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NONLINEAR DELAY INTEGRAL INEQUALITIES FOR MULTI-VARIABLE FUNCTIONS

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Abstract. In this article, we establish some nonlinear retarded integral inequalities in n independent variables. These inequalities represent a generalization of the results obtained in [1, 9, 12] for function of one and two variables. Our results can be used in the qualitative theory of delay partial differential equations and delay integral equations.

1. Introduction

In the study of ordinary differential and integral equations, one often deals with certain integral inequalities. The Gronwall-Bellman inequality and its various linear and nonlinear generalizations are crucial in the discussion of the existence, uniqueness, continuation, boundedness, oscillation, stability and other qualitative properties of the solutions of differential and integral equations. The literature on such inequalities and their applications is vast; see [3, 4, 7, 8, 15] and references therein.

During the past few years, investigators have established some useful and interesting delay integral inequalities in order to achieve various goals; see [2, 6, 10, 11, 14] and the references cited therein.

Let us first list the main results of [1, 9, 12], for functions with two variables for $u(x,y) \in (\Delta \in \mathbb{R}_+^2, \mathbb{R}_+)$:

Inequality by Ma and Pecaric [9, Theorem 2.1]:

$$\begin{aligned} u^p(x,y) = k &+ \sum_{i=1}^{m} \int_{\alpha_{1i}(x_0)}^{\alpha_{1i}(x)} \int_{\beta_{1i}(y_0)}^{\beta_{1i}(y)} a_i(s,t) u^q(s,t)\, dt\, ds \\ &+ \sum_{j=1}^{n} \int_{\alpha_{2j}(x_0)}^{\alpha_{2j}(x)} \int_{\beta_{2j}(y_0)}^{\beta_{2j}(y)} b_j(s,t) u^q(s,t) w(u(s,t))\, dt\, ds. \end{aligned} \tag{1.1}$$

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Pachpatte's inequality [12, Theorem 4];

$$\begin{aligned} u^p(x,y) = k &+ \int_{x_0}^{x}\int_{y_0}^{y} a(s,t)g_1(u(s,t))\,dt\,ds \\ &+ \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} b(s,t)g_2(u(s,t))\,dt\,ds. \end{aligned} \tag{1.2}$$

Cheung's inequality [1, Theorem 2.4]:

$$\begin{aligned} u^p(x,y) = k &+ \frac{p}{p-q}\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} a(s,t)u^q(s,t)\,dt\,ds \\ &+ \int_{\gamma(x_0)}^{\gamma(x)}\int_{\gamma(y_0)}^{\delta(y)} b(s,t)u^q(s,t)\varphi(u(s,t))\,dt\,ds. \end{aligned} \tag{1.3}$$

However, sometimes we need to study such inequalities with a function $c(x)$ in place of the constant term k. Our main result, for functions with n independent variables, is given in the inequality

$$\begin{aligned} \varphi(u(x)) \le c(x) &+ \sum_{j=1}^{n_1} d_j(x)\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(x)\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_2(u(t))dt, \end{aligned} \tag{1.4}$$

where $c(x)$ is a function and all the functions which appear in this inequality are assumed to be real valued of n variables.

It is interesting to note that the results (1.1)-(1.3) can be deduced from our inequality (1.4) in some special cases. As applications we give the estimate solution of retarded partial differential equation.

The main purpose of this article is to establish some nonlinear retarded integral inequalities for functions of n independent variables which can be used as handy tools in the theory of partial differential and integral equations with time delays. These new inequalities represent a generalization of the results obtained by Ma and Pecaric [9], Pachpatte [12] and by Cheung [1] in case of the functions with one and two variables. We note that the inequality (1.4) is also a generalization of the main results in [5, 16].

2. Main results

In this article, we denote $\mathbb{R}_+^n = [0,\infty)$ which is a subset of $\mathbb{R}^n$. All the functions which appear in the inequalities are assumed to be real valued of n-variables which are nonnegative and continuous. All integrals are assumed to exist on their domains of definitions.

For $x = (x_1, x_2, \dots, x_n)$, $t = (t_1, t_2, \dots, t_n)$, $x^0 = (x_1^0, x_2^0, \dots, x_n^0) \in \mathbb{R}_+^n$, we shall denote:

$$\begin{aligned} \int_{\widetilde{\alpha}_i(x^0)}^{\widetilde{\alpha}_i(x)} dt &= \int_{\alpha_{j1}(x_1^0)}^{\alpha_{j1}(x_1)}\int_{\alpha_{j2}(x_2^0)}^{\alpha_{j2}(x_2)}\dots\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} \dots dt_n\dots dt_1, \quad j = 1,2,\dots,n_1, \\ \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} dt &= \int_{\beta_{k1}(x_1^0)}^{\beta_{k1}(x_1)}\int_{\beta_{k2}(x_2^0)}^{\beta_{k2}(x_2)}\dots\int_{\beta_{kn}(x_n^0)}^{\beta_{kn}(x_n)} \dots dt_n\dots dt_1, \quad k = 1,2,\dots,n_2, \end{aligned}$$

with $n_1, n_2 \in \{1, 2, \dots, \}$. For $x, t \in \mathbb{R}_+^n$, we shall write $t \le x$ whenever $t_i \le x_i$, $i = 1, 2, \dots, n$ and $x \ge x_0 \ge 0$, for $x, x^0 \in \mathbb{R}_+^n$.

