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MULTIPLE POSITIVE PERIODIC SOLUTIONS TO A NON-AUTONOMOUS LOTKA-VOLTERRA PREDATOR-PREY SYSTEM WITH HARVESTING TERMS

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Abstract. Using Mawhin's continuation theorem of coincidence degree theory, we establish the existence of 2^{n+m} positive periodic solutions for a non-autonomous Lotka-Volterra network-like predator-prey system with harvesting terms. Here n and m denote the number of prey and predator species respectively. An example is given to illustrate our results.

1. Introduction and description of the model

In the usual predator-prey model, there is only one predator and one prey. However, in nature we encounter complex systems with several species as predators and several species as prey. In our model all predators form one layer, and all prey form another layer; to be called predator layer and prey layer, respectively. There is a competition relationship among each species lying in the same layer because they fight for food, living space and so on. Considering the above, in this paper, we introduce the following non-autonomous Lotka-Volterra network-like predator-prey system with harvesting terms

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)\Big(a_i(t) - b_i(t)x_i(t) - \sum_{r=1, r\neq i}^{n} c_{ir}(t)x_r(t) \\ &\quad - \sum_{k=1}^{m} d_{ik}(t)x_{n+k}(t)\Big) - h_i(t), \quad i = 1, 2, \dots, n, \\ \dot{x}_{n+j}(t) &= x_{n+j}(t)\Big(\alpha_j(t) - \beta_j(t)x_{n+j}(t) - \sum_{r=1, r\neq j}^{m} \gamma_{rj}(t)x_{n+r}(t) \\ &\quad + \sum_{k=1}^{n} \delta_{kj}(t)x_k(t)\Big) - e_j(t), \quad j = 1, 2, \dots, m \end{aligned} \tag{1.1}$$

where $x_i(t)$ and $x_{n+j}(t)$ $(j = 1, 2, \dots, m)$ are the ith prey species population density and the jth predator species population density, respectively; $a_i(t), b_i(t)$ and $h_i(t)$

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stand for the ith prey species birth rate, death rate and harvesting rate, respectively; $\alpha_j(t), \beta_j(t)$ and $e_j(t)$ stand for the jth predator species birth rate, death rate and harvesting rate, respectively; $c_{is}(t)(i \neq s)$ represents the competition rate between the sth prey species and the ith prey species, $d_{ik}(t)$ $(i = 1, 2, \dots, n; k = 1, 2, \dots, m)$ represents the kth predator species predation rate on the ith prey species, $\gamma_{sj}(t)(s \neq j)$ stands for the competition rate between the sth predator species and the jth predator species, $\delta_{kj}(t)$ $(j = 1, 2, \dots, m; k = 1, 2, \dots, n)$ stands for the transformation rate between the kth prey species and the $n + j$th predator species. In addition, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Therefore, the assumptions of periodicity of the parameters are a way of incorporating the periodicity of the environment (e.g, seasonal effects of weather, food supplies, mating habits, etc), which leads us to assume that $a_i(t), b_i(t), c_{is}(t), d_{ik}(t), h_i(t), \alpha_j(t), \beta_j(t), \gamma_{sj}(t), \delta_{kj}(t)$ and $e_j(t)$ $(i = 1, 2, \dots, n; j = 1, 2, \dots, m)$ are all positive continuous ω-periodic functions.

Since a very basic and important problem in the study of a population growth model with a periodic environment is the global existence and stability of a positive periodic solution, which plays a similar role as a globally stable equilibrium does in an autonomous model. This motivates us to investigate the existence of a positive periodic or multiple positive periodic solutions for system (1.1). In fact, it is more likely for some biological species to take on multiple periodic change regulations and have multiple local stable periodic phenomena. Therefore, it is essential for us to investigate the existence of multiple positive periodic solutions for population models. Our main purpose of this paper is by using Mawhin's continuation theorem of coincidence degree theory [2], to establish the existence of 2^{n+m} positive periodic solutions for system (1.1). For the work concerning the multiple existence of periodic solutions of periodic population models which was done by using coincidence degree theory, we refer the reader to [1, 3, 4, 5, 6].

The organization of the rest of this paper is as follows. In Section 2, by employing the continuation theorem of coincidence degree theory and the skills of inequalities, we establish the existence of 2^{n+m} positive periodic solutions of system (1.1). In Section 3, one example is given to illustrate the effectiveness of our results.

