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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A NAVIER BOUNDARY-VALUE PROBLEM INVOLVING THE p-BIHARMONIC WITH CRITICAL EXPONENT

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Abstract. By using the Nehari manifold and variational methods, we prove that a p-biharmonic system has at least two positive solutions when the pair the parameters satisfy certain inequality.

1. Introduction

In this article, we consider the multiplicity results of positive solutions of the semilinear p-biharmonic system

$$\begin{gathered}
\Delta(|\Delta u|^{p-2}\Delta u)=\frac{1}{p^{**}}\frac{\partial F(x,u,v)}{\partial u}+\lambda|u|^{q-2}u \quad \text{in } \Omega,\\
\Delta(|\Delta v|^{p-2}\Delta v)=\frac{1}{p^{**}}\frac{\partial F(x,u,v)}{\partial v}+\mu|v|^{q-2}v \quad \text{in } \Omega,\\
u>0,\quad v>0 \quad \text{in } \Omega,\\
u=v=\Delta u=\Delta v=0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}$$

where $x_0 \in \Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $F \in C^1(\overline{\Omega}\times(\mathbb{R}^+)^2,\mathbb{R}^+)$ is positively homogeneous of degree $p^{**}=\frac{pN}{N-2p}$ which is the Sobolev critical exponent; that is, $F(x,tu,tv)=t^{p^{**}}F(x,u,v)$ $(t>0)$ holds for all $(x,u,v)\in\overline{\Omega}\times(\mathbb{R}^+)^2$, $(\frac{\partial F(x,u,v)}{\partial u},\frac{\partial F(x,u,v)}{\partial v})=\nabla F$. We assume that $1<q<p<\frac{N}{2}$, $\lambda>0$, $\mu>0$.

In recent years, there have been many article concerned with the existence and multiplicity of positive solutions for p-biharmonic elliptic problems. Results relating to these problems can be found in [5, 7, 10, 12, 13, 14, 15, 16] and the references therein.

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Brown and Wu [2] considered the semilinear elliptic system

$$\begin{gathered}
-\Delta u + u = \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^{\beta} \quad \text{in } \Omega,\\
-\Delta v + v = \frac{\beta}{\alpha+\beta} f(x)|u|^{\alpha}|v|^{\beta-2}v \quad \text{in } \Omega,\\
\frac{\partial u}{\partial n} = \lambda g(x)|u|^{q-2}u, \quad \frac{\partial v}{\partial n} = \mu h(x)|v|^{q-2}v \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.2}$$

where $\alpha > 1$, $\beta > 1$ satisfying $2 < \alpha + \beta < 2^*$ and the weight functions f, g, h are satisfying the following conditions:

- (A) $f \in C(\overline{\Omega})$ with $\|f\|_\infty = 1$ and $f^+ = \max\{f, 0\} \not\equiv 0$;
- (B) $g, h \in C(\partial\Omega)$ with $\|g\|_\infty = \|h\|_\infty = 1$, $g^\pm = \max\{\pm g, 0\} \not\equiv 0$ and $h^\pm = \max\{\pm h, 0\} \not\equiv 0$.

They showed that (1.2) has at least two negative solutions if the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$.

Recently, Hsu [11] considered the case $F(x, u, v) = 2|u|^\alpha |v|^\beta, \alpha > 1, \beta > 1$ satisfying $\alpha + \beta = p^*$; i.e., the elliptic system:

$$\begin{gathered}
-\Delta_p u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \lambda |u|^{q-2}u \quad \text{in } \Omega,\\
-\Delta_p v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \mu |v|^{q-2}v \quad \text{in } \Omega,\\
u = v = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.3}$$

By variational methods, he proved that (1.2) has at least two positive solutions if the pair of the parameters (λ, μ) belongs to a certain subset of $\mathbb{R}^2$.

In this article, we give a simple variational method which is similar to the "fibering method" of Pohozaev's (see [8, 4]) to prove the existence of at least two positive solutions of problem (1.1). Throughout this paper, we let S be the best Sobolev embedding constant defined by

$$S = \inf_{u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\Delta u|^p dx}{(\int_\Omega |u|^{p^{**}} dx)^{\frac{p}{p^{**}}}},$$

and let

$$\begin{gathered}
C(p, q, N, K, S, |\Omega|) = \big(\frac{p-q}{K(p^{**}-q)}\big)^{\frac{p}{p^{**}-q}} \big(\frac{p^{**}-q}{p^{**}-p}|\Omega|^{\frac{p^{**}-q}{p^{**}}}\big)^{-\frac{p}{p-q}} S^{\frac{N}{2p}+\frac{q}{p-q}},\\
C_0 = \big(\frac{q}{p}\big)^{\frac{p}{p-q}} C(p, q, N, K, S, |\Omega|).
\end{gathered}$$

For our results, we need the following assumptions:

- (F1) $F : \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 function and $F(x, tu, tv) = t^{p^{**}} F(x, u, v)$ for all $t > 0$ and $x \in \overline{\Omega}$, $(u, v) \in (\mathbb{R}^+)^2$;
- (F2) $F(x, u, 0) = F(x, 0, v) = \frac{\partial F}{\partial u}(x, u, 0) = \frac{\partial F}{\partial v}(x, 0, v) = 0$, where $u, v \in \mathbb{R}^+$;
- (F3) $\frac{\partial F(x,u,v)}{\partial u}, \frac{\partial F(x,u,v)}{\partial v}$ are strictly increasing functions about u and v for all $u > 0$, $v > 0$.