We denote $D = D_1 D_2 \dots D_n$, where $D_i = \frac{\partial}{\partial x_i}$, for $i = 1, 2, \dots, n$, We use the usual convention of writing $\sum_{s\in\emptyset} u(s) = 0$ if $\emptyset$ is the empty set.

$$\begin{aligned}
\widetilde{\alpha}_j(t) &= \big(\alpha_{j1}(t_1), \alpha_{j2}(t_2), \dots, \alpha_{jn}(t_n)\big) \in \mathbb{R}_+^n \quad \text{for} j = 1, 2, \dots, n_1;\\
\widetilde{\beta}_k(t) &= \big(\alpha_{k1}(t_1), \alpha_{k2}(t_2), \dots, \alpha_{kn}(t_n)\big) \in \mathbb{R}_+^n \quad \text{for } k = 1, 2, \dots, n_1.
\end{aligned}$$

We denote $\widetilde{\alpha}_j(t) \le t$ for $j = 1, 2, \dots, n_1$ whenever $\alpha_{ji}(t_i) \le t_i$ for $i = 1, 2, \dots, n$ and $j = 1, 2, \dots, n_1$, and $\widetilde{\beta}_k(t) \le t$ for $k = 1, 2, \dots, n_2$ whenever $\beta_{ki}(t_i) \le t_i$ for $i = 1, 2, \dots, n$ and $k = 1, 2, \dots, n_2$

Our main results read as the follows.

Theorem 2.1. *Let $c \in C(\mathbb{R}_+^n, \mathbb{R}_+)$, $w_1, w_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $w_1(u), w_2(u) > 0$ on $(0, \infty)$ and let $a_j(x,t)$ and $b_k(x,t) \in C(\mathbb{R}_+^n \times \mathbb{R}_+^n, \mathbb{R}_+)$ be nondecreasing functions in x for every t fixed for any $j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$. Let $\alpha_{ji}, \beta_{ki} \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\alpha_{ji}(t_i) \le t_i$ and $\beta_{ki}(t_i) \le t_i$ on $\mathbb{R}_+$ for $i = 1, 2, \dots, n$; $j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$ and $p > q \ge 0$.*

(A1) *If $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$\begin{aligned}
u^p(x) \le c(x) &+ \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t) u^q(t) dt\\
&+ \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) u^q(t) w_1(u(t)) dt,
\end{aligned} \tag{2.1}$$

for any $x \in \mathbb{R}_+^n$ with $x^0 \le t \le x$, then there exists $x^* \in \mathbb{R}_+^n$, such as for all $x^0 \le t \le x^*$, we have

$$u(x) \le \Big(\Psi_1^{-1}\Big[\Psi_1(p(x)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) dt\Big]\Big)^{1/(p-q)}. \tag{2.2}$$

Where

$$p(x) = c^{(p-q)/p}(x) + \frac{p-q}{p} \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\, dt, \tag{2.3}$$

$$\Psi_1(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_1(s^{\frac{1}{p-q}})}, \quad \delta > \delta_0 > 0. \tag{2.4}$$

Here, Ψ^{-1} is the inverse function of Ψ, and the real numbers x^* are chosen so that $\Psi_1(p(x)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) dt \in \mathrm{dom}(\Psi_1^{-1})$.

(A2) *If $u \in C(\mathbb{R}_+^n, \mathbb{R}_+)$ and*

$$\begin{aligned}
u^p(x) \le c(x) &+ \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t) u^q(t) w_1(u(t)) dt\\
&+ \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t) u^q(t) w_2(u(t)) dt.
\end{aligned} \tag{2.5}$$

(i) In the case $w_2(u) \le w_1(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \le t \le x$, there exists $\xi_1 \in \mathbb{R}^n_+$, such as for all $x^0 \le t \le \xi_1$, we have

$$u(x) \le \Big(\Psi_1^{-1}\big(\Psi_1(c^{(p-q)/p}(x)) + e(x)\big)\Big)^{1/(p-q)}.$$

(ii) In the case $w_1(u) \le w_2(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \le t \le x$, there exists $\xi_2 \in \mathbb{R}^n_+$, such as for all $x^0 \le t \le \xi_2$, we have

$$u(x) \le \Big(\Psi_2^{-1}\big(\Psi_2(c^{(p-q)/p}(x)) + e(x)\big)\Big)^{1/(p-q)},$$

where

$$e(x) = \frac{p-q}{p}\Big[\sum_{j=1}^{n_1}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} a_j(x,t)dt + \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt\Big],$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta}\frac{ds}{w_i(s^{\frac{1}{p-q}})}, \quad \delta > \delta_0 > 0, \text{ for } i = 1,2.$$

Here, Ψ_i^{-1} is the inverse function of Ψ_i and the real numbers ξ_i are chosen so that $\Psi_2(c^{(p-q)/p}(x)) + e(x) \in \operatorname{dom}(\Psi_i^{-1})$ for $i = 1,2$ respectively.

The proof of the above theorem will be given in the next section.

