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OPERATOR TYPE EXPANSION-COMPRESSION FIXED POINT THEOREM

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Abstract. This article presents an alternative to the compression and expansion fixed point theorems of functional type by using operators and functions to replace the functionals and constants that are used in functional compression and expansion fixed point theorems. Only portions of the boundaries are required to be mapped outward or inward in the spirit of the original work of Leggett-Williams. We conclude with an application verifying the existence of a positive solution to a second-order boundary-value problem.

1. Introduction

Mavridis [7] published the first extension to the work of Leggett-Williams [6] that replaced the arguments involving functionals with arguments involving operators. An invariance condition was a key component of the arguments in that paper, that is, $T(K_{A,B}(u,v)\cap K_c)\subset K_{A,B}(u,v)$ (condition (i) of Theorem 2.8, the main result therein). A similar approach was taken in the topological generalizations of fixed point theorems presented by Kwong [5] which required boundaries to be mapped inward or outward (invariance-like conditions). The spirit of the Leggett-Williams fixed point theorems [6] and the functional extensions by Avery [1], Anderson-Avery-Henderson [2], and Sun-Zhang [9], to mention a few, is that at least one of the boundaries is void of any invariance like conditions. Anderson-Avery-Henderson [2] recently published the first results of this nature that do not require either boundary to have invariance like conditions.

The difficulty in replacing the arguments involving functionals with arguments involving operators lies in the ability to compare the output of an operator to a function using the comparison generated by an underlying cone P. That is, for an operator R and a specified function x_R, one needs to be able to say, for any $y\in P$, that either $R(y)<x_R$ or $x_R\le R(y)$. In this paper we accomplish this by restricting our attention to a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set $K\subset\mathbb{R}$. We introduce what it means to say that an operator R is comparable to a function x_R on a cone P relative to J_R which is a subset of K. This allows us to maintain the spirit of the original work of Leggett-Williams and the extensions to the outer boundary

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by Anderson-Avery-Henderson by avoiding any invariance-like conditions in our arguments.

The proof of the main results hinge on the comparability criteria. The Operator Type Expansion-Compression Fixed Point Theorem can be used to verify the existence of positive solutions to boundary value problems such as $x'' + g(t)f(x, x') = 0$ for $t \in [0,1]$ with $x(0) = x'(1) = 0$ (see Section 4). In the following example we illustrate the comparability criteria which will formally be defined in the next section. For x in the cone P of increasing, nonnegative functions of $C^1[0,1]$, define the operator

$$(Ax)(t) = x'(0)t$$

which is a continuous linear operator mapping P to P. Let $b \in (0, \infty)$, $\tau \in (0,1)$, $J_A = [\tau, 1]$, and $x_A(t) = bt$. Then for all $x \in P$, either

$$x'(0) < b \quad \text{or} \quad b \le x'(0).$$

Hence for all $t \in J_A$, either

$$(Ax)(t) = x'(0)t < bt = x_A(t) \quad \text{or} \quad x_A(t) = bt \le x'(0)t = (Ax)(t).$$

Therefore, for any $x \in P$, either

$$Ax(t) < x_A(t) \quad \text{or} \quad x_A(t) \le Ax(t) \quad \forall\, t \in J_A,$$

which we will denote as

$$Ax <_{J_A} x_A \quad \text{or} \quad x_A \le_{J_A} Ax,$$

and we say the operator A is comparable to x_A on P relative to J_A. The operator A is also an example of both a convex and a concave operator which we will formally define in the next section.

2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a *cone* if, for all $x \in P$ and $\lambda \ge 0$, $\lambda x \in P$, and if $x, -x \in P$ then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by $x \le y$ if and only if $y - x \in P$, and we say that $x < y$ whenever $x \le y$ and $x \ne y$. Let K be a subset of real numbers and $F(K)$ the set of all real valued functions defined over K. If $J \subset K$ and $x, y \in F(K)$ we will say that:

$$x <_J y \quad \text{if and only if } x(t) < y(t) \text{ for all } t \in J,$$

and

$$x \le_J y \quad \text{if and only if } x(t) \le y(t) \text{ for all } t \in J.$$

Furthermore, we will say that

$$x \leqq_J y \quad \text{if and only if } x \le_J y \quad \text{and there exists a } t_0 \in J \text{ such that } x(t_0) = y(t_0).$$

Definition 2.2. An operator is called *completely continuous* if it is continuous and maps bounded sets into precompact sets.