From assumption (F1), we have the so-called Euler identity

$$(u, v) \cdot \nabla F(x, u, v) = p^{**} F(x, u, v) \tag{1.4}$$

and, for a positive constant K,

$$F(x,u,v) \le K(|u|^p + |v|^p)^{\frac{p^{**}}{p}}. \tag{1.5}$$

Theorem 1.1. *If λ, μ satisfy $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C(p,q,N,K,S,|\Omega|)$, and (F1)–(F3) hold, then (1.1) has at least one positive solution.*

Theorem 1.2. *If λ, μ satisfy $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0^*$, (F1)–(F3) hold, where $C_0^* = \min\{C^*, C_0\}$, and $C^* = \min\{\delta_1, \rho_0^{\frac{N-2p}{p-1}}, \delta_2\}$, then (1.1) has at least two positive solutions.*

Remark 1.3. There are functions satisfying the conditions of Theorems 1.1 and 1.2. For example,

$$F(x,u,v) = \begin{cases} f_1^2(x)|u|^{3/2}|v|^{5/2} + f_2^2(x)\frac{u^3v^3}{u^2+v^2} & \text{if } (u,v) \ne (0,0), \\ 0 & \text{if } (u,v) = (0,0), \end{cases}$$

where $f_1, f_2 \in C(\overline{\Omega}) \cap L^\infty(\Omega)$ with $\max\{\pm f_1, \pm f_2, 0\} \not\equiv 0$. Obviously, F satisfy (F1), (F2) and (F3).

This article is organized as follows: In Section 2, we give some notation and preliminaries. In Section 3, we prove Theorems 1.1 and 1.2.

2. Notation and preliminaries

Problem (1.1) is posed in the framework of the Sobolev space $E = (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)) \times (W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ with the standard norm

$$\|(u,v)\|^p = \int_\Omega |\Delta u|^p dx + \int_\Omega |\Delta v|^p dx = \|\Delta u\|^p_{L^p(\Omega)} + \|\Delta v\|^p_{L^p(\Omega)}.$$

In addition, we define $\|u\|_{L^p(\Omega)} = (\int_\Omega |u|^p dx)^{\frac{1}{p}}$ as the norm of the Sobolev space $L^p(\Omega)$.

A pair of functions $(u^+, v^+) \in E$, with ($u^+ := \max\{u, 0\}$ and $v^+ := \max\{v, 0\}$), is said to be a weak solution of (1.1) if

$$\begin{aligned} &\int_\Omega (|\Delta u^+|^{p-2}\Delta u^+ \Delta\varphi_1 + |\Delta v^+|^{p-2}\Delta v^+\Delta\varphi_2)dx - \frac{1}{p^{**}}\int_\Omega \frac{\partial F(x,u^+,v^+)}{\partial u}\varphi_1 dx \\ &- \frac{1}{p^{**}}\int_\Omega \frac{\partial F(x,u^+,v^+)}{\partial v}\varphi_2 dx - \lambda\int_\Omega |u^+|^{q-2}u\varphi_1 dx - \mu\int_\Omega |v^+|^{q-2}v\varphi_2 dx = 0 \end{aligned}$$

for all $(\varphi_1, \varphi_2) \in E$. Thus, by (1.4) the corresponding energy functional of problem (1.1) is defined by

$$J_{\lambda,\mu}(u^+,v^+) = \frac{1}{p}\|(u^+,v^+)\|^p - \frac{1}{p^{**}}\int_\Omega F(x,u^+,v^+)dx - \frac{1}{q}K_{\lambda,\mu}(u^+,v^+)$$

for $(u^+, v^+) \in E$, where $K_{\lambda,\mu}(u^+,v^+) = \lambda\int_\Omega |u^+|^q dx + \mu\int_\Omega |v^+|^q dx$.

To verify $J_{\lambda,\mu} \in C^1(E,R)$, we need the following lemmas.

Lemma 2.1. *Suppose that (F3) holds. Assume that $F \in C^1(\overline{\Omega} \times (\mathbb{R}^+)^2, \mathbb{R}^+)$ is positively homogeneous of degree p^{**}, then $\frac{\partial F}{\partial u}, \frac{\partial F}{\partial v} \in C(\overline{\Omega} \times (\mathbb{R}^+)^2, \mathbb{R}^+)$ are positively homogeneous of degree $p^{**} - 1$.*

The proof of the above lemma is almost the same as that in Chu and Tang [6], and it is omitted.

From Lemma 2.1, we obtain the existence of a positive constant M such that for all $x \in \overline{\Omega}$,

$$\big|\frac{\partial F}{\partial u}(x,u,v)\big| \le M(|u|^{p^{**}-1} + |v|^{p^{**}-1}), \tag{2.1}$$

$$\big|\frac{\partial F}{\partial v}(x,u,v)\big| \le M(|u|^{p^{**}-1} + |v|^{p^{**}-1}), u, v \in \mathbb{R}^+. \tag{2.2}$$

As in Willem [16, Theorem A.2], we consider the continuity of the superposition operator

$$A : L^p(\Omega) \times L^p(\Omega) \to L^q(\Omega) : (u,v) \mapsto f(x,u,v).$$

Lemma 2.2. *Assume that $|\Omega| < \infty$, $1 \le p$, $r < \infty$, $f \in C(\overline{\Omega} \times \mathbb{R}^2, \mathbb{R})$ and*

$$|f(x,u,v)| \le c(1 + |u|^{\frac{p}{r}} + |v|^{\frac{p}{r}}).$$

Then, for every $(u,v) \in L^p(\Omega) \times L^p(\Omega)$, $f(\cdot,u,v) \in L^r(\Omega)$ and the operator A : $L^p(\Omega) \times L^p(\Omega) \to L^r(\Omega)$: $(u,v) \mapsto f(x,u,v)$ is continuous.

Now we consider the functional $\psi(u,v) = \int_\Omega F(x,u,v)dx$.

Lemma 2.3. *Assume that $|\Omega| < \infty$, $\frac{\partial F}{\partial u}$, $\frac{\partial F}{\partial v} \in C(\overline{\Omega} \times (\mathbb{R}^+)^2)$ satisfying* (2.1), (2.2)*, then the functional ψ is of class $C^1(E, \mathbb{R}^+)$ and*

$$\langle \psi'(u,v), (a,b)\rangle = \int_\Omega (\frac{\partial F(x,u,v)}{\partial u} a + \frac{\partial F(x,u,v)}{\partial v} b)dx,$$

where $(u,v), (a,b) \in E$.