Corollary 2.2. *Let the functions u, c, w_1, a_j, b_k ($j = 1,2,\dots,n_1$; $k = 1,2,\dots,n_1$) and the constants p, q be defined as in Theorem 2.1 and*

$$\begin{aligned} u^p(x,y) \le c(x,y) &+ \sum_{j=1}^{n_1}\int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\alpha_j(y_0)}^{\alpha_j(y)} a_j(x,y,s,t)u^q(s,t)\,ds\,dt \\ &+ \sum_{k=1}^{n_2}\int_{\beta_k(x_0)}^{\beta_k(x)}\int_{\beta_k(y_0)}^{\beta_k(y)} b_k(x,y,s,t)u^q(t)w_1(u(t))dt, \end{aligned} \tag{2.6}$$

for any $(x,y) \in \mathbb{R}^2_+$ with $x_0 \le s \le x$ and $y_0 \le t \le y$, then there exists $(x^*, y^*) \in \mathbb{R}^n_+$, such as for all $x_0 \le s \le x^*$ and $y_0 \le s \le y^*$, then

$$u(x,y) \le \Big(\Psi^{-1}\big[\Psi(p_1(x,y)) + \frac{p-q}{p}B_1(x,y)\big]\Big)^{1/(p-q)}, \tag{2.7}$$

where

$$p_1(x,y) = c^{(p-q)/p}(x,y) + \frac{p-q}{p}A_1(x,y),$$

$$A_1(x,y) = \sum_{j=1}^{n_1}\int_{\alpha_j(x_0)}^{\alpha_j(x)}\int_{\alpha_j(y_0)}^{\alpha_j(y)} a_j(x,y,s,t)\,ds\,dt,$$

$$B_1(x,y) = \sum_{k=1}^{n_2}\int_{\beta_k(x_0)}^{\beta_k(x)}\int_{\beta_k(y_0)}^{\beta_k(y)} b_k(x,y,s,t)\,ds\,dt,$$

and

$$\Psi(\delta) = \int_{\delta_0}^{\delta}\frac{ds}{w_1(s^{1/(p-q)})}, \quad \delta > \delta_0 > 0. \tag{2.8}$$

Here, Ψ^{-1} is the inverse function of Ψ, and the real numbers (x^*, y^*) are chosen so that $\Psi(p_1(x,y)) + \frac{p-q}{p}B_1(x,y) \in \operatorname{dom}(\Psi^{-1})$.

Remark 2.3. Setting $a_j(x,y,s,t) = a_j(s,t)$, $b_k(x,y,s,t) = b_k(s,t)$ and $c(x,y) = k \geq 0$ in Corollary 2.2, we obtain Ma and Pecaric's result [9, Theorem 2.1].

Remark 2.4. *Defining* $a_j(x,y,s,t) = \frac{p}{p-q}a_j(s,t)$, $b_k(x,y,s,t) = \frac{p}{p-q}b_k(s,t)$ $c(x,y) = k > 0$ *(Constant) and* $j = k = 1$ *in Corollary 2.2, we obtain Cheung's result* [1, Theorem 2.4].

Obviously, (1.1)–(1.3) *are special cases of Theorem 2.1. So our result includes the main results in* [9, 12, 1].

Using Theorem 2.1, we can get some more generalized results as follow:

Theorem 2.5. *Let the functions* u, c, w_i, a_j, b_k $(i = 1,2,\ j = 1,2,\dots,n_1,\ k = 1,2,\dots,n_1)$ *be defined as in Theorem 2.1. Moreover, let* $\varphi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *be a strictly increasing function so that* $\lim_{x\to\infty}\varphi(x) = \infty$*, and let* $\Phi \in C(\mathbb{R}_+,\mathbb{R}_+)$ *be nondecreasing function with* $\Phi(x) > 0$ *for all* $x \in \mathbb{R}^n_+$*.*

(B1) *If* $u \in C(\mathbb{R}^n_+,\mathbb{R}_+)$ *and*

$$\begin{aligned}\varphi(u(x)) \leq c(x) &+ \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))dt \\ &+ \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_1(u(t))dt,\end{aligned} \tag{2.9}$$

for any $x \in \mathbb{R}^n_+$ *with* $x^0 \leq t \leq x$*, then there exists* $x^* \in \mathbb{R}^n_+$*, so that for all* $x^0 \leq t \leq x^*$*, we have*

$$u(x) \leq \varphi^{-1}\big(G^{-1}[\Psi_1^{-1}\left(\Psi_1(\pi(x)) + B(x)\right)]\big), \tag{2.10}$$

where

$$\pi(x) = G(c(x)) + A(x), \tag{2.11}$$

$$A(x) = \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)dt, \tag{2.12}$$

$$B(x) = \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt, \tag{2.13}$$

$$G(x) = \int_{x_0}^{x}\frac{ds}{\Phi(\varphi^{-1}(s))}, \quad x > x_0 > 0, \tag{2.14}$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta}\frac{ds}{w_i(\varphi^{-1}(G^{-1}(s)))}, \quad \delta > \delta_0 > 0,\ i = 1,2. \tag{2.15}$$

The real number x^* *is chosen so that* $\Psi_1(\pi(x)) + B(x) \in \operatorname{dom}(\Psi_1^{-1})$*.*

(B2) *If* $u \in C(\mathbb{R}^n_+,\mathbb{R}_+)$ *and*

$$\begin{aligned}\varphi(u(x)) \leq c(x) &+ \sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_2(u(t))dt.\end{aligned}$$