2. Existence of 2^{n+m} positive periodic solutions

In this section, by using Mawhin's continuation theorem and some inequalities, we shall show the existence of positive periodic solutions of (1.1). To do so, we need to make some preparations.

Let X and Z be real normed vector spaces. Let $L : \operatorname{Dom} L \subset X \to Z$ be a linear mapping and $N : X \times [0, 1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Z. If L is a Fredholm mapping of index zero, then there exists continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $\operatorname{Im} P = \ker L$ and $\ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, and $X = \ker L \oplus \ker P, Z = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L \cap \ker P} : (I - P)X \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_P. If Ω is a bounded open subset of X, the mapping N is called L-compact on $\bar{\Omega} \times [0, 1]$, if $QN(\bar{\Omega} \times [0, 1])$ is bounded and $K_P(I - Q)N : \bar{\Omega} \times [0, 1] \to X$ is compact. Because $\operatorname{Im} Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \operatorname{Im} Q \to \ker L$.

The Mawhin's continuous theorem [2, p.40] is as follows.

Lemma 2.1 ([2]). *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar\Omega \times [0,1]$. Assume*

(a) *for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega \cap \operatorname{Dom} L$;*
(b) *$QN(x,0)x \neq 0$ for each $x \in \partial\Omega \cap \ker L$;*
(c) *$\deg(JQN(x,0), \Omega \cap \ker L, 0) \neq 0$.*

Then $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

For the sake of convenience, we denote $f^l = \min_{t\in[0,\omega]} f(t)$, $f^M = \max_{t\in[0,\omega]} f(t)$, $\bar f = \frac{1}{\omega}\int_0^\omega f(t)\,\mathrm{d}t$, respectively, here $f(t)$ is a continuous ω-periodic function.

For simplicity, we need to introduce some notations as follows.

$$
\begin{gathered}
l_i^{\pm} = \frac{a_i^M \pm \sqrt{(a_i^M)^2 - 4b_i^l h_i^l}}{2b_i^l},\\
l_{n+j}^{\pm} = \frac{(\alpha_j^M + \sum_{k=1}^n \delta_{kj}^M l_k^+) \pm \sqrt{(\alpha_j^M + \sum_{k=1}^n \delta_{kj}^M l_k^+)^2 - 4\beta_j^l e_j^l}}{2\beta_j^l},\\
A_i^{\pm} = \Big(a_i^l - \sum_{r=1, r\neq i}^n c_{ir}^M l_r^+ - \sum_{k=1}^m d_{ik}^l l_{n+k}^+ \pm \Big((a_i^l - \sum_{r=1,r\neq i}^n c_{ir}^M l_r^+ \\
- \sum_{k=1}^m d_{ik}^l l_{n+k}^+)^2 - 4b_i^M h_i^M\Big)^{1/2}\Big)/(2b_i^M),\\
A_{n+j}^{\pm} = \frac{\alpha_j^l - \sum_{r=1,r\neq j}^m \gamma_{rj}^M l_{n+r}^+ \pm \sqrt{(\alpha_j^l - \sum_{r=1,r\neq j}^m \gamma_{rj}^M l_{n+r}^+)^2 - 4\beta_j^M e_j^M}}{2\beta_j^M}
\end{gathered}
$$

where $i = 1,2,\dots,n$; $j = 1,2,\dots,m$.

In this paper, we use the following assumptions.

(H) $a_i^l - \sum_{r=1,r\neq i}^n c_{ir}^M l_r^+ - \sum_{k=1}^m d_{ik}^l l_{n+k}^+ > 2\sqrt{b_i^M h_i^M}$, $i = 1,2,\dots,n$ and $\alpha_j^l - \sum_{r=1,r\neq j}^m \gamma_{rj}^M l_{n+r}^+ > 2\sqrt{\beta_j^M e_j^M}$, $j = 1,2,\dots,m$.

By elementary calculus, one can easily show the following result.

Lemma 2.2. *Let $x > 0$, $y > 0$, $z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x,y,z) = \frac{x+\sqrt{x^2-4yz}}{2z}$ and $g(x,y,z) = \frac{x-\sqrt{x^2-4yz}}{2z}$, the following assertions hold.*

(1) *$f(x,y,z)$ and $g(x,y,z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0,\infty)$, respectively.*
(2) *$f(x,y,z)$ and $g(x,y,z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0,\infty)$, respectively.*
(3) *$f(x,y,z)$ and $g(x,y,z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0,\infty)$, respectively.*

By assumption (H) and Lemma 2.2, one can prove the following statement.