Proof. First, we proof the existence of the Gateaux derivative. Given $x \in \Omega$ and $0 < |t| < 1$, by the mean value theorem and (2.1), (2.2), there exists $\lambda_1 \in [0,1]$ such that

$$\begin{aligned}
&\frac{|F(x,u+ta,v+tb) - F(x,u,v)|}{|t|} \\
&= \big|\frac{\partial F(x,u+t\lambda_1 a, v+t\lambda_1 b)}{\partial u} a\big| + \big|\frac{\partial F(x,u+t\lambda_1 a, v+t\lambda_1 b)}{\partial v} b\big| \\
&\le M(|u+a|^{p^{**}-1} + |v+b|^{p^{**}-1})|a| + M(|u+a|^{p^{**}-1} + |v+b|^{p^{**}-1})|b| \\
&\le 2^{p^{**}-2} M(|u|^{p^{**}-1} + |v|^{p^{**}-1} + |a|^{p^{**}-1} + |b|^{p^{**}-1})(|a| + |b|).
\end{aligned}$$

The Hölder inequality and the Sobolev imbedding theorem imply that

$$(|u|^{p^{**}-1} + |v|^{p^{**}-1} + |a|^{p^{**}-1} + |b|^{p^{**}-1})(|a| + |b|) \in L^1(\Omega).$$

It follows from the Lebesgue theorem that

$$\langle \psi'(u,v), (a,b)\rangle = \int_\Omega (\frac{\partial F(x,u,v)}{\partial u} a + \frac{\partial F(x,u,v)}{\partial v} b)\, dx.$$

Next, we proof the continuity of the Gateaux derivative. Assume that $(u_n, v_n) \to (u,v)$ in E. By Sobolev imbedding theorem, $(u_n, v_n) \to (u,v)$ in $L^{p^{**}}(\Omega) \times L^{p^{**}}(\Omega)$. By Lemma 2.2, we obtain that $\nabla F(x,u_n,v_n) \to \nabla F(x,u,v)$ in $L^\beta(\Omega)$ where $\beta := \frac{p^{**}}{p^{**}-1}$. By the Hölder inequality and Sobolev imbedding theorem,

$$|\langle \psi'(u_n,v_n) - \psi'(u,v), (a,b)\rangle| \le \|\frac{\partial F(x,u_n,v_n)}{\partial u} - \frac{\partial F(x,u,v)}{\partial u}\|_{L^\beta(\Omega)} \|a\|_{L^{p^{**}}(\Omega)}$$

$$
\begin{aligned}
&+ \|\frac{\partial F(x,u_n,v_n)}{\partial v} - \frac{\partial F(x,u,v)}{\partial v}\|_{L^\beta(\Omega)}\|b\|_{L^{p^{**}}(\Omega)} \\
&\le S^{-\frac{1}{p}}(\|\frac{\partial F(x,u_n,v_n)}{\partial u} - \frac{\partial F(x,u,v)}{\partial u}\|_{L^\beta(\Omega)} \\
&+ \|\frac{\partial F(x,u_n,v_n)}{\partial v} - \frac{\partial F(x,u,v)}{\partial v}\|_{L^\beta(\Omega)})\|(a,b)\|
\end{aligned}
$$

and so

$$
\begin{aligned}
\|\psi'(u_n,v_n) - \psi'(u,v)\| &\le S^{-1/p}(\|\frac{\partial F(x,u_n,v_n)}{\partial u} - \frac{\partial F(x,u,v)}{\partial u}\|_{L^\beta(\Omega)} \\
&+ \|\frac{\partial F(x,u_n,v_n)}{\partial v} - \frac{\partial F(x,u,v)}{\partial v}\|_{L^\beta(\Omega)}) \to 0 \quad \text{as } n \ \to \infty.
\end{aligned}
$$

□

From the above lemmas, we have $J_{\lambda,\mu} \in C^1(E,R)$.

As the energy functional $J_{\lambda,\mu}$ is not bounded below on E, it is useful to consider the functional on the Nehari manifold

$$
N_{\lambda,\mu} = \{(u,v) \in E\backslash\{(0,0)\} | \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}.
$$

Thus, $(u,v) \in N_{\lambda,\mu}$ if and only if

$$
\langle J'_{\lambda,\mu}(u,v),(u,v)\rangle = \|(u,v)\|^p - \int_\Omega F(x,u,v)dx - K_{\lambda,\mu}(u,v) = 0. \tag{2.3}
$$

Note that $N_{\lambda,\mu}$ contains every nonzero solution of problem (1.1). Moreover, we have the following results.

Lemma 2.4. *The energy functional $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$.*

Proof. If $(u,v) \in N_{\lambda,\mu}$, then by the Hölder inequality and the Sobolev imbedding theorem,

$$
\begin{aligned}
J_{\lambda,\mu}(u,v) &= \frac{p^{**}-p}{p^{**}p}\|(u,v)\|^p - \frac{p^{**}-q}{p^{**}q}K_{\lambda,\mu}(u,v) \\
&\ge \frac{p^{**}-p}{p^{**}p}\|(u,v)\|^p - \frac{p^{**}-q}{p^{**}q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^{**}-q}{p^{**}}}(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}\|(u,v)\|^q.
\end{aligned} \tag{2.4}
$$

Thus, $J_{\lambda,\mu}$ is coercive and bounded below on $N_{\lambda,\mu}$. □

Define $\Phi_{\lambda,\mu}(u,v) = \langle J'_{\lambda,\mu}(u,v),(u,v)\rangle$. Then for $(u,v) \in N_{\lambda,\mu}$,

$$
\langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = p\|(u,v)\|^p - p^{**}\int_\Omega F(x,u,v)dx - qK_{\lambda,\mu}(u,v) \tag{2.5}
$$

$$
= (p-p^{**})\int_\Omega F(x,u,v)dx - (q-p)K_{\lambda,\mu}(u,v) \tag{2.6}
$$

$$
= (p-q)\|(u,v)\|^p - (p^{**}-q)\int_\Omega F(x,u,v)dx \tag{2.7}
$$

$$
= (p-p^{**})\|(u,v)\|^p - (q-p^{**})K_{\lambda,\mu}(u,v). \tag{2.8}
$$

Now, we split $N_{\lambda,\mu}$ into three parts:

$$
\begin{aligned}
N^+_{\lambda,\mu} &= \{(u,v) \in N_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle > 0\}; \\
N^0_{\lambda,\mu} &= \{(u,v) \in N_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle = 0\}; \\
N^-_{\lambda,\mu} &= \{(u,v) \in N_{\lambda,\mu} | \langle \Phi'_{\lambda,\mu}(u,v),(u,v)\rangle < 0\}.
\end{aligned}
$$

Then, we have the following results.