(i) When $w_2(u) \le w_1(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \le t \le x$, there exists $\xi_1 \in \mathbb{R}^n_+$, so that for all $x^0 \le t \le \xi_1$, we have

$$u(x) \le \varphi^{-1}\big(G^{-1}[\Psi_1^{-1}\big(\Psi_1(G(c(x))) + A(x) + B(x)\big)]\big).$$

(ii) When $w_1(u) \le w_2(u)$, for any $x \in \mathbb{R}^n_+$ with $x^0 \le t \le x$, there exists $\xi_2 \in \mathbb{R}^n_+$, so that for all $x^0 \le t \le \xi_2$, we have

$$u(x) \le \varphi^{-1}\big(G^{-1}[\Psi_2^{-1}\big(\Psi_2(G(c(x))) + A(x) + B(x)\big)]\big).$$

Where A, B, G and $\Psi_i (i = 1, 2)$ are defined in (2.12)-(2.15), Ψ_i^{-1} is the inverse function of Ψ_i and the real numbers ξ_i are chosen so that $\Psi_i(G(c(x))) + A(x) + B(x) \in \mathrm{dom}(\Psi_i^{-1})$ for $i = 1, 2$ respectively.

Many interesting corollaries can also be obtained from the above theorems (in the case of one or n independent variables).

Corollary 2.6 (Inequality in one variable)**.** *Let $p > q \ge 0$, $c > 0$ be constant and w_1, w_2 be defined as in Theorem 2.1. Moreover, let $a_j(x,t)$ and $b_k(x,t) \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions in x for every t fixed and $\alpha_j, \beta_k \in C^1(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing functions with $\alpha_j(t) \le t$ and $\beta_k(t) \le t_i$ on $\mathbb{R}_+$ for $j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$ for any $j = 1, 2, \dots, n_1$, $k = 1, 2, \dots, n_2$.*

(C1) *Let $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ and*

$$\begin{aligned} u(x)^p \le c^{p/(p-q)} &+ \frac{p}{p-q} \sum_{j=1}^{n_1} \int_0^{\alpha_j(x)} a_j(x,t) u(t)^q dt \\ &+ \frac{p}{p-q} \sum_{k=1}^{n_2} \int_0^{\beta_k(x)} b_k(x,t) u(t)^q w_1(u(t)) dt, \end{aligned}$$

for any $x \in \mathbb{R}_+$ with $0 \le t \le x$. Then there exists $(x^*) \in \mathbb{R}_+$, so that for all $0 \le t \le x^*$, we have

$$u(x) \le \big([\Psi_1^{-1}\big(\Psi_1(\pi(x)) + B(x)\big)]\big)^{1/(p-q)}. \tag{2.16}$$

Where $\pi(x) = c + A(x)$ and

$$A(x) = \sum_{j=1}^{n_1} \int_0^{\alpha_j(x)} a_j(x,t) dt, \tag{2.17}$$

$$B(x) = \sum_{k=1}^{n_2} \int_0^{\beta_k(x)} b_k(x,t) dt, \tag{2.18}$$

$$\Psi_i(\delta) = \int_{\delta_0}^{\delta} \frac{ds}{w_i(s^{\frac{1}{p-q}})} \quad \delta > \delta_0 > 0,\ i = 1, 2. \tag{2.19}$$

Where the real number x^* is chosen so that $\Psi_1(\pi(x)) + B(x) \in \mathrm{dom}(\Psi_1^{-1})$.

(C2) *If $u \in C(\mathbb{R}_+, \mathbb{R}_+)$ and*

$$\begin{aligned} u(x)^p \le c^{p/(p-q)} &+ \frac{p}{p-q} \sum_{j=1}^{n_1} \int_0^{\alpha_j(x)} a_j(x,t) u(t)^q w_1(u(t)) dt \\ &+ \frac{p}{p-q} \sum_{k=1}^{n_2} \int_0^{\beta_k(x)} b_k(x,t) u(t)^q w_2(u(t)) dt. \end{aligned}$$

(i) In the case $w_2(u) \le w_1(u)$, for any $x, t \in \mathbb{R}_+$ with $0 \le t \le x$, we have

$$u(x) \le u(x) \le \left([\Psi_1^{-1}(\Psi_1(c) + A(x) + B(x))]\right)^{1/(p-q)}.$$

(ii) In the case $w_1(u) \le w_2(u)$, for any $x, t \in \mathbb{R}_+$ with $0 \le t \le x$, we have

$$u(x) \le u(x) \le \left([\Psi_2^{-1}(\Psi_2(c) + A(x) + B(x))]\right)^{1/(p-q)}.$$

Where Ψ_i, A, B $(i = 1, 2)$ are defined in (2.17)-(2.19).

Remark 2.7. (i) Corollary 2.6 (C1) reduces to Sun's inequality [16, Theorem 2.1] in case of one variable $(n = 1)$ when $a_j(x,t) = a_j(t)$, $b_k(x,t) = b_k(t)$, $\beta_k(x) = \alpha_j(x)$ and $j = k = 1$.

(ii) Corollary 2.6 (C2) reduces to Sun's inequality [16, Theorem 2.2] in case of one variable $(n = 1)$ when $a_j(x,t) = a_j(t)$,$b_k(x,t) = b_k(t)$, $\beta_k(x) = x$ and $j = k = 1$ and $w_1 = w_2$.