Definition 2.3. Let P be a cone in a real Banach space E. Then we say that $A : P \to P$ is a *continuous concave operator* on P if $A : P \to P$ is continuous and
$$tA(x) + (1-t)A(y) \leq A(tx + (1-t)y)$$
for all $x, y \in P$ and $t \in [0,1]$. Similarly we say that $B : P \to P$ is a *continuous convex operator* on P if $B : P \to P$ is continuous and
$$B(tx + (1-t)y) \leq tB(x) + (1-t)B(y)$$
for all $x, y \in P$ and $t \in [0,1]$.

Let R and S be operators on a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set K. For $J_R, J_S \subset K$ and $x_R, x_S \in E$ we define the sets,
$$\begin{gathered} P_{J_R}(R, x_R) = \{y \in P : R(y) <_{J_R} x_R\}, \\ P(R, S, x_R, x_S, J_R, J_S) = P_{J_S}(S, x_S) - \overline{P_{J_R}(R, x_R)}. \end{gathered}$$

Definition 2.4. Let R be an operator on a cone P of a real Banach space E which is a subset of $F(K)$, the set of real valued functions defined on a set K. For $J_R \subset K$ and $x_R \in E$, we say that *R is comparable to x_R on P relative to J_R* if, given any $y \in P$, either
$$R(y) <_{J_R} x_R \quad \text{or} \quad x_R \leq_{J_R} R(y).$$

Definition 2.5. Let D be a subset of a real Banach space E. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then D is a *retract* of E, and the map r is a *retraction*. The *convex hull* of a subset D of a real Banach space X is given by
$$\operatorname{conv}(D) = \Big\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D,\ \lambda_i \in [0,1],\ \sum_{i=1}^{n} \lambda_i = 1,\ \text{and } n \in \mathbb{N} \Big\}.$$

The following theorem is due to Dugundji and its proof can be found in [8, p. 22].

Theorem 2.6. *Let E and X be Banach spaces and let $f : C \to K$ be a continuous mapping, where C is closed in E and K is convex in X. there exists a continuous mapping $\tilde{f} : E \to K$ such that $\tilde{f}(u) = f(u)$, $u \in C$.*

Yet in establishing our main results, we will use the following form of Dugundji's theorem [3, p. 44].

Corollary 2.7. *For Banach spaces X and Y, let $D \subset X$ be closed and let $F : D \to Y$ be continuous. Then F has a continuous extension $\tilde{F} : X \to Y$ such that $\tilde{F}(X) \subset \overline{\operatorname{conv}(F(D))}$.*

Corollary 2.8. *Every closed convex set in a Banach space is a retract of the Banach space.*

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [4, pp. 82-86]; an elementary proof can be found in [3, pp. 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 2.9. *Let X be a retract of a real Banach space E. Then, for every bounded relatively open subset U of X and every completely continuous operator $A : \overline{U} \to X$ which has no fixed points on ∂U (relative to X), there exists an integer $i(A, U, X)$ satisfying the following conditions:*

(G1) *Normality:* $i(A, U, X) = 1$ *if* $Ax \equiv y_0 \in U$ *for any* $x \in \overline{U}$*;*
(G2) *Additivity:* $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ *whenever* U_1 *and* U_2 *are disjoint open subsets of* U *such that* A *has no fixed points on* $\overline{U} - (U_1 \cup U_2)$*;*
(G3) *Homotopy Invariance:* $i(H(t, \cdot), U, X)$ *is independent of* $t \in [0, 1]$ *whenever* $H : [0, 1] \times \overline{U} \to X$ *is completely continuous and* $H(t, x) \neq x$ *for any* $(t, x) \in [0, 1] \times \partial U$*;*
(G4) *Solution: If* $i(A, U, X) \neq 0$*, then* A *has at least one fixed point in* U*.*

Moreover, $i(A, U, X)$ *is uniquely defined.*

3. Main Results

Anderson, Avery, and Henderson [2] proved an expansion-compression fixed point theorem of Leggett-Williams type; imbedded in the proof were two lemmas which were the primary means of generalizing the fixed point theorems of Leggett-Williams. The two lemmas in this section are the operator versions of those lemmas and they will be the essential components in our expansion-compression fixed point theorem of operator type. The key to Leggett-Williams type arguments is in using concavity to remove an invariance condition (an inward condition). Note that if one of $A(y) <_{J_A} x_A$ or $x_A \leq_{J_A} A(y)$ holds for each $y \in P$ (this is the condition that A is comparable to x_A on P relative to J_A), then an invariance condition of the form

(F0) if $y \in P$ with $B(y) \leqq_{J_B} x_B$, then $B(Ty) <_{J_B} x_B$

is equivalent to the following two conditions

(F1) if $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $x_A \leq_{J_A} A(y)$, then $B(Ty) <_{J_B} x_B$.
(F2) if $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $A(y) <_{J_A} x_A$, then $B(Ty) <_{J_B} x_B$.

