad \forall t \in [0,1].
\end{gathered}
$$

Then there exists a positive number λ^* such that the system

$$
\begin{gathered}
u'' + \frac{N-1}{t}u' + \lambda a(t)f(v) = 0, \quad 0 < t < 1, \\
v'' + \frac{N-1}{t}v' + \lambda b(t)g(u) = 0, \quad 0 < t < 1, \\
u'(0) = u(1) = 0, \quad v'(0) = v(1) = 0
\end{gathered}
\tag{1.5}
$$

has a positive solution for $0 < \lambda < \lambda^*$.

Remark 1.6. *It is worth remarking that Hai* [11] *considered only the single equation*

$$
\begin{gathered}
u'' + \frac{N-1}{t}u' + \lambda a(t)f(u) = 0, \quad 0 < t < 1, \\
u'(0) = u(1) = 0.
\end{gathered}
$$

Here we extend [7, Corollary 1.2] *to system* (1.5)*.*

2. Proof of main results

Let

$$
C(\overline{\Omega}) \times C(\overline{\Omega}) := \big\{(u, v) : u, v \text{ are continuous on } \overline{\Omega}\big\},
$$

with the norm $\|(u, v)\| = \max\{\|u\|_\infty, \|v\|_\infty\}$, where $\|u\|_\infty = \max_{x\in\overline{\Omega}} |u(x)|$. Then $\big(C(\overline{\Omega}) \times C(\overline{\Omega}), \|(\cdot, \cdot)\|\big)$ is a Banach space.

In this article, we assume that

$$
f(v) = f(0), \quad v \leq 0; \quad g(u) = g(0), \quad u \leq 0.
$$

To prove our main result, we need the following lemma.

Lemma 2.1. *Let $0 < \delta < 1$. Then there exists a positive number $\overline{\lambda}$ such that for $0 < \lambda < \overline{\lambda}$,*

$$
\begin{gathered}
-\Delta u = \lambda a^+(x)f(v), \quad \text{in } \Omega, \\
-\Delta v = \lambda b^+(x)g(u), \quad \text{in } \Omega, \\
u = 0 = v, \quad \text{on } \partial\Omega
\end{gathered}
\tag{2.1}
$$

has a positive solution $(\tilde{u}_\lambda, \tilde{v}_\lambda)$ *with* $\|(\tilde{u}_\lambda, \tilde{v}_\lambda)\| \to 0$ *as* $\lambda \to 0$*, and*

$$\tilde{u}_\lambda(x) \geq \lambda\delta f(0)p_1(x), \quad x \in \Omega; \quad \tilde{v}_\lambda(x) \geq \lambda\delta g(0)p_2(x), \quad x \in \Omega,$$

where

$$p_1(x) = \int_\Omega G(x,y)a^+(y)dy, \quad p_2(x) = \int_\Omega G(x,y)b^+(y)dy,$$

and $G(x,y)$ *is the Green's function of* $-\Delta$ *associated with Dirichlet boundary condition.*

Proof. Let $A : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ be defined by

$$A(u,v)(x) = \big(\lambda \int_\Omega G(x,y)a^+(y)f(v)dy, \lambda \int_\Omega G(x,y)b^+(y)g(u)dy\big).$$

Then $A : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is completely continuous, and the fixed points of A are solutions of system (2.1). We shall apply Lemma 1.2 to prove that A has a fixed point for λ small.