Lemma 2.5. *Suppose that (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ and that $(u_0,v_0)\notin N^0_{\lambda,\mu}$. Then $J'_{\lambda,\mu}(u_0,v_0)=0$ in E^{-1} (the dual space of the Sobolev space E).*

Proof. If (u_0,v_0) is a local minimizer for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$, then (u_0,v_0) is a solution of the optimization problem minimize $J_{\lambda,\mu}(u,v)$ subject to $\Phi_{\lambda,\mu}(u,v)=0$. Hence, by the theory of Lagrange multiplies, there exists $\theta\in R$, such that

$$J'_{\lambda,\mu}(u_0,v_0)=\theta\ \Phi'_{\lambda,\mu}(u_0,v_0)\quad\text{in } E^{-1}(\Omega),$$

Thus,

$$\langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle_E=\theta\langle\Phi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle_E. \tag{2.9}$$

Since $(u_0,v_0)\in N_{\lambda,\mu}$, we have $\langle J'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle_E=0$. Moreover, $\langle\Phi'_{\lambda,\mu}(u_0,v_0),(u_0,v_0)\rangle_E\neq 0$, by (2.9), $\theta=0$. Thus, $J'_{\lambda,\mu}(u_0,v_0)=0$ in E^{-1} (the dual space of the Sobolev space E). □

Lemma 2.6. *If*

$$0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<C(p,q,N,K,S,|\Omega|),$$

then $N^0_{\lambda,\mu}=\emptyset$.

Proof. Suppose otherwise, that is there exists $\lambda>0,\mu>0$ with

$$0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<C(p,q,N,K,S,|\Omega|)$$

such that $N^0_{\lambda,\mu}\neq\emptyset$. Then for $(u,v)\in N^0_{\lambda,\mu}$, by (2.7), (2.8) we have

$$\begin{aligned}0&=\langle\Phi'_{\lambda,\mu}(u,v),(u,v)\rangle=(p-q)\|(u,v)\|^p-(p^{**}-q)\int_\Omega F(x,u,v)dx\\&=(p-p^{**})\|(u,v)\|^p-(q-p^{**})K_{\lambda,\mu}(u,v).\end{aligned}$$

By the Minkowski inequality, the Sobolev imbedding theorem and (1.5),

$$\begin{aligned}\int_\Omega F(x,u,v)dx&\le K\Big(\int_\Omega(|u|^p+|v|^p)^{\frac{p^{**}}{p}}dx\Big)^{\frac{p}{p^{**}}\cdot\frac{p^{**}}{p}}\\&\le K\Big(\Big(\int_\Omega|u|^{p^{**}}dx\Big)^{\frac{p}{p^{**}}}+\Big(\int_\Omega|v|^{p^{**}}dx\Big)^{\frac{p}{p^{**}}}\Big)^{\frac{p^{**}}{p}}\\&\le KS^{-\frac{p^{**}}{p}}\Big(\int_\Omega|\Delta u|^pdx+\int_\Omega|\Delta v|^pdx\Big)^{\frac{p^{**}}{p}}\\&=KS^{-\frac{p^{**}}{p}}\|(u,v)\|^{p^{**}}.\end{aligned}$$

Thus,

$$\|(u,v)\|\ge\Big(\frac{p-q}{K(p^{**}-q)}S^{\frac{p^{**}}{p}}\Big)^{\frac{1}{p^{**}-p}}$$

and

$$\|(u,v)\|\le\Big(\frac{p^{**}-q}{p^{**}-p}S^{-\frac{q}{p}}|\Omega|^{\frac{p^{**}-q}{p^{**}}}\Big)^{\frac{1}{p-q}}\big(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\big)^{\frac{1}{p}}.$$

This implies

$$\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\ge C(p,q,N,K,S,|\Omega|),$$

which is a contradiction. Thus, we conclude that if

$$0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<C(p,q,N,K,S,|\Omega|),$$

we have $N^0_{\lambda,\mu}=\emptyset$. □

By Lemma 2.6, we write $N_{\lambda,\mu} = N^+_{\lambda,\mu} \cup N^-_{\lambda,\mu}$ and define

$$\begin{aligned}\theta_{\lambda,\mu} &= \inf_{(u,v)\in N_{\lambda,\mu}} J_{\lambda,\mu}(u,v)\\ \theta^+_{\lambda,\mu} &= \inf_{(u,v)\in N^+_{\lambda,\mu}} J_{\lambda,\mu}(u,v);\\ \theta^-_{\lambda,\mu} &= \inf_{(u,v)\in N^-_{\lambda,\mu}} J_{\lambda,\mu}(u,v).\end{aligned}$$

Then we have the following result.

Lemma 2.7. (i) *If $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C(p,q,N,K,S,|\Omega|)$, then we have $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$;*

(ii) *if $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0$, then $\theta^-_{\lambda,\mu} > d_0$ for some constant*

$$d_0 = d_0(p,q,N,K,S,|\Omega|,\lambda,\mu) > 0.$$

Proof. (i) Let $(u,v) \in N^+_{\lambda,\mu}$. By (2.7),

$$\frac{p-q}{p^{**}-q}\|(u,v)\|^p > \int_\Omega F(x,u,v)dx$$

and so

$$\begin{aligned}J_{\lambda,\mu}(u,v) &= (\frac{1}{p}-\frac{1}{q})\|(u,v)\|^p + (\frac{1}{q}-\frac{1}{p^{**}})\int_\Omega F(x,u,v)dx\\ &< [(\frac{1}{p}-\frac{1}{q}) + (\frac{1}{q}-\frac{1}{p^{**}})\frac{p-q}{p^{**}-q}]\|(u,v)\|^p\\ &= -\frac{2(p-q)}{qN}\|(u,v)\|^p < 0.\end{aligned}$$

Thus, from the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu} \le \theta^+_{\lambda,\mu} < 0$.