Lemma 2.3. *For the equations*

$$
\begin{gathered}
a_i(t) - b_i(t)e^{u_i(t)} - h_i(t)e^{-u_i(t)} = 0, \quad \forall\, t \in \mathbb{R},\ i = 1,2,\dots,n;\\
\alpha_j(t) - \beta_j(t)e^{u_{n+j}(t)} - e_j(t)e^{-u_{n+j}(t)} = 0, \quad \forall\, t \in \mathbb{R},\ j = 1,2,\dots,m,
\end{gathered}
$$

if assumption (H) holds, then we have the inequalities

$$\ln l_i^- < \ln u_i^- < \ln A_i^- < \ln A_i^+ < \ln u_i^+ < \ln l_i^+, \quad \forall t \in \mathbb{R};$$
$$\ln l_{n+j}^- < \ln u_{n+j}^- < \ln A_{n+j}^- < \ln A_{n+j}^+ < \ln u_{n+j}^+ < \ln l_{n+j}^+, \quad \forall t \in \mathbb{R},$$

where

$$u_i^\pm = \frac{a_i(t) \pm \sqrt{(a_i(t))^2 - 4b_i(t)h_i(t)}}{2b_i(t)}, \quad i = 1,2,\dots,n;$$
$$u_{n+j}^\pm = \frac{\alpha_j(t) \pm \sqrt{(\alpha_j(t))^2 - 4\beta_j(t)e_j(t)}}{2\beta_j(t)}, \quad j = 1,2,\dots,m.$$

Theorem 2.4. *Assume that* (H) *holds. Then* (1.1) *has at least* 2^{n+m} *positive* ω*-periodic solutions.*

Proof. By making the substitutions

$$x_i(t) = \exp\{u_i(t)\}, \quad x_{n+j}(t) = \exp\{u_{n+j}(t)\}, \quad i = 1,2,\dots,n;\ j = 1,2,\dots,m, \tag{2.1}$$

system (1.1) can be reformulated as

$$\begin{aligned}
\dot{u}_i(t) &= a_i(t) - b_i(t)e^{u_i(t)} - \sum_{r=1,r\neq i}^{n} c_{ir}(t)e^{u_r(t)} \\
&\quad - \sum_{k=1}^{m} d_{ik}(t)e^{u_{n+k}(t)} - h_i(t)e^{-u_i(t)}, \\
\dot{u}_{n+j}(t) &= \alpha_j(t) - \beta_j(t)e^{u_{n+j}(t)} - \sum_{r=1,r\neq j}^{m} \gamma_{rj}(t)e^{u_{n+r}(t)} \\
&\quad + \sum_{k=1}^{n} \delta_{kj}(t)e^{u_k(t)} - e_j(t)e^{-u_{n+j}(t)},
\end{aligned} \tag{2.2}$$

where $i = 1,2,\dots,n$; $j = 1,2,\dots,m$. Let

$$X = Z = \big\{u = (u_1, u_2, \dots, u_{n+m})^T \in C(R, R^{n+m}) : u(t+\omega) = u(t)\big\}$$

and define

$$\|u\| = \sum_{i=1}^{n+m} \max_{t\in[0,\omega]} |u_i(t)|, \quad u \in X \mathrm{or} Z.$$

Equipped with the above norm $\|\cdot\|$, X and Z are Banach spaces. Let

$$Lu = \dot{u} = \frac{\mathrm{d}u(t)}{\mathrm{d}t}$$

and $N(u,\lambda)$ be column vector

$$\begin{pmatrix}
a_1(t) - b_1(t)e^{u_1(t)} - \lambda\Big(\sum_{r=2}^{n} c_{1r}(t)e^{u_r(t)} + \sum_{k=1}^{m} d_{1k}(t)e^{u_{n+k}(t)}\Big) - h_1(t)e^{-u_1(t)} \\
\vdots \\
a_n(t) - b_n(t)e^{u_n(t)} - \lambda\Big(\sum_{r=1}^{n-1} c_{nr}(t)e^{u_r(t)} + \sum_{k=1}^{m} d_{nk}(t)e^{u_{n+k}(t)}\Big) - h_n(t)e^{-u_n(t)} \\
\alpha_1(t) - \beta_1(t)e^{u_{n+1}(t)} - \lambda\Big(\sum_{r=2}^{m} \gamma_{r1}(t)e^{u_{n+r}(t)} - \sum_{k=1}^{n} \delta_{k1}(t)e^{u_k(t)}\Big) - e_1(t)e^{-u_{n+1}(t)} \\
\vdots \\
\alpha_m(t) - \beta_m(t)e^{u_{n+m}(t)} - \lambda\Big(\sum_{r=1}^{m-1} \gamma_{rm}(t)e^{u_{n+r}(t)} - \sum_{k=1}^{n} \delta_{km}(t)e^{u_k(t)}\Big) - e_m(t)e^{-u_{n+m}(t)}
\end{pmatrix}$$