Remark 2.8. Under some suitable conditions in (B1), the inequality (2.9) gives a new estimate for the inequality (2.1) in (A1).

Theorem 2.9. *Let the functions $u, c, , \varphi, \Phi, w_i, a_j, b_k$ $(i = 1, 2,\ j = 1, 2, \dots, n_1,$ $k = 1, 2, \dots, n_1)$ be defined as in Theorem 2.5 and If*

$$\begin{aligned}\varphi(u(x)) \le c(x) &+ \sum_{j=1}^{n_1} d_j(x) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)\Phi(u(t))w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(x) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)\Phi(u(t))w_2(u(t))dt,\end{aligned}$$

for any $x \in \mathbb{R}_+^n$, we have

$$u(x) \le \varphi^{-1}\big(G^{-1}[\Psi^{-1}\Big(\Psi(G(c(x))) + \widetilde{A}(x) + \widetilde{B}(x)\Big)]\big),$$

where

$$\widetilde{A}(x) = \sum_{j=1}^{n_1} d_j(x) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(x,t)dt,$$

$$\widetilde{B}(x) = \sum_{k=1}^{n_2} l_k(x) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(x,t)dt.$$

Corollary 2.10. *If*

$$u^p(x) \le c(x) + \int_0^{\widetilde{\alpha}(x)} a(t)u^q(t) + b(t)u^p(t)dt$$

for any $x \in \mathbb{R}_+^n$ with $x^0 \le t \le x$, then there exists $x^* \in \mathbb{R}_+^n$, so that for all $x^0 \le t \le x^*$, we have

$$u(x) \le \frac{p}{p-q} c^{\frac{p-q}{p}}(x) \exp\Big[\frac{p}{p-q} \int_0^{\widetilde{\alpha}(x)} a(t) + b(t)dt\Big]$$

Remark 2.11. (i) Theorem 2.9 reduced to [5, Theorem 2.2] in the case of one variable, when $\varphi(x) = x$, $b_k(x,t) = 0, w_1(t) = 1$, $j = 1$ and $n = 1$

(ii) Theorem 2.9 is also a generalization of the main result in Lipovan [5, Theorem 2.1] in case of one variable, when $\varphi(x) = x$, $b_k(x,t) = 0$, $w_1(t) = 1$, $\Phi(t) = 1$, for any $x, t \in \mathbb{R}_+(n = 1)$ and for $j = 1$.

Remark 2.12. (i) Under some suitable conditions, Theorem 2.9 reduced to Theorem 2.3 and Theorem 2-4 in case of two variables of the main results in Zhang and Meng [17].

(ii) Under some suitable conditions in Theorem 2.9, we can also obtain other estimations of the Ma and Pecaric's inequality (1.1) and the main results in [9].

Remark 2.13. Theorem 2.9 further reduces to the main results in [1, Theorem 2.1, 2.2, 2.4] and the results in [13].

3. Proof of theorems

Since the proofs resemble each other, we give the details for (A1) in Theorem 2.9 only; the proofs of the remaining inequalities can be completed by following the proofs of the above-mentioned inequalities.

Proof of Theorem 2.1 (A1). Fixing arbitrary numbers $y = (y_1, \dots, y_n) \in \mathbb{R}_+^n$ with $x^0 < y \le x^*$, we define on $[x^0; y]$ a function $z(x)$ by

$$z(x) = c(y) + \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(y,t) u^q(t) dt + \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) u^q(t) w_1(u(t)) dt. \quad (3.1)$$

Then $z(x)$ is a positive and nondecreasing function with $z(x^0) = c(y)$, and

$$u(x) \le z(x)^{1/p}, \quad x \in [x^0; y]. \quad (3.2)$$

We know that

$$\begin{aligned} D_1 D_2 \dots D_n z(x) &= \sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) u^q(\widetilde{\alpha}_j(x)) \alpha'_{j1} \alpha'_{j2} \dots \alpha'_{jn} \\ &\quad + \sum_{k=1}^{n_2} b_j(y, \widetilde{\beta}_j(x)) u^q(\widetilde{\beta}_j(x)) w_1(u(\widetilde{\beta}_j(x))) \beta'_{k1} \beta'_{k2} \dots \beta'_{kn} \\ &\le z^{q/p}(x) \Big[\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1}(x_1) \alpha'_{j2}(x_2) \dots \alpha'_{jn}(x_n) \\ &\quad + \sum_{k=1}^{n_2} b_j(y, \widetilde{\beta}_j(x)) w_1(z^{1/p}(\widetilde{\beta}_j(x))) \beta'_{k1} \beta'_{k2} \dots \beta'_{kn} \Big]. \end{aligned} \quad (3.3)$$

Using the above inequality, we have

$$\begin{aligned} \frac{D_1 D_2 \dots D_n z(x)}{z^{q/p}(x)} &\le \Big[\sum_{j=1}^{n_1} a_j(y, \widetilde{\alpha}_j(x)) \alpha'_{j1}(x_1) \alpha'_{j2}(x_2) \dots \alpha'_{jn}(x_n) \\ &\quad + \sum_{k=1}^{n_2} b_j(y, \widetilde{\beta}_j(x)) w_1(z^{1/p}(\widetilde{\beta}_j(x))) \beta'_{k1} \beta'_{k2} \dots \beta'_{kn} \Big]. \end{aligned} \quad (3.4)$$