(ii) Let $(u,v) \in N^-_{\lambda,\mu}$. By (2.7),

$$\frac{p-q}{p^{**}-q}\|(u,v)\|^p < \int_\Omega F(x,u,v)dx.$$

Moreover, by the Minkowski inequality, the Sobolev imbedding theorem, and (1.5),

$$\int_\Omega F(x,u,v)dx \le KS^{-\frac{p^{**}}{p}}\|(u,v)\|^{p^{**}}. \tag{2.10}$$

This implies

$$\|(u,v)\| > (\frac{p-q}{K(p^{**}-q)})^{\frac{1}{p^{**}-p}} S^{\frac{N}{2p^2}} \quad \text{for all } (u,v) \in N^-_{\lambda,\mu}. \tag{2.11}$$

By (2.4) in the proof of Lemma 2.4

$$\begin{aligned}J_{\lambda,\mu}(u,v) &\ge \|(u,v)\|^q[\frac{p^{**}-p}{p^{**}p}\|(u,v)\|^{p-q} - \frac{p^{**}-q}{p^{**}q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^{**}-q}{p^{**}}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}]\\ &> (\frac{p-q}{K(p^{**}-q)})^{\frac{q}{p^{**}-p}} S^{\frac{qN}{2p^2}}[\frac{p^{**}-p}{p^{**}p}S^{\frac{(p-q)N}{2p^2}}(\frac{p-q}{K(p^{**}-q)})^{\frac{p-q}{p^*-p}}\\ &\quad - \frac{p^{**}-q}{p^{**}q}S^{-\frac{q}{p}}|\Omega|^{\frac{p^{**}-q}{p^{**}}}(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}})^{\frac{p-q}{p}}].\end{aligned}$$

Thus, if $0 < |\lambda|^{\frac{p}{p-q}} + |\mu|^{\frac{p}{p-q}} < C_0$, then

$$J_{\lambda,\mu}(u,v) > d_0 \quad \text{for all } (u,v) \in N^-_{\lambda,\mu},$$

for some $d_0 = d_0(p, q, N, K, S, |\Omega|, \lambda, \mu) > 0$. This completes the proof. $\square$

For each $(u, v) \in E$ with $\int_\Omega F(x, u, v)dx > 0$, set

$$t_{\max} = (\frac{(p-q)\|(u,v)\|^p}{(p**-q)\int_\Omega F(x,u,v)dx})^{\frac{1}{p**-p}} > 0.$$

Then the following lemma holds, which is similar to the one in Brown and Wu [2, Lemma 2.6].

Lemma 2.8. *For each $(u, v) \in E$ with $\int_\Omega F(x, u, v)dx > 0$, there are unique $0 < t^+ < t_{\max} < t^-$ such that $(t^+u, t^+v) \in N^+_{\lambda,\mu}$, $(t^-u, t^-v) \in N^-_{\lambda,\mu}$ and*

$$J_{\lambda,\mu}(t^+u, t^+v) = \inf_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu, tv); \quad J_{\lambda,\mu}(t^-u, t^-v) = \sup_{t\ge 0} J_{\lambda,\mu}(tu, tv).$$

3. Proof of Theorems 1.1 and 1.2

We will need the following lemma.

Lemma 3.1.
(i) *If $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C(p, q, N, K, S, |\Omega|)$, then there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$;*
(ii) *if $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0$, then there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$.*

The proof of the above lemma is almost the same as that in Wu [17]; we omit it. First, we establish the existence of a local minimum for $J_{\lambda,\mu}$ on $N^+_{\lambda,\mu}$.

Theorem 3.2. *If $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C(p, q, N, K, S, |\Omega|)$ and* (F1)-(F3) *hold, then $J_{\lambda,\mu}$ has a minimizer (u_0^+, v_0^+) in $N^+_{\lambda,\mu}$ and it satisfies*
(i) *$J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu}$;*
(ii) *(u_0^+, v_0^+) is a positive solution of* (1.1).

Proof. By the Lemma 3.1(i), there exists a minimizing sequence $\{(u_n, v_n)\}$ for $J_{\lambda,\mu}$ on $N_{\lambda,\mu}$ such that

$$J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu} + o(1), \quad J'_{\lambda,\mu}(u_n, v_n) = o(1) \quad \text{in} \quad E^{-1} \tag{3.1}$$

Then by Lemma 2.4 and the compact imbedding theorem, there exist a subsequence $\{(u_n, v_n)\}$ and $(u_0^+, v_0^+) \in E$ such that

$$\begin{gathered} u_n \rightharpoonup u_0^+ \quad \text{weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ u_n \to u_0^+ \quad \text{strongly in } L^q(\Omega), \\ v_n \rightharpoonup v_0^+ \quad \text{weakly in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ v_n \to v_0^+ \quad \text{strongly in } L^q(\Omega). \end{gathered} \tag{3.2}$$

This implies that $K_{\lambda,\mu}(u_n, v_n) \to K_{\lambda,\mu}(u_0^+, v_0^+)$ as $n \to \infty$. By (3.1) and (3.2), it is easy to prove that (u_0^+, v_0^+) is a weak solution of (1.1). Since

$$\begin{aligned} J_{\lambda,\mu}(u_n, v_n) &= \frac{2}{N}\|(u_n, v_n)\|^p - \frac{p^{**}-q}{p^{**}q}K_{\lambda,\mu}(u_n, v_n) \\ &\ge -\frac{p^{**}-q}{p^{**}q}K_{\lambda,\mu}(u_n, v_n) \end{aligned}$$

and by Lemma 2.7 (i),

$$J_{\lambda,\mu}(u_n, v_n) \to \theta_{\lambda,\mu} < 0 \quad \text{as } n \to \infty.$$

Letting $n \to \infty$, we see that $K_{\lambda,\mu}(u_0^+, v_0^+) > 0$. Thus, (u_0^+, v_0^+) is a nontrivial solution of (1.1).