We put

$$Pu = \frac{1}{\omega}\int_0^\omega u(t)\,\mathrm{d}t, \quad u \in X, \quad Qz = \frac{1}{\omega}\int_0^\omega z(t)\,\mathrm{d}t, \quad z \in Z.$$

Thus it follows that $\ker L = R^{n+m}$, $\operatorname{Im} L = \{z \in Z : \int_0^\omega z(t)\,\mathrm{d}t = 0\}$ is closed in Z, $\dim\ker L = n + m = \operatorname{codim}\operatorname{Im} L$, and P, Q are continuous projectors such that

$$\operatorname{Im} P = \ker L, \quad \ker Q = \operatorname{Im} L = \operatorname{Im}(I - Q).$$

Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : \operatorname{Im} L \to \ker P \bigcap \operatorname{Dom} L$ is given by

$$K_P(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^\omega \int_0^s z(s)\,\mathrm{d}s.$$

Then

$$QN(u,\lambda) = \begin{pmatrix} \frac{1}{\omega}\int_0^\omega F_1(s,\lambda)\,\mathrm{d}s \\ \vdots \\ \frac{1}{\omega}\int_0^\omega F_{n+m}(s,\lambda)\,\mathrm{d}s \end{pmatrix}_{(n+m)\times 1}$$

and

$$\begin{aligned} &K_P(I-Q)N(u,\lambda) \\ &= \begin{pmatrix} \int_0^t F_1(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t F_1(s,\lambda)\,\mathrm{d}s\mathrm{d}t + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega F_1(s,\lambda)\,\mathrm{d}s \\ \vdots \\ \int_0^t F_n(s,\lambda)\mathrm{d}s - \frac{1}{\omega}\int_0^\omega\int_0^t F_n(s,\lambda)\,\mathrm{d}s\mathrm{d}t + (\frac{1}{2} - \frac{t}{\omega})\int_0^\omega F_n(s,\lambda)\,\mathrm{d}s \end{pmatrix}_{n\times 1}, \end{aligned}$$

where $F(u,\lambda)$ is the column vector

$$\begin{pmatrix} a_1(s) - b_1(s)e^{u_1(s)} - \lambda\sum_{r=2}^n c_{1r}(t)e^{u_r(t)} - \lambda\sum_{k=1}^m d_{1k}(s)e^{u_{n+k}(s)} - h_1(s)e^{-u_1(s)} \\ \vdots \\ a_n(s) - b_n(s)e^{u_n(s)} - \lambda\sum_{r=1}^{n-1} c_{nr}(s)e^{u_r(s)} - \lambda\sum_{k=1}^m d_{nk}(s)e^{u_{n+k}(s)} - h_n(s)e^{-u_n(s)} \\ \alpha_1(s) - \beta_1(s)e^{u_{n+1}(s)} - \lambda\sum_{r=2}^m \gamma_{r1}(s)e^{u_{n+r}(s)} + \lambda\sum_{k=1}^n \delta_{k1}(s)e^{u_k(s)} - e_1(s)e^{-u_{n+1}(s)} \\ \vdots \\ \alpha_m(s) - \beta_m(s)e^{u_{n+m}(s)} - \lambda\sum_{r=1}^{m-1} \gamma_{rm}(s)e^{u_{n+r}(s)} + \lambda\sum_{k=1}^n \delta_{km}(s)e^{u_k(s)} - e_m(s)e^{-u_{n+m}(s)} \end{pmatrix}.$$

Obviously, QN and $K_P(I-Q)N$ are continuous. Using the Arzela-Ascoli theorem, it is not difficult to show that $K_P(I-Q)N(\overline{\Omega})$ is compact for any open bounded set $\Omega \subset X$. Moreover, $QN(\overline{\Omega})$ is clearly bounded. Thus, N is L-compact on $\overline{\Omega}$ with any open bounded set $\Omega \subset X$.