In the spirit of the original work of Leggett-Williams, using the properties of a concave operator A, one can replace (F2) with

(F2′) if $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $A(Ty) <_{J_A} x_A$, then $B(Ty) <_{J_B} x_B$.

These inward conditions (F1) and (F2′) are used in the next Lemma 3.1. A similar technique was employed by Anderson, Avery and Henderson to remove outward conditions in the expansion-compression arguments in [2] and the operator version appears below as Lemma 3.2. Although the technique appears unmotivated, the beauty comes in applications when a clever choice of the operator A effortlessly verifies (F2′). Then verification of (F0) is replaced by verification of (F1) in our fixed point arguments. Condition (F1) requires fewer $y \in P$ to be checked compared to (F0), therefore it is easier to use in applications. The operator A is only a tool for obtaining a fixed point of T, consequently it is not part of the conclusion of any lemma or theorem.

Lemma 3.1. *Let* $F(K)$ *be the set of real valued functions defined on* $K \subset \mathbb{R}$*,* J_A *and* J_B *be subsets of* K *with* J_B *being compact, and* P *be a cone of non-negative functions in a real Banach space* E *which is a subset of* $F(K)$*. Suppose that* A *is a concave operator on* P*,* B *is a continuous convex operator on* P*, and* $T : P \to P$ *is a completely continuous operator. Suppose there exist* $x_A, x_B \in E$ *such that*

(B0) A *is comparable to* x_A *on* P *relative to* J_A*;*
(B1) $\{y \in P : x_A <_{J_A} A(y)$ *and* $B(y) <_{J_B} x_B\} \neq \emptyset$*;*
(B2) *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $x_A \leq_{J_A} A(y)$*, then* $B(Ty) <_{J_B} x_B$*;*
(B3) *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $A(Ty) <_{J_A} x_A$*, then* $B(Ty) <_{J_B} x_B$*.*

If $\overline{P_{J_B}(B, x_B)}$ is bounded, then $i(T, P_{J_B}(B, x_B), P) = 1$.

Proof. By Corollary 2.8, P is a retract of the Banach space E since it is convex and closed.

Claim 1: $Ty \neq y$ for all $y \in \partial P_{J_B}(B, x_B)$. Let $z_0 \in \partial P_{J_B}(B, x_B)$. By the continuity of B and the compactness of J_B, $B(z_0) \leqq_{J_B} x_B$. We want to show that z_0 is not a fixed point of T; so suppose to the contrary that $T(z_0) = z_0$. Since A is comparable to x_A on P relative to J_A, either $A(Tz_0) <_{J_A} x_A$ or $x_A \leq_{J_A} A(Tz_0)$. If $A(Tz_0) <_{J_A} x_A$, then $B(Tz_0) <_{J_B} x_B$ by condition (B3), and if $x_A \leq_{J_A} A(Tz_0) = A(z_0)$, then $B(Tz_0) <_{J_B} x_B$ by condition (B2). Hence, in either case we have that $B(Tz_0) <_{J_B} x_B$ and $B(z_0) \leqq_{J_B} x_B$. Thus $Tz_0 \neq z_0$ and we have verified that T does not have any fixed points on $\partial P_{J_B}(B, x_B)$.

Let $z_1 \in \{y \in P : x_A <_{J_A} A(y) \text{ and } B(y) <_{J_B} x_B\}$ (see condition (B1)), and let $H_1 : [0,1] \times \overline{P_{J_B}(B, x_B)} \to P$ be defined by $H_1(t, y) = (1-t)Ty + tz_1$. Clearly, H_1 is continuous and $H_1([0,1] \times \overline{P_{J_B}(B, x_B)})$ is relatively compact.

Claim 2: $H_1(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_{J_B}(B, x_B)$. Suppose not; that is, suppose there exists $(t_1, y_1) \in [0,1] \times \partial P_{J_B}(B, x_B)$ such that $H(t_1, y_1) = y_1$. Since $y_1 \in \partial P_{J_B}(B, x_B)$ we have that $B(y_1) \leqq_{J_B} x_B$. Again, since A is comparable to x_A on P relative to J_A, either $A(Ty_1) <_{J_A} x_A$ or $x_A \leq_{J_A} A(Ty_1)$.

Case 1: $A(Ty_1) <_{J_A} x_A$. By condition (B3), we have $B(Ty_1) <_{J_B} x_B$. Since B is convex on P, then

$$B(y_1) = B((1-t_1)Ty_1 + t_1 z_1) \leq (1-t_1)B(Ty_1) + t_1 B(z_1) <_{J_B} x_B,$$

which contradicts $B(y_1) \leqq_{J_B} x_B$.