Using $D_1D_2\cdots D_{n-1}z(x)\geq 0$, $\frac{q}{p}z^{(q-p)/p}(x)\geq 0$, $D_n(x)\geq 0$ and(3.4), we have

$$\begin{aligned}
&D_n\Big(\frac{D_1D_2\dots D_{n-1}z(x)}{z^{q/p}(x)}\Big)\\
&\leq \frac{D_1D_2\cdots D_nz(x)}{z^{q/p}(x)}\\
&\leq \sum_{j=1}^{n_1} a_j(y,\widetilde{\alpha}_j(x))\alpha'_{j1}(x_1)\alpha'_{j2}(x_2)\dots\alpha'_{jn}(x_n)\\
&\quad+\sum_{k=1}^{n_2} b_k(y,\widetilde{\beta}_k(x))w_1(z^{1/p}(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2}\cdots\beta'_{kn}.
\end{aligned}\tag{3.5}$$

Fixing $x_1,x_2,\dots,x_{n-1}$, setting $x_n=t_n$ and integrating (3.5) from x_n^0 to x_n, we obtain

$$\begin{aligned}
&\frac{D_1D_2\cdots D_{n-1}z(x)}{z^{q/p}(x)}\\
&\leq \sum_{j=1}^{n_1}\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} a_j\big(y,\alpha_{j1}(x_1),\alpha_{j2}(x_2),\cdots,\alpha_{jn-1}(x_{n-1}),\alpha_{jn}(t_n)\big)\\
&\quad\times \alpha'_{j1}\alpha'_{j2}\cdots\alpha'_{jn-1}dt_n\\
&\quad+\sum_{k=1}^{n_2}\int_{\beta_{kn}(x_n^0)}^{\beta_{kn}(x_n)} b_k(y,\beta_{k1}(x_1),\beta_{k2}(x_2),\cdots,\beta_{kn-1}(x_{n-1}),t_n)\\
&\quad\times w_1(z^{1/p}(\beta_{k1},\beta_{k2},\cdots,\beta_{kn-1},t_n))\beta'_{k1}(x_1)\beta'_{k2}(x_2)\cdots\beta'_{kn-1}(x_{n-1})\,dt_n.
\end{aligned}$$

Using the same method, we obtain

$$\begin{aligned}
&\frac{D_1z(x)}{z^{q/p}(x)}\\
&\leq \sum_{j=1}^{n_1}\Big[\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)}\dots\int_{\alpha_{jn}(x_n^0)}^{\alpha_{jn}(x_n)} a_j(y,\alpha_{j1}(x_1),t_2,\dots,t_n)\alpha'_{j1}(x_1)dt_n\dots dt_2\Big]\\
&\quad+\sum_{k=1}^{n_2}\quad[\int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)}\dots\int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} b_k(y,\beta_{k1}(x_1),t_2,\dots,t_n)\\
&\quad\times w_1(z^{1/p}(\beta_{k1}(x_1),t_2,\dots,t_n))\beta'_{k1}(x_1)dt_n\dots dt_2\Big].
\end{aligned}\tag{3.6}$$

Integrating (3.6) form x_1^0 to x_1, we obtain

$$\begin{aligned}
\frac{p}{p-q}z^{(p-q)/p}(x)\leq \frac{p}{p-q}c^{(p-q)/p}(y)&+\sum_{j=1}^{n_1}\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(y)} a_j(y,t)dt\\
&+\sum_{k=1}^{n_2}\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t)w_1(z^{1/p}(t))dt,
\end{aligned}$$

for all $x \in [x^0; y]$, which implies that

$$\begin{aligned} z^{(p-q)/p}(x) &\leq c^{(p-q)/p}(y) + \frac{p-q}{p} \sum_{j=1}^{n_1} \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(y)} a_j(y,t)dt \\ &\quad + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(z^{1/p}(t))dt. \end{aligned} \tag{3.7}$$

Setting $r_1(x) = z^{(p-q)/p}(x)$, (3.7) can be rewritten as

$$r_1(x) \leq p(y) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(r_1^{1/(p-q)}(t))dt.$$

Defining $v(x)$ on $[x^0; y]$, by

$$v(x) = p(y) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t) w_1(r_1^{1/(p-q)}(t))dt, \tag{3.8}$$

by (3.8), we have $v(x^0) = p(y)$ and

$$z^{(p-q)/p}(x) \leq v(x), \tag{3.9}$$

and

$$\begin{aligned} D_1 D_2 \cdots D_n v(x) &= \frac{p-q}{p} \sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_1(r_1^{1/(p-q)}(\widetilde{\beta}_k(x))) \beta'_{k1} \beta'_{k2} \dots \beta'_{kn} \\ &\leq \frac{p-q}{p} \sum_{k=1}^{n_2} b_k(y, \widetilde{\beta}_k(x)) w_1(v^{1/(p-q)}(\widetilde{\beta}_k(x))) \beta'_{k1} \beta'_{k2} \dots \beta'_{kn}. \end{aligned}$$

Using the same method as above, we obtain

$$\begin{aligned} &\frac{D_1 v(x)}{w_1(v(x)^{1/p-q})} \\ &\leq \frac{p-q}{p} \sum_{k=1}^{n_2} \Big[\int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} \dots \int_{\beta_{jn}(x_n^0)}^{\beta_{jn}(x_n)} b_k(y, \beta_{k1}(x_1), t_2, \dots, t_n) \beta'_{k1}(x_1) dt_n \dots dt_2 \Big]. \end{aligned}$$