Now it follows that $u_n \to u_0^+$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $v_n \to v_0^+$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and $J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}$. By $(u_0^+, v_0^+) \in N_{\lambda,\mu}$ and applying Fatou's lemma, we obtain

$$\begin{aligned}
\theta_{\lambda,\mu} &\le J_{\lambda,\mu}(u_0^+, v_0^+) \\
&= \frac{2}{N}\|(u_0^+, v_0^+)\|^p - \frac{p^{**} - q}{p^{**}q} K_{\lambda,\mu}(u_0^+, v_0^+) \\
&\le \liminf_{n\to\infty} \Big(\frac{2}{N}\|(u_n, v_n)\|^p - \frac{p^{**} - q}{p^{**}q} K_{\lambda,\mu}(u_n, v_n)\Big) \\
&\le \liminf_{n\to\infty} J_{\lambda,\mu}(u_n, v_n) = \theta_{\lambda,\mu}.
\end{aligned}$$

This implies

$$J_{\lambda,\mu}(u_0^+, v_0^+) = \theta_{\lambda,\mu}, \quad \lim_{n\to\infty} \|(u_n, v_n)\|^p = \|(u_0^+, v_0^+)\|^p.$$

Let $(\widetilde{u}_n, \widetilde{v}_n) = (u_n, v_n) - (u_0^+, v_0^+)$, then by Brézis-Lieb lemma [1],

$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p = \|(u_n, v_n)\|^p - \|(u_0^+, v_0^+)\|^p.$$

Therefore, $u_n \to u_0^+$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $v_n \to v_0^+$ strongly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Moreover, we have $(u_0^+, v_0^+) \in N_{\lambda,\mu}^+$. In fact, if $(u_0^+, v_0^+) \in N_{\lambda,\mu}^-$, by Lemma 2.8, there are unique t_0^+ and t_0^- such that $(t_0^+ u_0^+, t_0^+ v_0^+) \in N_{\lambda,\mu}^+$ and $(t_0^- u_0^+, t_0^- v_0^+) \in N_{\lambda,\mu}^-$. In particular, we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d}{dt} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) = 0 \quad \text{and} \quad \frac{d^2}{dt^2} J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) > 0,$$

there exists $t_0^+ < \bar{t} \le t_0^-$ such that $J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+)$. By Lemma 2.8,

$$J_{\lambda,\mu}(t_0^+ u_0^+, t_0^+ v_0^+) < J_{\lambda,\mu}(\bar{t} u_0^+, \bar{t} v_0^+) \le J_{\lambda,\mu}(t_0^- u_0^+, t_0^- v_0^+) = J_{\lambda,\mu}(u_0^+, v_0^+),$$

which is a contradiction. It follows from the maximum principle that (u_0^+, v_0^+) is a positive solution of (1.1). This completes the proof. □

The following two lemmas are similar to those in Hsu [11].

Lemma 3.3. *If $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $(u_n, v_n) \rightharpoonup (u, v)$ in E, then $J'_{\lambda,\mu}(u, v) = 0$, and there exists a positive constant Λ depending on p, q, N, S and $|\Omega|$, such that $J_{\lambda,\mu}(u, v) \ge -\Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})$.*

Lemma 3.4. *If $\{(u_n, v_n)\} \subset E$ is a $(PS)_c$-sequence for $J_{\lambda,\mu}$, then $\{(u_n, v_n)\}$ is bounded in E.*

Define

$$S_F := \inf_{(u,v)\in E} \Big\{ \frac{\|(u, v)\|^p}{(\int_\Omega F(x, u, v)dx)^{\frac{p}{p^{**}}}} : \int_\Omega F(x, u, v)dx > 0 \Big\}.$$

We need also the following version of Brézis-Lieb lemma [1].

Lemma 3.5. *Consider $F \in C^1(\overline{\Omega}, (\mathbb{R}^+)^2)$ with $F(x,0,0) = 0$ and*
$$|\frac{\partial F(x,u,v)}{\partial u}|, |\frac{\partial F(x,u,v)}{\partial v}| \leq C_1(|u|^{p-1} + |v|^{p-1})$$
for some $1 \leq p < \infty, C_1 > 0$. Let (u_k, v_k) be a bounded sequence in $L^p(\overline{\Omega}, (\mathbb{R}^+)^2)$, and such that $(u_k, v_k) \rightharpoonup (u,v)$ weakly in E. Then as $k \to \infty$,
$$\int_\Omega F(x, u_k, v_k)dx \to \int_\Omega F(x, u_k - u, v_k - v)dx + \int_\Omega F(x,u,v)dx.$$

Lemma 3.6. *$J_{\lambda,\mu}$ satisfies the $(PS)_c$ condition with c satisfying*
$$-\infty < c < c_\infty = \frac{2}{N} S_F^{N/(2p)} - \Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}).$$

Proof. Let $\{(u_n, v_n)\} \subset E$ be a $(PS)_c$-sequence for $J_{\lambda,\mu}$ with $c \in (-\infty, c_\infty)$. It follows from Lemma 3.4 that $\{(u_n, v_n)\}$ is bounded in E, and then $(u_n, v_n) \rightharpoonup (u,v)$ up to a subsequence, (u,v) is a critical point of $J_{\lambda,\mu}$. Furthermore, we may assume
$$\begin{gathered} u_n \rightharpoonup u, \quad v_n \rightharpoonup v \quad \text{in } W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \\ u_n \to u, \quad v_n \to v \quad \text{in } L^q(\Omega), \\ u_n \to u, \quad v_n \to v \quad \text{a.e. on } \Omega. \end{gathered}$$
Hence we have $J'_{\lambda,\mu}(u,v) = 0$ and
$$\int_\Omega (\lambda|u_n|^q + \mu|v_n|^q)dx \to \int_\Omega (\lambda|u|^q + \mu|v|^q)dx. \tag{3.3}$$
Let $\widetilde{u}_n = u_n - u, \widetilde{v}_n = v_n - v$. Then by Brézis-Lieb lemma [1],
$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p \to \|(u_n, v_n)\|^p - \|(u,v)\|^p \quad \text{as } n \to \infty. \tag{3.4}$$
and by Lemma 3.5,
$$\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx \to \int_\Omega F(x, u_n, v_n)dx - \int_\Omega F(x,u,v)dx. \tag{3.5}$$
Since $J_{\lambda,\mu}(u_n, v_n) = c + o(1), J'_{\lambda,\mu}(u_n, v_n) = o(1)$ and (3.3)-(3.5), we deduce that
$$\frac{1}{p}\|(\widetilde{u}_n, \widetilde{v}_n)\|^p - \frac{1}{p^{**}} \int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx = c - J_{\lambda,\mu}(u,v) + o(1). \tag{3.6}$$
and
$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p - \int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx = o(1).$$
Hence, we may assume that
$$\|(\widetilde{u}_n, \widetilde{v}_n)\|^p \to l, \quad \int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx \to l. \tag{3.7}$$
If $l = 0$, the proof is complete. Assume $l > 0$, then from (3.7), we obtain
$$S_F l^{\frac{p}{p^{**}}} = S_F \lim_{n\to\infty} (\int_\Omega F(x, \widetilde{u}_n, \widetilde{v}_n)dx)^{p/p^{**}} \leq \lim_{n\to\infty} \|(\widetilde{u}_n, \widetilde{v}_n)\|^p = l,$$
which implies $l \geq S_F^{N/(2p)}$. In addition, from Lemma 3.3, (3.6) and (3.7), we obtain
$$c = (\frac{1}{p} - \frac{1}{p^{**}})l + J_{\lambda,\mu}(u,v) \geq \frac{2}{N} S_F^{N/(2p)} - \Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}),$$
which contradicts $c < \frac{2}{N} S_F^{N/(2p)} - \Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})$. $\square$