Case 2: $x_A \leq_{J_A} A(Ty_1)$. Since A is concave on P,

$$(1-t_1)A(Ty_1) + t_1 A(z_1) \leq A((1-t_1)Ty_1 + t_1 z_1) = A(y_1).$$

Since P is a cone of non-negative functions,

$$x_A \leq_{J_A} (1-t_1)A(Ty_1) + t_1 A(z_1) \leq_{J_A} A(y_1),$$

and thus by condition (B2) we have $B(Ty_1) <_{J_B} x_B$. This is the same contradiction we reached in the previous case.

Therefore, we have shown that $H_1(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_{J_B}(B, x_B)$, and thus by the homotopy invariance property $(G3)$ of the fixed point index, $i(T, P_{J_B}(B, x_B), P) = i(z_1, P_{J_B}(B, x_B), P)$. And by the normality property $(G1)$ of the fixed point index, $i(T, P_{J_B}(B, x_B), P) = i(z_1, P_{J_B}(B, x_B), P) = 1$. □

Lemma 3.2. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, J_C and J_D be subsets of K with J_C being compact, and P be a cone of non-negative functions in a real Banach space E which is a subset of $F(K)$. Suppose that C is a continuous concave operator on P, D is a convex operator on P, and $T : P \to P$ is a completely continuous operator. Suppose there exist $x_C, x_D \in E$ such that*

- (A0) *D is comparable to x_D on P relative to J_D;*
- (A1) *$\{y \in P : x_C <_{J_C} C(y) \text{ and } D(y) <_{J_D} x_D\} \neq \emptyset$;*
- (A2) *if $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $D(y) \leq_{J_D} x_D$, then $x_C <_{J_C} C(Ty)$;*
- (A3) *if $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $x_D <_{J_D} D(Ty)$, then $x_C <_{J_C} C(Ty)$.*

If $\overline{P_{J_C}(C, x_C)}$ is bounded, then $i(T, P_{J_C}(C, x_C), P) = 0$.

Proof. By Corollary 2.8, P is a retract of the Banach space E since it is convex and closed.

Claim 1: $Ty \neq y$ for all $y \in \partial P_{J_C}(C, x_C)$. Let $w_0 \in \partial P_{J_C}(C, x_C)$. By the continuity of C and the compactness of J_C, $C(w_0) \leqq_{J_C} x_C$. We want to show that w_0 is not a fixed point of T; so suppose to the contrary that $T(w_0) = w_0$. Since D is comparable to x_D on P relative to J_D, either $x_D <_{J_D} D(Tw_0)$ or $D(Tw_0) \leq_{J_D} x_D$. If $x_D <_{J_D} D(Tw_0)$, then $x_C <_{J_C} C(Tw_0)$ by condition (A3), and if $D(w_0) = D(Tw_0) \leq_{J_D} x_D$, then $x_C <_{J_C} C(Tw_0)$ by condition (A2). Hence, in either case, we have that $x_C <_{J_C} C(Tw_0)$ and $C(w_0) \leqq_{J_C} x_C$. Thus $Tw_0 \neq w_0$ and we have verified that T does not have any fixed points on $\partial P_{J_C}(C, x_C)$.

Let $w_1 \in \{y \in P : x_C <_{J_C} C(y) \text{ and } D(y) <_{J_D} x_D\}$ (see condition (A1)), and let $H_2 : [0,1] \times \overline{P_{J_C}(C, x_C)} \to P$ be defined by $H_2(t, y) = (1-t)Ty + tw_1$. Clearly, H_2 is continuous and $H_2([0,1] \times \overline{P_{J_C}(C, x_C)})$ is relatively compact.

Claim 2: $H_2(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_{J_C}(C, x_C)$. Suppose not; that is, there exists $(t_2, y_2) \in [0,1] \times \partial P_{J_C}(C, x_C)$ such that $H_2(t_2, y_2) = y_2$. Since $y_2 \in \partial P_{J_C}(C, x_C)$, we have that $C(y_2) \leqq_{J_C} x_C$. Also, since D is comparable to x_D on P relative to J_D, either $x_D <_{J_D} D(Ty_2)$ or $D(Ty_2) \leq_{J_D} x_D$.

Case 1: $x_D <_{J_D} D(Ty_2)$. By condition (A3) we have $x_C <_{J_C} C(Ty_2)$, which is a contradiction, since

$$x_C <_{J_C} (1-t_2)C(Ty_2) + t_2 C(w_1) \leq_{J_C} C((1-t_2)Ty_2 + t_2 w_1) = C(y_2) \leqq_{J_C} x_C.$$

Case 2: $D(Ty_2) \leq_{J_D} x_D$. We have that $D(y_2) \leq_{J_D} x_D$, since

$$D(y_2) = D((1-t_2)Ty_2 + t_2 w_1) \leq_{J_D} (1-t_2)D(Ty_2) + t_2 D(w_1) \leq_{J_D} x_D,$$

where $D((1-t_2)Ty_2 + t_2 w_1) \leq_{J_D} (1-t_2)D(Ty_2) + t_2 D(w_1)$ is guaranteed by the fact that D is convex on P and P is a cone of non-negative functions. Thus by condition (A2), we have $x_C <_{J_C} C(Ty_2)$. This is the same contradiction $x_C <_{J_C} x_C$ we reached in the previous case.