Let $\varepsilon > 0$ be such that

$$f(x) \geq \delta f(0), \quad g(x) \geq \delta g(0), \quad \text{for } 0 \leq x \leq \varepsilon. \tag{2.2}$$

In fact, it follows from (H1') that there exist two positive constants $\varepsilon_1, \varepsilon_2$ small such that

$$f(x) \geq \delta f(0), \quad 0 \leq x \leq \varepsilon_1; \quad g(x) \geq \delta g(0), \quad 0 \leq x \leq \varepsilon_2.$$

Choosing $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, then (2.2) holds. Define

$$\widetilde{f}(t) = \max_{s\in[0,t]} f(s), \quad \widetilde{g}(t) = \max_{s\in[0,t]} g(s), \tag{2.3}$$

then $\widetilde{f}$ and $\widetilde{g}$ are continuous and nondecreasing. Let

$$\widetilde{h}(t) = \max\{\widetilde{f}(t), \widetilde{g}(t)\}, \tag{2.4}$$

then $\widetilde{h}$ is continuous.

Suppose that $\lambda < \frac{\varepsilon}{2\|p\|_\infty \widetilde{h}(\varepsilon)}$, thus

$$\frac{\widetilde{h}(\varepsilon)}{\varepsilon} < \frac{1}{2\lambda\|p\|_\infty}, \tag{2.5}$$

where $\|p\|_\infty = \max\{\|p_1\|_\infty, \|p_2\|_\infty\}$.

(H1'), (2.3) and (2.4) imply that $\widetilde{h}(0) > 0$, and therefore

$$\lim_{t\to 0+} \frac{\widetilde{h}(t)}{t} = +\infty. \tag{2.6}$$

Inequalities (2.5)and (2.6) imply that there exists $A_\lambda \in (0, \varepsilon)$ such that

$$\frac{\widetilde{h}(A_\lambda)}{A_\lambda} = \frac{1}{2\lambda\|p\|_\infty}. \tag{2.7}$$

Now, let $(u,v)\in C(\overline{\Omega})\times C(\overline{\Omega})$ and $\theta\in(0,1)$ be such that $(u,v)=\theta A(u,v)$. Then we have

$$\begin{aligned}\|(u,v)\| &= \max\{\|u\|_\infty,\|v\|_\infty\}\\ &\le \max\big\{\lambda\|p_1\|_\infty\widetilde{f}(\|v\|_\infty),\lambda\|p_2\|_\infty\widetilde{g}(\|u\|_\infty)\big\}\\ &\le \max\big\{\lambda\|p_1\|_\infty\widetilde{f}(\|(u,v)\|),\lambda\|p_2\|_\infty\widetilde{g}(\|(u,v)\|)\big\}\\ &\le \max\big\{\lambda\|p\|_\infty\widetilde{f}(\|(u,v)\|),\lambda\|p\|_\infty\widetilde{g}(\|(u,v)\|)\big\}\\ &\le \lambda\|p\|_\infty\widetilde{h}(\|(u,v)\|),\end{aligned}\tag{2.8}$$

which implies that $\|(u,v)\|\neq A_\lambda$. Note that $A_\lambda\to 0$ as $\lambda\to 0$. By Lemma 1.2, A has a fixed point $(\tilde u_\lambda,\tilde v_\lambda)$ with $\|(\tilde u_\lambda,\tilde v_\lambda)\|\le A_\lambda<\varepsilon$. Consequently, from (2.2) it follows that

$$\tilde u_\lambda(x)\ge\lambda\delta f(0)p_1(x),\quad x\in\Omega;\quad \tilde v_\lambda(x)\ge\lambda\delta g(0)p_2(x),\quad x\in\Omega.\tag{2.9}$$

The proof is complete. □

Proof of Theorem 1.3. Let

$$q_1(x)=\int_\Omega G(x,y)a^-(y)dy,\quad q_2(x)=\int_\Omega G(x,y)b^-(y)dy.$$

It follows from (H2), (H3) and Lemma 2.1 that there exist four positive constants $\alpha_1,\alpha_2,\gamma_1,\gamma_2\in(0,1)$ such that

$$\begin{gathered}q_1(x)|f(s)|\le\gamma_1p_1(x)f(0),\quad\text{for } s\in[0,\alpha_1],\ x\in\Omega;\\ q_2(x)|g(s)|\le\gamma_2p_2(x)g(0),\quad\text{for } s\in[0,\alpha_2],\ x\in\Omega.\end{gathered}$$