Integrating form x_1^0 to x_1, we obtain

$$\Psi_1(v(x)) \leq \Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(y,t)dt, \tag{3.10}$$

from (3.10) and for any arbitrary y, we obtain

$$v(y) \leq \Psi_1^{-1} \Big[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y,t)dt \Big]. \tag{3.11}$$

From (3.11) and (3.9),

$$z(y) \leq \Big(\Psi_1^{-1} \Big[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y,t)dt \Big] \Big)^{p/(p-q)}. \tag{3.12}$$

By (3.12) and (3.2),

$$u(y) \leq \Big(\Psi_1^{-1} \Big[\Psi_1(p(y)) + \frac{p-q}{p} \sum_{k=1}^{n_2} \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(y)} b_k(y,t)dt \Big] \Big)^{1/(p-q)}.$$

Since $y \le x^*$ is arbitrary, the proof is complete. $\square$

Proof of Theorem 2.9. Fixing arbitrary numbers $\tau = (\tau_1, \dots, \tau_n) \in \mathbb{R}^n_+$ with $x^0 < \tau \le \xi$, we define on $[x^0; \tau]$ a function $z(x)$ by

$$\begin{aligned} z(x) = c(\tau) &+ \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t)\Phi(u(t))w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t)\Phi(u(t))w_2(u(t))dt. \end{aligned}$$

Then $z(x)$ is a positive and nondecreasing function with $z(x^0) = c(\tau)$, and

$$u(x) \le \varphi^{-1}(z(x)); \quad x \in [x^0; \tau].$$

We know that

$$\begin{aligned} &D_1 D_2 \cdots D_n z(x) \\ &= \sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x))\Phi(u(\widetilde{\alpha}_j(x)))w_1(u(\widetilde{\alpha}_j(x)))\alpha'_{j1}\alpha'_{j2} \dots \alpha'_{jn} \\ &\quad + \sum_{k=1}^{n_2} l_k(\tau) b_k(\tau, \widetilde{\beta}_k(x))\Phi(u(\widetilde{\beta}_k(x)))w_2(u(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2} \dots \beta'_{kn}, \\ &\le \Phi(\varphi^{-1}(z(x)))\Big[\sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x))w_1(\varphi^{-1}(z(\widetilde{\alpha}_j(x)))\alpha'_{j1}\alpha'_{j2} \dots \alpha'_{jn} \\ &\quad + \sum_{k=1}^{n_2} l_k(\tau) b_k(\tau, \widetilde{\beta}_k(x))w_2(\varphi^{-1}(z(\widetilde{\beta}_k(x)))\beta'_{k1}\beta'_{k2} \dots \beta'_{kn}\Big]. \end{aligned}$$

Using the same method in proof of the Theorem 2.1, and for all $x \in [x^0; \tau]$, which implies that

$$\begin{aligned} z(x) \le G^{-1}\Big[G(c(\tau)) &+ \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t)w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t)w_2(u(t))dt\Big]. \end{aligned}$$

Defining $v(x)$ on $[x^0; \tau]$ by

$$\begin{aligned} v(x) = G(c(\tau)) &+ \sum_{j=1}^{n_1} d_j(\tau) \int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau, t)w_1(u(t))dt \\ &+ \sum_{k=1}^{n_2} l_k(\tau) \int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau, t)w_2(u(t))dt. \end{aligned}$$

We have $v(x^0) = G(c(\tau))$, $z(x) \le G^{-1}(v(x))$, and

$$u(x) \le \varphi^{-1}(G^{-1}(v(x))). \tag{3.13}$$

Then we obtain

$$\frac{D_1 D_2 \cdots D_n v(x)}{w_1(\varphi^{-1}(G^{-1}(v(x))))} \le \Big[\sum_{j=1}^{n_1} d_j(\tau) a_j(\tau, \widetilde{\alpha}_j(x))\alpha'_{j1}(x_1)\alpha'_{j2}(x_2) \dots \alpha'_{jn}(x_n)$$

$$+\sum_{k=1}^{n_2} l_k(\tau)b_k(\tau,\widetilde{\beta}_k(x))\beta'_{k1}(x_1)\beta'_{k2}(x_2)\dots\beta'_{kn}(x_n)\Big].$$

Using the same method as above, we obtain

$$\Psi_1(v(x)) \le \Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau)\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(x)} a_j(\tau,t)dt + \sum_{k=1}^{n_2} l_k(\tau)\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(x)} b_k(\tau,t)dt.$$

From which we have

$$\begin{aligned} v(\tau) \le \Psi_1^{-1}\Big[&\Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau)\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(\tau)} a_j(\tau,t)dt \\ &+ \sum_{k=1}^{n_2} l_k(\tau)\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(\tau)} b_k(\tau,t)dt\Big], \end{aligned} \tag{3.14}$$

for any arbitrary numbers $\tau \in \mathbb{R}_+^n$, with $x^0 < \tau \le \xi$. From (3.13) and (3.14), we obtain

$$\begin{aligned} u(\tau) \le \varphi^{-1}\Big\{G^{-1}\Big(\Psi_1^{-1}\Big[&\Psi_1(G(c(\tau))) + \sum_{j=1}^{n_1} d_j(\tau)\int_{\widetilde{\alpha}_j(x^0)}^{\widetilde{\alpha}_j(\tau)} a_j(\tau,t)dt \\ &+ \sum_{k=1}^{n_2} l_k(\tau)\int_{\widetilde{\beta}_k(x^0)}^{\widetilde{\beta}_k(\tau)} b_k(\tau,t)dt\Big]\Big)\Big\}. \end{aligned}$$