Therefore, we have shown that $H_2(t, y) \neq y$ for all $(t, y) \in [0,1] \times \partial P_{J_C}(C, x_C)$, and thus by the homotopy invariance property (G3) of the fixed point index, $i(T, P_{J_C}(C, x_C), P) = i(w_1, P_{J_C}(C, x_C), P)$. And by the solution property (G4) of the fixed point index (since $w_1 \notin P_{J_C}(C, x_C)$ the index cannot be nonzero), we have $i(T, P_{J_C}(C, x_C), P) = i(w_1, P_{J_C}(C, x_C), P) = 0$. □

Theorem 3.3. *Let $F(K)$ be the set of real valued functions defined on $K \subset \mathbb{R}$, let J_A, J_B, J_C and J_D be subsets of K such that J_B and J_C are compact, and let P be a cone of non-negative functions in a real Banach space E that is a subset of $F(K)$. Suppose that A and C are concave operators on P and that B and D are convex operators on P such that B and C are continuous, and that $T : P \to P$ is a completely continuous operator. For $J \subset K$ and $x, y \in F(K)$ let $x <_J y$ ($x \leq_J y$) if and only if $x(t) < y(t)$ ($x(t) \leq y(t)$) for all $t \in J$, whereas $x \leqq_J y$ if and only if $x \leq_J y$ and there exists a $t_0 \in J$ such that $x(t_0) = y(t_0)$.*

Suppose there exist $x_A, x_B, x_C, x_D \in E$ such that

- (D1) *T is LW-outward with respect to $P_{J_C}(C, x_C) := \{y \in P : C(y) <_{J_C} x_C\}$, that is, the following conditions are satisfied:*
  - (A0) *either $D(y) <_{J_D} x_D$ or $x_D \leq_{J_D} D(y)$ for any $y \in P$;*
  - (A1) *$\{y \in P : x_C <_{J_C} C(y) \text{ and } D(y) <_{J_D} x_D\} \neq \emptyset$;*
  - (A2) *if $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $D(y) \leq_{J_D} x_D$, then $x_C <_{J_C} C(Ty)$;*

(A3) *if* $y \in P$ *with* $C(y) \leqq_{J_C} x_C$ *and* $x_D <_{J_D} D(Ty)$*, then* $x_C <_{J_C} C(Ty)$*; and* $\text{closure}\{y \in P : C(y) <_{J_C} x_C\}$ *is bounded.*

(D2) T *is LW-inward with respect to* $P_{J_B}(B, x_B) := \{y \in P : B(y) <_{J_B} x_B\}$*, that is, the following conditions are satisfied:*

(B0) *either* $A(y) <_{J_A} x_A$ *or* $x_A \leq_{J_A} A(y)$ *for any* $y \in P$*;*

(B1) $\{y \in P : x_A <_{J_A} A(y)$ *and* $B(y) <_{J_B} x_B\} \neq \emptyset$*;*

(B2) *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $x_A \leq_{J_A} A(y)$*, then* $B(Ty) <_{J_B} x_B$*;*

(B3) *if* $y \in P$ *with* $B(y) \leqq_{J_B} x_B$ *and* $A(Ty) <_{J_A} x_A$*, then* $B(Ty) <_{J_B} x_B$*; and* $\text{closure}\{y \in P : B(y) <_{J_B} x_B\}$ *is bounded.*

If

(H1) $\text{closure}\{y \in P : B(y) <_{J_B} x_B\} \subsetneq \{y \in P : C(y) <_{J_C} x_C\}$*, then* T *has a fixed point* $y \in P$ *such that* $C(y) <_{J_C} x_C$ *with* $y \notin \text{closure}\{u \in P : B(u) <_{J_B} x_B\}$*,*

whereas, if

(H2) $\text{closure}\{y \in P : C(y) <_{J_C} x_C\} \subsetneq \{y \in P : B(y) <_{J_B} x_B\}$*, then* T *has a fixed point* $y \in P$ *such that* $B(y) <_{J_B} x_B$ *with* $y \notin \text{closure}\{u \in P : C(u) <_{J_C} x_C\}$*.*

Proof. We will prove the expansive result (H1), as the proof of the compressive result (H2) is nearly identical. First, define the sets

$$P_{J_R}(R, x_R) := \{y \in P : R(y) <_{J_R} x_R\}$$

and

$$P(B, C, x_B, x_C, J_B, J_C) := P_{J_C}(C, x_C) - \overline{P_{J_B}(B, x_B)}.$$

To prove the existence of a fixed point for our operator T in $P(B, C, x_B, x_C, J_B, J_C)$, it is enough for us to show that $i(T, P(B, C, x_B, x_C, J_B, J_C), P) \neq 0$.