Let $\alpha=\min\{\alpha_1,\alpha_2\}$. Then

$$q_1(x)|f(s)|\le\gamma_1p_1(x)f(0),\quad\text{for } s\in[0,\alpha],\ x\in\Omega;\tag{2.10}$$
$$q_2(x)|g(s)|\le\gamma_2p_2(x)g(0),\quad\text{for } s\in[0,\alpha],\ x\in\Omega.\tag{2.11}$$

Fix $\delta\in(\gamma,1)$, where $\gamma=\max\{\gamma_1,\gamma_2\}$. Let $h(0)=\max\{f(0),g(0)\}$ and let λ_1^*,λ_2^* be so small such that

$$\begin{gathered}\|\tilde u_\lambda\|_\infty+\lambda\delta h(0)\|p\|_\infty\le\alpha,\quad\text{for }\lambda\in(0,\lambda_1^*),\\ \|\tilde v_\lambda\|_\infty+\lambda\delta h(0)\|p\|_\infty\le\alpha,\quad\text{for }\lambda\in(0,\lambda_2^*),\end{gathered}$$

where $\tilde u_\lambda$ and $\tilde v_\lambda$ are given by Lemma 2.1, and

$$\begin{gathered}|f(t)-f(s)|\le f(0)\frac{\delta-\gamma_1}{2},\quad\text{for } t,s\in[-\alpha,\alpha],\ |t-s|\le\lambda_1^*\delta h(0)\|p\|_\infty,\\ |g(t)-g(s)|\le g(0)\frac{\delta-\gamma_2}{2},\quad\text{for } t,s\in[-\alpha,\alpha],\ |t-s|\le\lambda_2^*\delta h(0)\|p\|_\infty.\end{gathered}$$

Let $\lambda^*=\min\{\lambda_1^*,\lambda_2^*\}$. Then for $\lambda\in(0,\lambda^*)$, we have

$$\|\tilde u_\lambda\|_\infty+\lambda\delta h(0)\|p\|_\infty\le\alpha,\quad \|\tilde v_\lambda\|_\infty+\lambda\delta h(0)\|p\|_\infty\le\alpha,\tag{2.12}$$

and for $t,s\in[-\alpha,\alpha]$, $|t-s|\le\lambda^*\delta h(0)\|p\|_\infty$, we have

$$|f(t)-f(s)|\le f(0)\frac{\delta-\gamma_1}{2},\quad |g(t)-g(s)|\le g(0)\frac{\delta-\gamma_2}{2}.\tag{2.13}$$

Now, let $\lambda<\lambda^*$. We look for a solution (u_λ,v_λ) of (1.1) of the form $(\tilde u_\lambda+m_\lambda,\tilde v_\lambda+w_\lambda)$. Thus (m_λ,w_λ) solves the system

$$\Delta m_\lambda=-\lambda a^+(x)(f(\tilde v_\lambda+w_\lambda)-f(\tilde v_\lambda))+\lambda a^-(x)f(\tilde v_\lambda+w_\lambda),\quad\text{in }\Omega,$$

$$\Delta w_\lambda = -\lambda b^+(x)(g(\tilde u_\lambda + m_\lambda) - g(\tilde u_\lambda)) + \lambda b^-(x) g(\tilde u_\lambda + m_\lambda), \quad \text{in } \Omega,$$
$$m_\lambda = 0 = w_\lambda. \quad \text{on } \partial\Omega.$$

For each $(\psi, \varphi) \in C(\overline{\Omega}) \times C(\overline{\Omega})$, let $(m, w) = A(\psi, \varphi)$ be the solution of the system

$$\begin{gathered} \Delta m = -\lambda a^+(x)(f(\tilde v_\lambda + \varphi) - f(\tilde v_\lambda)) + \lambda a^-(x) f(\tilde v_\lambda + \varphi), \quad \text{in } \Omega, \\ \Delta w = -\lambda b^+(x)(g(\tilde u_\lambda + \psi) - g(\tilde u_\lambda)) + \lambda b^-(x) g(\tilde u_\lambda + \psi), \quad \text{in } \Omega, \\ m = 0 = w, \quad \text{on } \partial\Omega. \end{gathered}$$