To use Lemma 2.1, we have to find at least 2^{n+m} appropriate open bounded subsets in X. Considering the operator equation $Lu = \lambda N(u,\lambda), \lambda \in (0,1)$, we have

$$\begin{aligned}
\dot{u}_i(t) &= \lambda\Big(a_i(t) - b_i(t)e^{u_i(t)} - \lambda \sum_{r=1, r\neq i}^{n} c_{ir}(t)e^{u_r(t)} \\
&\quad - \lambda \sum_{k=1}^{m} d_{ik}(t)e^{u_{n+k}(t)} - h_i(t)e^{-u_i(t)}\Big), \quad i = 1, 2, \dots, n, \\
\dot{u}_{n+j}(t) &= \lambda\Big(\alpha_j(t) - \beta_j(t)e^{u_{n+j}(t)} - \lambda \sum_{r=1, r\neq j}^{m} \gamma_{rj}(t)e^{u_{n+r}(t)} \\
&\quad + \lambda \sum_{k=1}^{n} \delta_{kj}(t)e^{u_k(t)} - e_j(t)e^{-u_{n+j}(t)}\Big), \quad j = 1, 2, \dots, m.
\end{aligned} \tag{2.3}$$

Assume that $u \in X$ is an ω-periodic solution of (2.3) for some $\lambda \in (0, 1)$. Then there exist $\xi_i, \eta_i, \xi_{n+j}, \eta_{n+j} \in [0, \omega]$ such that $u_i(\xi_i) = \max_{t\in[0,\omega]} u_i(t)$, $u_i(\eta_i) = \min_{t\in[0,\omega]} u_i(t)$, $u_{n+j}(\xi_{n+j}) = \max_{t\in[0,\omega]} u_{n+j}(t)$, $u_{n+j}(\eta_{n+j}) = \min_{t\in[0,\omega]} u_{n+j}(t)$. It is clear that $\dot{u}_i(\xi_i) = 0$, $\dot{u}_i(\eta_i) = 0$, $\dot{u}_{n+j}(\xi_{n+j}) = 0$, $\dot{u}_{n+j}(\eta_{n+j}) = 0$. From this and (2.3), we have

$$\begin{aligned}
&a_i(\xi_i) - b_i(\xi_i)e^{u_i(\xi_i)} - \lambda \sum_{r=1, r\neq i}^{n} c_{ir}(\xi_i)e^{u_r(\xi_i)} \\
&- \lambda \sum_{k=1}^{m} d_{ik}(\xi_i)e^{u_{n+k}(\xi_i)} - h_i(\xi_i)e^{-u_i(\xi_i)} = 0, \\
&\alpha_j(\xi_{n+j}) - \beta_j(\xi_{n+j})e^{u_{n+j}(\xi_{n+j})} - \lambda \sum_{r=1, r\neq j}^{m} \gamma_{rj}(t)e^{u_{n+r}(\xi_{n+j})} \\
&+ \lambda \sum_{k=1}^{n} \delta_{kj}(\xi_{n+j})e^{u_k(\xi_{n+j})} - e_j(\xi_{n+j})e^{-u_{n+j}(\xi_{n+j})} = 0
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
&a_i(\eta_i) - b_i(\eta_i)e^{u_i(\eta_i)} - \lambda \sum_{r=1, r\neq i}^{n} c_{ir}(\eta_i)e^{u_r(\eta_i)} \\
&- \lambda \sum_{k=1}^{m} d_{ik}(\eta_i)e^{u_{n+k}(\eta_i)} - h_i(\eta_i)e^{-u_i(\eta_i)} = 0, \\
&\alpha_j(\eta_{n+j}) - \beta_j(\eta_{n+j})e^{u_{n+j}(\eta_{n+j})} - \lambda \sum_{r=1, r\neq j}^{m} \gamma_{rj}(t)e^{u_{n+r}(\eta_{n+j})} \\
&+ \lambda \sum_{k=1}^{n} \delta_{kj}(\eta_{n+j})e^{u_k(\eta_{n+j})} - e_j(\eta_{n+j})e^{-u_{n+j}(\eta_{n+j})} = 0,
\end{aligned} \tag{2.5}$$

where $i = 1, 2, \dots, n$; $j = 1, 2, \dots, m$. On the one hand, according to the first equation of (2.4), we have