Since T is LW-inward with respect to $P_{J_B}(B, x_B)$, we have by Lemma 3.1 that $i(T, P_{J_B}(B, x_B), P) = 1$, and since T is LW-outward with respect to $P_{J_C}(C, x_C)$, we have by Lemma 3.2 that $i(T, P_{J_C}(C, x_C), P) = 0$.

T has no fixed points in $\overline{P_{J_C}(C, x_C)} - (P_{J_B}(B, x_B) \cup P(B, C, x_B, x_C, J_B, J_C))$, since if $y \in \overline{P_{J_C}(C, x_C)} - (P_{J_B}(B, x_B) \cup P(B, C, x_B, x_C, J_B, J_C))$, then either $B(y) \leqq_{J_B} x_B$ or $C(y) \leqq_{J_C} x_C$. Now, if $B(y) \leqq_{J_B} x_B$, then we showed in Lemma 3.1 that y was not a fixed point of T, and if $C(y) \leqq_{J_C} x_C$, then we showed in Lemma 3.2 that y was not a fixed point of T. Also, the sets $P_{J_B}(B, x_B)$ and $P(B, C, x_B, x_C, J_B, J_C)$ are nonempty, disjoint, open subsets of $\overline{P_{J_C}(C, x_C)}$, since $\overline{P_{J_B}(B, x_B)} \subsetneq P_{J_C}(C, x_C)$ implies that $P(B, C, x_B, x_C, J_B, J_C) = P_{J_C}(C, x_C) - \overline{P_{J_B}(B, x_B)} \neq \emptyset$. Therefore, by the additivity property $(G2)$ of the fixed point index

$$i(T, P_{J_C}(C, x_C), P) = i(T, P_{J_B}(B, x_B), P) + i(T, P(B, C, x_B, x_C, J_B, J_C), P).$$

Consequently, we have $i(T, P(B, C, x_B, x_C, J_B, J_C), P) = -1$, and thus by the solution property $(G4)$ of the fixed point index, the operator T has a fixed point $y \in P(B, C, x_B, x_C, J_B, J_C)$. $\square$

4. Application

As an application of our main results, we consider the following second order nonlinear right focal boundary value problem,

$$x'' + g(t)f(x, x') = 0, \quad t \in [0,1], \tag{4.1}$$

$$x(0) = x'(1) = 0, \tag{4.2}$$

where $g : [0,1] \to [0,\infty)$ and $f : \mathbb{R}^2 \to [0,\infty)$ are continuous.

Let the Banach space $E = C^1[0,1]$ with the norm of $\|x\| = \max_{t\in[0,1]} |x(t)| + \max_{t\in[0,1]} |x'(t)|$, and define the cone $P \subset E$ by

$$P := \{x \in E : x(t) \geq 0,\ x'(t) \geq 0, \text{ for } t \in [0,1],\ x \text{ is concave, and } x(0) = 0\}.$$

Then for any $x \in P$, we have $\|x\| = x(1) + x'(0)$. And from the concavity of any $x \in P$, we have that $x(t) \geq tx(1)$ and $x(t) \leq x'(0)t$ for $t \in [0,1]$.

It is well known that the Green's function for $-x'' = 0$ and satisfying (4.2) is given by

$$G(t,s) = \min\{t,s\}, \quad (t,s) \in [0,1] \times [0,1].$$

We note that, for any $s \in [0,1]$, $G(t,s) \geq tG(1,s)$ and $G(t,s)$ is nondecreasing in t.

By using properties of the Green's function, solutions of (4.1), (4.2) are the fixed points of the completely continuous operator $T : P \to E$ defined by

$$Tx(t) = \int_0^1 G(t,s)g(s)f(x(s),x'(s))ds.$$

Since $(Tx)''(t) = -g(t)f(x,x') \leq 0$ on $[0,1]$ and $(Tx)(0) = (Tx)'(1) = 0$, we have $T : P \to P$.