Then $A : C(\overline{\Omega}) \times C(\overline{\Omega}) \to C(\overline{\Omega}) \times C(\overline{\Omega})$ is completely continuous. Let $(m, w) \in C(\overline{\Omega}) \times C(\overline{\Omega})$ and $\theta \in (0, 1)$ be such that $(m, w) = \theta A(m, w)$. Then

$$\begin{gathered} \Delta m = -\lambda\theta a^+(x)(f(\tilde v_\lambda + w) - f(\tilde v_\lambda)) + \lambda\theta a^-(x) f(\tilde v_\lambda + w), \quad \text{in } \Omega, \\ \Delta w = -\lambda\theta b^+(x)(g(\tilde u_\lambda + m) - g(\tilde u_\lambda)) + \lambda\theta b^-(x) g(\tilde u_\lambda + m), \quad \text{in } \Omega, \\ m = 0 = w, \quad \text{on } \partial\Omega. \end{gathered}$$

Now, we claim that $\|(m, w)\| \neq \lambda\delta h(0)\|p\|_\infty$. Suppose to the contrary that $\|(m, w)\| = \lambda\delta h(0)\|p\|_\infty$, then there are three possible cases.

Case 1. $\|m\|_\infty = \|w\|_\infty = \lambda\delta h(0)\|p\|_\infty$. Then we have from (2.12) that $\|\tilde v_\lambda + w\|_\infty \le \|\tilde v_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \le \alpha$, and so $\|\tilde v_\lambda\|_\infty \le \alpha$. Thus by (2.13) we obtain

$$|f(\tilde v_\lambda + w) - f(\tilde v_\lambda)| \le f(0)\frac{\delta - \gamma_1}{2}. \tag{2.14}$$

On the other hand, (2.14) implies

$$\begin{aligned} |m(x)| &\le \lambda p_1(x) f(0)\frac{\delta - \gamma_1}{2} + \lambda\gamma_1 p_1(x) f(0) \\ &= \lambda p_1(x) f(0)\frac{\delta + \gamma_1}{2} \\ &< \lambda p_1(x) f(0)\delta \\ &\le \lambda\delta h(0)\|p\|_\infty, \quad \text{for } x \in \Omega, \end{aligned}$$

which implies that $\|m\|_\infty < \lambda\delta h(0)\|p\|_\infty$, a contradiction.

Case 2. $\|w\|_\infty < \|m\|_\infty = \lambda\delta h(0)\|p\|_\infty$. Then $\|\tilde v_\lambda + w\|_\infty < \|\tilde v_\lambda\|_\infty + \lambda\delta h(0)\|p\|_\infty \le \alpha$, and so $\|\tilde v_\lambda\|_\infty \le \alpha$. Thus

$$|f(\tilde v_\lambda + w) - f(\tilde v_\lambda)| \le f(0)\frac{\delta - \gamma_1}{2}.$$

By the same method used to prove Case 1, we can show that $\|m\|_\infty < \lambda\delta h(0)\|p\|_\infty$, which is a desired contradiction.

Case 3. $\|m\|_\infty < \|w\|_\infty = \lambda\delta h(0)\|p\|_\infty$. As in Case 2, we obtain $\|w\|_\infty < \lambda\delta h(0)\|p\|_\infty$, a contradiction.