Lemma 3.7. *There exist a nonnegative function* $(u,v) \in E\backslash\{(0,0)\}$ *and* $C^*>0$ *such that for* $0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<C^*$, *we have*

$$\sup_{t\geq 0} J_{\lambda,\mu}(tu,tv)<c_\infty.$$

In particular, $\theta^-_{\lambda,\mu}<c_\infty$ *for all* $0<\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}<C^*$.

Proof. Since $x_0\in\Omega$, there is $\rho_0>0$ such that $B^N(x_0;2\rho_0)\subset\Omega$. Now, we consider the functional $I:E\to R$ defined by

$$I(u,v)=\frac{1}{p}\|(u,v)\|^p-\frac{1}{p^{**}}\int_\Omega F(x,u,v)dx$$

and define a cut-off function $\eta(x)\in C_0^\infty(\Omega)$ such that $\eta(x)=1$ for $|x-x_0|<\rho_0, \eta(x)=0$ for $|x-x_0|>2\rho_0, 0\leq\eta\leq 1$ and $|\nabla\eta|\leq C$. For $\varepsilon>0$, let

$$u_\varepsilon(x)=\eta(x)U(\frac{x}{\varepsilon}),$$

where $U(\cdot)$ is a radially symmetric minimizer of $\{\frac{\|\Delta u\|^p_{L^p}}{\|u\|^p_{L^{p^{**}}}}\}_{u\in W^{2,p}(\mathbb{R}^N)\backslash\{0\}}$. Similar to the work of Brown and Wu [3], we have the following estimates:

$$\begin{gathered}
\Big(\int_\Omega |u_\varepsilon|^{p^{**}}dx\Big)^{\frac{p}{p^{**}}}=\varepsilon^{-\frac{N-2p}{p}}\|U\|^p_{L^{p^{**}}(\mathbb{R}^N)}+O(\varepsilon),\\
\int_\Omega |\Delta u_\varepsilon|^p dx=\varepsilon^{-\frac{N-2p}{p}}\|\Delta U\|^p_{L^p(\mathbb{R}^N)}+O(1),\\
\frac{\int_\Omega|\Delta u_\varepsilon|^p dx}{(\int_\Omega|u_\varepsilon|^{p^{**}}dx)^{\frac{p}{p^{**}}}}=S+O(\varepsilon^{\frac{N-2p}{p}}),
\end{gathered}\tag{3.8}$$

Thus, we obtain

$$\frac{\|\Delta U\|^p_{L^p(\mathbb{R}^N)}}{\|U\|^p_{L^{p^{**}}(\mathbb{R}^N)}}=S=\inf_{u\in W^{2,p}(\mathbb{R}^N)\backslash\{0\}}\frac{\|\Delta u\|^p_{L^p(\mathbb{R}^N)}}{\|u\|^p_{L^{p^{**}}(\mathbb{R}^N)}}.$$

Set $u_0(x)=e_1u_\varepsilon(x-x_0), v_0(x)=e_2u_\varepsilon(x-x_0)$ and $(u_0,v_0)\in E$, where $x_0\in\Omega$, $(e_1,e_2)\in(\mathbb{R}^+)^2$, and $e_1^p+e_2^p=1$ are such that

$$F(x_0,e_1,e_2)=\max_{x\in\overline{\Omega},g_1^p+g_2^p=1,g_1,g_2>0}F(x,g_1,g_2)=:K.$$

Then, by (F1), (1.5), the definition of S_F and (3.8), we obtain

$$\begin{aligned}
\sup_{t\geq 0} I(tu_0,tu_0)&\leq\frac{2}{N}\Big(\frac{(e_1^p+e_2^p)\int_\Omega|\Delta u_\varepsilon|^p dx}{(\int_\Omega F(x,e_1u_\varepsilon(x-x_0),e_2u_\varepsilon(x-x_0))dx)^{\frac{p}{p^{**}}}}\Big)^{N/(2p)}\\
&=\frac{2}{N}\Big(\frac{\int_\Omega|\Delta u_\varepsilon|^p dx}{\int_\Omega(|u_\varepsilon(x-x_0)|^{p^{**}}F(x,e_1,e_2)dx)^{\frac{p}{p^{**}}}}\Big)^{N/(2p)}\\
&\leq\frac{2}{N}\Big(\frac{1}{K^{\frac{p}{p^{**}}}}\Big)^{N/(2p)}(S+O(\varepsilon^{\frac{N-2p}{p}}))^{N/(2p)}\\
&=\frac{2}{N}\Big(\frac{1}{K^{\frac{p}{p^{**}}}}\Big)^{N/(2p)}(S^{N/(2p)}+O(\varepsilon^{\frac{N-2p}{p}}))\\
&\leq\frac{2}{N}S_F^{N/(2p)}+O(\varepsilon^{\frac{N-2p}{p}}),
\end{aligned}\tag{3.9}$$

where we have used that

$$\sup_{t\geq 0}(\frac{t^p}{p}A - \frac{t^{p^{**}}}{p^{**}}B) = \frac{2}{N}(\frac{A}{B^{\frac{p}{p^{**}}}})^{N/(2p)}, \quad A, B > 0.$$