Let $\tau \in (0,1)$. For $x \in P$, we define the following operators:

$$(Ax)(t) = (Cx)(t) = x'(0)t, \quad (Bx)(t) = \Big(\frac{x'(0)+x(1)}{2}\Big)t, \qquad (Dx)(t) = \Big(\frac{x(\tau)}{\tau}\Big)t.$$

All the above operators are continuous linear operators mapping P to P, and are convex or concave continuous operators as well. In the following theorem, we demonstrate how to apply the expansive condition of Theorem 3.3 to prove the existence of at least one positive solution to (4.1), (4.2).

Theorem 4.1. *Suppose there is some $\tau \in (0,1)$ and $0 < d < b$ such that g and f satisfy*

(a) $f(u_1,u_2) > \frac{d}{\int_0^\tau g(s)ds}$, *for* $(u_1,u_2) \in [0,d\tau] \times [0,d]$,

(b) $f(u_1,u_2) < \frac{2b}{\int_0^1 (1+s)g(s)ds}$, *for* $(u_1,u_2) \in [0,b] \times [0,2b)$.

Then the right focal problem (4.1), (4.2) *has at least one positive solution $y \in P$ with $y'(0) > d$ and $y'(0) + y(1) < 2b$.*

Proof. We choose $x_A(t) = bt$, $x_B(t) = b$, $x_C(t) = dt$, $x_D(t) = d\tau$ defined on $[0,1]$, and $J_A = J_C = [\tau,1]$, $J_B = \{1\}$ and $J_D = \{\tau\}$. Then it is easy to see that J_A, J_B, J_C, J_D are compact subsets of $[0,1]$ and $x_A, x_B, x_C, x_D \in E$.

Claim 1: T is LW-inward with respect to $P_{J_B}(B, x_B)$. A is comparable to x_A on P relative to J_A, since for any $y \in P$, we have $(Ay)(t) = y'(0)t \geq x_A(t) = bt$, or $(Ay)(t) = y'(0)t < x_A(t) = bt$, for $t \in [\tau,1]$. Thus $A(y) <_{J_A} x_A$ or $x_A \leq_{J_A} A(y)$. Also, $\{y \in P : x_A <_{J_A} A(y) \text{ and } B(y) <_{J_B} x_B\} \neq \emptyset$, since $y_0(t) := at(2-t) \in P$

on $[0,1]$, with $a \in (\frac{b}{2}, \frac{2b}{3})$, and $(Ay_0)(t) = y_0'(0)t = 2at > bt = x_A(t)$, for $t \in [\tau, 1] = J_A$, and $(By_0)(1) = \frac{y_0'(0)+y_0(1)}{2} = \frac{2a+a}{2} < b = x_B(1)$.

Subclaim 1.1: If $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $x_A \leq_{J_A} A(y)$, then $B(Ty) <_{J_B} x_B$. Let $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $x_A \leq_{J_A} A(y)$. From $B(y) \leqq_{J_B} x_B$, we have $y'(0) + y(1) = 2b$. From $x_A \leq_{J_A} A(y)$, we have $y'(0) \geq b$. Hence, $0 < y(1) \leq b$ and $b \leq y'(0) < 2b$, which implies $0 \leq y(t) \leq b$ and $0 \leq y'(t) < 2b$ for $t \in [0,1]$. Then by property (b),

$$f(y(t), y'(t)) < \frac{2b}{\int_0^1 (1+s)g(s)ds}, \quad t \in [0,1],$$

and so

$$\begin{aligned} (BTy)(1) &= \frac{(Ty)'(0) + (Ty)(1)}{2} \\ &= \frac{1}{2}\int_0^1 g(s)f(y(s), y'(s))ds + \frac{1}{2}\int_0^1 sg(s)f(y(s), y'(s))ds \\ &< \frac{1}{2}\int_0^1 (1+s)g(s)ds \cdot \frac{2b}{\int_0^1 (1+s)g(s)ds} \\ &= b = x_B(1); \end{aligned}$$

i.e., $B(Ty) <_{J_B} x_B$.

Subclaim 1.2: If $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $A(Ty) <_{J_A} x_A$, then $B(Ty) <_{J_B} x_B$. Let $y \in P$ with $B(y) \leqq_{J_B} x_B$ and $A(Ty) <_{J_A} x_A$. From $A(Ty) <_{J_A} x_A$, we get $(Ty)'(0) < b$. By the concavity of $T(y)$ on $[0,1]$, we know that $(Ty)(1) \leq (Ty)'(0) < b$, which implies

$$(BTy)(1) = \frac{(Ty)(1) + (Ty)'(0)}{2} < b;$$

i.e., $B(Ty) <_{J_B} x_B$.

It is easy to see that $\overline{P_{J_B}(B, x_B)}$ is bounded, thus T is LW-inward with respect to $P_{J_B}(B, x_B)$.