Then the claim is proved. By Lemma 1.2, A has a fixed point (m_λ, w_λ) with $\|(m_\lambda, w_\lambda)\| \le \lambda\delta h(0)\|p\|_\infty$. Using Lemma 2.1, we obtain

$$\begin{aligned} u_\lambda(x) &\ge \tilde u_\lambda(x) - |m_\lambda(x)| \\ &\ge \lambda\delta p_1(x) f(0) - \lambda\frac{\delta + \gamma_1}{2} f(0) p_1(x) \\ &= \lambda\frac{\delta - \gamma_1}{2} f(0) p_1(x) \\ &> 0, \quad x \in \Omega. \end{aligned}$$

Similarly, we can prove that $v_\lambda(x) > 0, x \in \Omega$. The proof is complete. $\square$

Proof of Corollary 1.5. Multiplying the both sides of the equation

$$u'' + \frac{N-1}{t}u' = -a^{\pm}(t), \quad u'(0) = u(1) = 0 \tag{2.15}$$

by t^{N-1}, we obtain

$$(t^{N-1}u')' = -a^{\pm}(t)t^{N-1}. \tag{2.16}$$

Integrating the both sides of (2.16) from 0 to t, we have

$$t^{N-1}u'(t) = -\int_0^t a^{\pm}(s)s^{N-1}ds.$$

Integrating the both sides of above equation from t to 1, we have

$$u^{\pm}(t) = \int_t^1 \frac{1}{s^{N-1}}\Big(\int_0^s a^{\pm}(\tau)\tau^{N-1}d\tau\Big)ds. \tag{2.17}$$

Therefore the solution of problem (2.15) is given by (2.17). This implies that $u^+ \geq k_1 u^-$. By the same method, we can show that $v^+ \geq k_2 v^-$, and the result follows from Theorem 1.3. $\square$

3. $n \times n$ SYSTEMS

In this section, we consider the existence of positive solutions of the $n \times n$ system

$$\begin{gathered}
-\Delta u_1 = \lambda a_1(x) f_1(u_2), \quad \text{in } \Omega, \\
-\Delta u_2 = \lambda a_2(x) f_2(u_3), \quad \text{in } \Omega, \\
\dots \\
-\Delta u_{n-1} = \lambda a_{n-1}(x) f_{n-1}(u_n), \quad \text{in } \Omega, \\
-\Delta u_n = \lambda a_n(x) f_n(u_1), \quad \text{in } \Omega, \\
u_1 = u_2 = \dots = u_n = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.1}$$

where $a_i \in L^\infty(\Omega)$ $(i = 1, 2, \dots, n)$ may be sign-changing in Ω and $\lambda > 0$ is a parameter.

We assume the following conditions:

- (H4) $f_i : [0, \infty) \to \mathbb{R}$ which is continuous and $f_i(0) > 0$ $(i = 1, 2, \dots, n)$;
- (H5) a_i $(i = 1, 2, \dots, n)$ is continuous on $\overline{\Omega}$ and there exists $k_i > 1$ $(i = 1, 2, \dots, n)$ such that
$$\int_\Omega G(x,y)a_i^+(y)dy \geq k_i \int_\Omega G(x,y)a_i^-(y)dy, \quad \forall x \in \Omega,$$
where $G(x, y)$ is defined as in Section 2.

Define the integral equation

$$(u_1, u_2, \dots, u_n) = A(u_1, u_2, \dots, u_n),$$

where $A : (C(\overline{\Omega}))^n \to (C(\overline{\Omega}))^n$ is defined by

$$\begin{aligned}
&A(u_1, u_2, \dots, u_n)(x) \\
&= \Big(\lambda \int_\Omega G(x,y)a_1(y)f_1(u_2)dy, \dots, \lambda \int_\Omega G(x,y)a_n(y)f_n(u_1)dy\Big).
\end{aligned}$$

Theorem 3.1. *Let* (H4), (H5) *hold. Then there exists a positive number* λ^* *such that* (3.1) *has a positive solution for* $0 < \lambda < \lambda^*$.

As a consequence of the above theorem we have the following corollary.