Since τ is arbitrary and $\tau \le \xi$, we obtain the result in the Theorem 2.9. □

4. An application

In this section we present an immediate application of our results (Theorem 2.1 and Corollary 2.10) to study boundedness of solutions of delay partial differential equations. First we consider the nonlinear partial delay differential equation in $\mathbb{R}^n$:

$$\begin{gathered} Du^p(x) = h(x,u(x),u(x-\widetilde{\alpha}(x))), \\ u^p(0,x_2,x_3,\dots,x_n) = c_1(x_1), \\ u^p(0,x_2,x_3,\dots x_{n-1},x_n) = c_n(x_n) \\ u^p(\dots,x_{i-1},0,x_{i+1},\dots) = c_i(x_i) \text{ for } i=2,3,\dots,n-1, \\ c_i(0)=0 \quad \text{for } i=1,2,\dots,n. \end{gathered} \tag{4.1}$$

For $x=(x_1;x_2,\dots,x_n)\in\mathbb{R}_+^n$ and $\widetilde{\alpha}(x)=\big(\alpha_1(x_1),\alpha_2(x_2),\dots,\alpha_n(x_n)\big)\in\mathbb{R}_+^n$ for $\alpha_i,c_i\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ for $i=1,2,\dots,n$. Where $h:\mathbb{R}_+^n\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$, is a continuous function. Assume that these functions are defined and continuous on their respective domains of definition such that

$$\widetilde{\alpha}(x) \le x, \quad \text{for all } x=(x_1;x_2,\dots,x_n)\in\mathbb{R}_+^n, \tag{4.2}$$

$$|h(x,u,v)| \le a(x)|v(x)|^q + b(x)|v(x)|^p, \tag{4.3}$$

for $x\in\mathbb{R}_+^n$, where $p>q\ge 0$ is a constants and $a(x),b(x)$ are nonnegative, continuous functions defined for $x\in\mathbb{R}_+^n$. For any solution $u(x)$ of the boundary value problem (4.1),

$$u^p(x) = \sum_{i=1}^n c_i(x_i) + \int_0^x h(t,u(t),u(t-\widetilde{\alpha}(t)))dt, \tag{4.4}$$

For all $x, t \in \mathbb{R}_+^n$ with $0 \le t \le x$. Using (4.1), (4.3) and a suitable change of variables in (4.4), we have

$$|u^p(x)| \le c(x) + \int_0^{\widetilde{\alpha}(x)} \widetilde{a}(t)\,|u(t)|^q + \widetilde{b}(t)\,|u(t)|^p\,dt, \tag{4.5}$$

with $c(x) = \sum_{i=1}^n |c_i(x_i)|$, $\widetilde{a}, \widetilde{b} \in C^1(\mathbb{R}_+^n, \mathbb{R}_+)$.

(E1) Applying (A1) in Theorem 2.1 to (4.5), when $\widetilde{\alpha}_j = \widetilde{\beta}_k$, $a_j(x,t) = \widetilde{a}(t)$, $b_k(x,t) = \widetilde{b}(t)$ with $j = k = 1$ and $w_1(u) = u^{p-q}$, we obtain a bound for the solution $u(x)$:

$$u(x) \le \Big(c^{(p-q)/p}(x) + \frac{p-q}{p}\int_0^{\widetilde{\alpha}(x)} \widetilde{a}(t)dt\Big)^{1/(p-q)} \exp\Big(\frac{1}{p}\int_0^{\widetilde{\alpha}(x)} \widetilde{b}(t)dt\Big). \tag{4.6}$$

(E2) Or by a direct application of Corollary 2.10 to (4.5),

$$u(x) \le \frac{p}{p-q} c^{\frac{p-q}{p}} \exp\Big[\frac{p}{p-q}\int_0^{\widetilde{\alpha}(x)} [\widetilde{a}(t) + \widetilde{b}(t)]dt\Big]. \tag{4.7}$$

Remark 4.1. In the special case ($p = 2$ and $q = 1$) in the boundary value problem (4.1), we have

(i) By (4.6), we obtain

$$u(x) \le \Big(\sqrt{c(x)} + \frac{1}{2}\int_0^{\widetilde{\alpha}(x)} \widetilde{a}(t)dt\Big) \exp\Big(\frac{1}{2}\int_0^{\widetilde{\alpha}(x)} \widetilde{b}(t)dt\Big).$$

(ii) Or by using (4.7),

$$u(x) \le 2\sqrt{c(x)} \exp\Big[2\int_0^{\widetilde{\alpha}(x)} [\widetilde{a}(t) + \widetilde{b}(t)]dt\Big].$$

Remark 4.2. *Note that the results given here can be very easily generalized to obtain explicit bounds on integral inequalities involving several retarded arguments.*

Using similar method of those in the proof of the Theorems above, we can also obtain a new reversed inequalities of our results.

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