Claim 2: T is LW-outward with respect to $P_{J_C}(C, x_C)$. D is comparable to x_D on P relative to J_D, since for any $y \in P$, we have $(Dy)(\tau) = y(\tau) \geq x_D(\tau) = d\tau$, or $(Dy)(\tau) = y(\tau) < x_D(\tau) = d\tau$. Thus $Dy <_{J_D} x_D$ or $x_D \leq_{J_D} Dy$. Also, $\{y \in P : x_C <_{J_C} C(y) \text{ and } D(y) <_{J_D} x_D\} \neq \emptyset$, since $y_0(t) := at(2-t) \in P$ on $[0,1]$ with $a \in (\frac{d}{2}, \frac{d}{2-\tau})$, and $(Cy_0)(t) = y_0'(0)t = 2at > dt = x_C(t)$, for $t \in [\tau, 1] = J_C$, and $(Dy_0)(\tau) = a\tau(2-\tau) < d\tau = x_D(\tau)$.

Subclaim 2.1: If $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $D(y) \leq_{J_D} x_D$, then $x_C <_{J_C} C(Ty)$. Let $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $D(y) \leq_{J_D} x_D$. From $C(y) \leqq_{J_C} x_C$, we have that $y'(0) = d$. From $D(y) \leq_{J_D} x_D$, we have that $y(\tau) \leq d\tau$. Hence, for $t \in [0, \tau]$, $0 \leq y(t) \leq d\tau$ and $0 \leq y'(t) \leq d$. Then by property (a),

$$f(y(t), y'(t)) > \frac{d}{\int_0^\tau g(s)ds}, \quad t \in [0, \tau],$$

and so for $t \in [\tau, 1]$,

$$\begin{aligned}(CTy)(t) = (Ty)'(0)t &= \int_0^1 g(s)f(y(s), y'(s))ds \cdot t \\ &\geq \int_0^\tau g(s)f(y(s), y'(s))ds \cdot t \\ &> \int_0^\tau g(s)ds \cdot \frac{dt}{\int_0^\tau g(s)ds} = dt = x_C(t);\end{aligned}$$

i.e., $x_C <_{J_C} C(Ty)$.

Subclaim 2.2: If $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $x_D <_{J_D} D(Ty)$, then $x_C <_{J_C} C(Ty)$. Let $y \in P$ with $C(y) \leqq_{J_C} x_C$ and $x_D <_{J_D} D(Ty)$. From $x_D <_{J_D} D(Ty)$, we have that $(Ty)(\tau) > d\tau$. Hence, $(Ty)'(0) \geq \frac{(Ty)(\tau)}{\tau} > d$. Therefore, $(CTy)(t) = (Ty)'(0)t > dt = x_C(t)$, for $t \in [\tau, 1]$; i.e., $x_C <_{J_C} C(Tx)$.

It is easy to see that $\overline{P_{J_C}(C, x_C)}$ is bounded, thus T is LW-outward with respect to $P_{J_C}(C, x_C)$.

Claim 3: $\overline{P_{J_C}(C, x_C)} \subset P_{J_B}(B, x_B)$ and $P(C, B, x_C, x_B, J_C, J_B) \neq \emptyset$. Let $y \in \overline{P_{J_C}(C, x_C)}$. Then, $y'(0) \leq d$. From $y(1) \leq y'(0) \leq d$, we have

$$(By)(1) = \frac{y'(0) + y(1)}{2} \leq \frac{d + d}{2} = d < b = x_B(1).$$

Hence, $\overline{P_{J_C}(C, x_C)} \subset P_{J_B}(B, x_B)$. Also, $P(C, B, x_C, x_B, J_C, J_B) \neq \emptyset$, since $y_0(t) = at(2 - t) \in P$ on $[0, 1]$, with $a \in (\frac{d}{2}, \frac{2b}{3})$, and $(Cy_0)(t) = y'(0)t = 2at > dt = x_C(t)$, for $t \in [\tau, 1]$, and $(By_0)(1) = \frac{y_0'(0)+y_0(1)}{2} = \frac{3a}{2} < b$.

Therefore, by Theorem 3.3, T has a fixed point y in $P(C, B, x_C, x_B, J_C, J_B)$. $\square$

Example. Consider the right focal boundary value problem

$$\begin{gathered}x''(t) + \frac{1}{t+1}(x-1)^2 e^x (1 - \sin(x')) = 0, \quad t \in [0, 1], \\ x(0) = x'(1) = 0.\end{gathered}$$

Choose $\tau = 0.9$, $d = 0.3$, $b = 0.6$. Then it is easy to verify that this problem satisfies Theorem 4.1 and hence it has at least one positive solution x on $[0, 1]$ with $x'(0) > 0.3$ and $x'(0) + x(1) < 1.2$.

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