Corollary 3.2. *Let f_i $(i = 1, 2, \dots, n)$ satisfy* (H4). *Let a_i $(i = 1, 2, \dots, n)$ be nonzero integrable functions on $[0, 1]$. Suppose that there exist positive constants $k_i > 1$ such that*

$$\int_0^t s^{N-1} a_i^+(s)ds \geq k_i \int_0^t s^{N-1} a_i^-(s)ds, \quad \textit{for } t \in [0,1],\ (i = 1, 2, \dots, n).$$

Then there exists a positive number λ^* such that the system

$$\begin{gathered}
u_1'' + \frac{N-1}{t} u_1' + \lambda a_1(t) f_1(u_2) = 0, \quad 0 < t < 1, \\
u_2'' + \frac{N-1}{t} u_2' + \lambda a_2(t) f_2(u_3) = 0, \quad 0 < t < 1, \\
\dots \\
u_n'' + \frac{N-1}{t} u_n' + \lambda a_n(t) f_n(u_1) = 0, \quad 0 < t < 1, \\
u_i'(0) = u_i(1) = 0
\end{gathered}$$

has a positive solution for $0 < \lambda < \lambda^*$.

References

[1] D. Aronson, M. G. Crandall, L. A. Peletier; *Stabilization of solutions of a degenerate nonlinear diffusion problem*, Nonlinear Anal. 6 (1982), 1001-1022.

[2] J. W. Bebernes, D. R. Kassoy; *A mathematical analysis of blow up for thermal reactions-the spatially inhomogeneous case*, SIAM J. Appl. Math. 40(1) (1981), 476-484.

[3] H. Berestycki, P. L. Lions; *Existence of a Groud State in Nonlinear Equations of the Type Klein-Gordon*, in *Variational Inequalities*, Cottle Gianessi, J. L. Lions, Eds., J. Wiley, New York, 1980.

[4] K. J. Brown, A. Tertikas; *On the bifurcation of radially symmetric steady-state solutions arising in population genetics*, SIAM J. Math. Anal. 22(2) (1991), 400-413.

[5] K. J. Brown, N. M. Stavrakakis; *On the construction of sup and subsolutions for elliptic equations on all of* $\mathbb{R}^{\mathbb{N}}$, Nonlinear Anal. 32 (1998), 87-95.

[6] N. P. Cac, A. M. Fink, J. A. Gatica; *Nonnegative solutions of the radial Laplacian with nonlinearity that changes sign*, Proc. Amer. Math. Soc. 123(5) (1995), 1393-1398.

[7] R. Dalmasso; *Existence and uniqueness of positive solutions of semilinear elliptic systems*, Nonlinear Anal. 39 (2000), 559-568.

[8] K. Deimling; *Nonlinear Functional Analysis*, New York, Springer-Verlag, 1985.

[9] D. G. de Figueiredo, J. Marcos do Ó. B. Ruf; *An Orlicz-space approach to superlinear elliptic system*, J. Funct. Anal. 224 (2005), 471-496.

[10] A. M. Fink; *The radial Laplacian Gelfand problem, in: Delay and Differential Equations*, World Scientific, New Jersey, 1992, pp.93-98.

[11] D. D. Hai; *Positive solutions to a class of elliptic boundary value problems*, J. Math. Anal. Appl. 227 (1998), 195-199.

[12] D. D. Hai, R. Shivaji; *An existence result on positive solutions for a class of semilinear elliptic systems*, Proc. Roy. Soc. Edinburgh. 134 A (2004), 137-141.

[13] D. Hulshof, R. Van der Vorst; *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. 114 (1993), 32-58.

[14] P. L. Lions; *On the existence of positive solutions of semilinear elliptic equations*, SIAM Rev. 24 (1982), 441-467.

[15] W. A. Strauss; *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), 149-162.

[16] J. Tyagi; *Existence of nonnegative solutions for a class of semilinear elliptic systems with indefinite weight*, Nonlinear Anal. 73 (2010), 2882-2889.

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