We can choose $\delta_1 > 0$ such that for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_1$, so we have

$$c_\infty = \frac{2}{N}S_F^{N/(2p)} - \Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}) > 0.$$

Using the definitions of $J_{\lambda,\mu}$ and (u_0, v_0), we obtain

$$J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{t^p}{p}\|(u_0, v_0)\|^p \quad \text{for all } t \geq 0,\ \lambda, \mu > 0,$$

which implies that there exists $t_0 \in (0, 1)$ satisfying

$$\sup_{0\leq t\leq t_0} J_{\lambda,\mu}(t_0u_0, t_0v_0) < c_\infty, \quad \text{for all } 0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_1.$$

Using the definitions of $J_{\lambda,\mu}$ and (u_0, v_0), we obtain

$$\begin{aligned}
\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) &= \sup_{t\geq t_0}(I_{\lambda,\mu}(tu_0, tv_0) - \frac{t^q}{q}K_{\lambda,\mu}(u_0, v_0)) \\
&\leq \frac{2}{N}S_F^{N/(2p)} + O(\varepsilon^{\frac{N-2p}{p}}) - \frac{t_0^q}{q}(e_1^q\lambda + e_2^q\mu)\int_{B^N(0;\rho_0)} |u_\varepsilon|^q dx \\
&\leq \frac{2}{N}S_F^{N/(2p)} + O(\varepsilon^{\frac{N-2p}{p}}) - \frac{t_0^q}{q}m(\lambda + \mu)\int_{B^N(0;\rho_0)} |u_\varepsilon|^q dx,
\end{aligned} \tag{3.10}$$

where $m = \min\{e_1^q, e_2^q\}$. Let $0 < \varepsilon \leq \rho_0^{\frac{p}{p-1}}$, we obtain

$$\begin{aligned}
\int_{B^N(0;\rho_0)} |u_\varepsilon|^q dx &= \int_{B^N(0;\rho_0)} \frac{1}{(\varepsilon + |x|^{\frac{p}{p-1}})^{\frac{N-2p}{p}q}} dx \\
&\geq \int_{B^N(0;\rho_0)} \frac{1}{(2\rho_0^{\frac{p}{p-1}})^{\frac{N-2p}{p}q}} dx = C_2(N, p, q, \rho_0).
\end{aligned}$$

Combining with (3.10) and the above inequality, for all $\varepsilon = (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p}{N-2p}} \in (0, \rho_0^{\frac{p}{p-1}})$, we have

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) \leq \frac{2}{N}S_F^{N/(2p)} + O(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}) - \frac{t_0^q}{q}mC_2(\lambda + \mu). \tag{3.11}$$

Hence, we can choose $\delta_2 > 0$ such that for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \delta_2$, we obtain

$$O(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}) - \frac{t_0^q}{q}mC_2(\lambda + \mu) < -\Lambda(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}).$$

If we set $C^* = \min\{\delta_1, \rho_0^{\frac{N-2p}{p-1}}, \delta_2\}$ and $\varepsilon = (\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}})^{\frac{p}{N-2p}}$ then for $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C^*$, we have

$$\sup_{t\geq t_0} J_{\lambda,\mu}(tu_0, tv_0) < c_\infty. \tag{3.12}$$

Finally, we prove that $\theta^-_{\lambda,\mu} < c_\infty$ for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C^*$. Recall that $(u_0, v_0) = (e_1u_\varepsilon, e_2u_\varepsilon)$. It is easy to see that

$$\int_\Omega F(x, u_0, v_0)dx > 0.$$

Combining this with Lemma 2.8, from the definition of $\theta^-_{\lambda,\mu}$ and (3.11), we obtain that there exists $t_0 > 0$ such that $(t_0u_0, t_0v_0) \in N^-_{\lambda,\mu}$ and

$$\theta^-_{\lambda,\mu} \le J_{\lambda,\mu}(t_0u_0, t_0v_0) \le \sup_{t\ge 0} J_{\lambda,\mu}(tu_0, tv_0) < c_\infty$$

for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C^*$. □

Theorem 3.8. *If* $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0^*$ *and* (F1)–(F3) *hold, then* $J_{\lambda,\mu}$ *has a minimizer* (u_0^-, v_0^-) *in* $N^-_{\lambda,\mu}$ *and it satisfies*

(i) $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta^-_{\lambda,\mu}$;

(ii) (u_0^-, v_0^-) *is a positive solution of* (1.1).

where $C_0^* = \min\{C^*, C_0\}$.

Proof. By lemma 3.1 (ii), there is a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset N^-_{\lambda,\mu}$ in E for $J_{\lambda,\mu}$ for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0$. From Lemmas 3.6, 3.7 and 2.7 (ii), for $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C^*$, $J_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$ condition and $\theta^-_{\lambda,\mu} > 0$. Since $J_{\lambda,\mu}$ is coercive on $N_{\lambda,\mu}$, we obtain that (u_n, v_n) is bounded in E. Therefore, there exist a subsequence still denoted by (u_n, v_n) and $(u_0^-, v_0^-) \in N^-_{\lambda,\mu}$ such that $(u_n, v_n) \to (u_0^-, v_0^-)$ strongly in E and $J_{\lambda,\mu}(u_0^-, v_0^-) = \theta^-_{\lambda,\mu} > 0$ for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0^*$. Finally, by the same arguments as in the proof of Theorem 3.2, for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0^*$, we have that (u_0^-, v_0^-) is a positive solution of (1.1). □

Now, we complete the proof of Theorems 1.1 and 1.2. By Theorem 3.2, we obtain that for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C(p, q, N, K, S, |\Omega|)$, problem (1.1) has a positive solution $(u_0^+, v_0^+) \in N^+_{\lambda,\mu}$. On the other hand, from Theorem 3.8, we obtain the second positive solution $(u_0^-, v_0^-) \in N^-_{\lambda,\mu}$ for all $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < C_0^* < C(p, q, N, K, S, |\Omega|)$. Since $N^+_{\lambda,\mu} \cap N^-_{\lambda,\mu} = \emptyset$, this implies that (u_0^+, v_0^+) and (u_0^-, v_0^-) are distinct. This completes the proof of Theorems 1.1 and 1.2.

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