$$\begin{aligned}
b_i^l e^{2u_i(\xi_i)} - a_i^M e^{u_i(\xi_i)} + h_i^l &\le b_i(\xi_i)e^{2u_i(\xi_i)} - a_i(\xi_i)e^{u_i(\xi_i)} + h_i(\xi_i) \\
&= -\lambda e^{u_i(\xi_i)}\Big(\sum_{r=1, r\neq i}^{n} c_{ir}(\xi_i)e^{u_r(\xi_i)} + \sum_{k=1}^{m} d_{ik}(\xi_i)e^{u_{n+k}(\xi_i)}\Big)
\end{aligned}$$

$$< 0, \quad i = 1, 2, \dots, n;$$

namely,

$$b_i^l e^{2u_i(\xi_i)} - a_i^M e^{u_i(\xi_i)} + h_i^l < 0, \quad i = 1, 2, \dots, n,$$

which implies

$$\ln l_i^- < u_i(\xi_i) < \ln l_i^+, \quad i = 1, 2, \dots, n. \tag{2.6}$$

From this inequality and the second equation in (2.4), we obtain

$$\begin{aligned}
&\beta_j^l e^{2u_{n+j}(\xi_{n+j})} - \alpha_j^M e^{u_{n+j}(\xi_{n+j})} + e_j^l \\
&\le \beta_j(\xi_{n+j}) e^{2u_{n+j}(\xi_{n+j})} - \alpha_j(\xi_{n+j}) e^{u_{n+j}(\xi_{n+j})} + e_j(\xi_{n+j}) \\
&= \lambda e^{u_{n+j}(\xi_{n+j})} \Big(- \sum_{r=1, r\neq j}^{m} \gamma_{ir}(\xi_{n+j}) e^{u_{n+r}(\xi_{n+j})} + \sum_{k=1}^{n} \delta_{ik}(\xi_{n+j}) e^{u_k(\xi_{n+j})} \Big) \\
&< e^{u_{n+j}(\xi_{n+j})} \sum_{k=1}^{n} \delta_{ik}^M l_k^+, \quad j = 1, 2, \dots, m;
\end{aligned}$$

that is,

$$\beta_j^l e^{2u_{n+j}(\xi_{n+j})} - \Big(\alpha_j^M + \sum_{k=1}^{n} \delta_{ik}^M l_k^+ \Big) e^{u_{n+j}(\xi_{n+j})} + e_j^l < 0, \quad j = 1, 2, \dots, m,$$

which implies

$$\ln l_{n+j}^- < u_{n+j}(\xi_{n+j}) < \ln l_{n+j}^+, \quad j = 1, 2, \dots, m. \tag{2.7}$$

From (2.5), we analogously have

$$\ln l_i^- < u_i(\eta_i) < \ln l_i^+, \quad i = 1, 2, \dots, n \tag{2.8}$$

$$\ln l_{n+j}^- < u_{n+j}(\eta_{n+j}) < \ln l_{n+j}^+, j = 1, 2, \dots, m. \tag{2.9}$$

On the other hand, by (2.4), (2.6) and (2.7), we obtain

$$\begin{aligned}
a_i^l &\le a_i(\xi_i) \\
&= b_i(\xi_i) e^{u_i(\xi_i)} + \lambda \sum_{r=1, r\neq i}^{n} c_{ir}(\xi_i) e^{u_r(\xi_i)} + \lambda \sum_{k=1}^{m} d_{ik}(\xi_i) e^{u_{n+k}(\xi_i)} + h_i(\xi_i) e^{-u_i(\xi_i)} \\
&< b_i^M e^{u_i(\xi_i)} + \sum_{r=1, r\neq i}^{n} c_{ir}^M l_r^+ + \sum_{k=1}^{m} d_{ik}^M l_{n+k}^+ + h_i^M e^{-u_i(\xi_i)}, \quad i = 1, 2, \dots, n,
\end{aligned}$$

and

$$\begin{aligned}
\alpha_j^l \le \alpha_j(\xi_{n+j}) &= \beta_j(\xi_{n+j}) e^{u_{n+j}(\xi_{n+j})} + \lambda \sum_{r=1, r\neq j}^{m} \gamma_{rj}(t) e^{u_{n+r}(\xi_{n+j})} \\
&\quad - \lambda \sum_{k=1}^{n} \delta_{kj}(\xi_{n+j}) e^{u_k(\xi_{n+j})} + e_j(\xi_{n+j}) e^{-u_{n+j}(\xi_{n+j})} \\
&< \beta_j^M e^{u_{n+j}(\xi_{n+j})} + \sum_{r=1, r\neq j}^{m} \gamma_{rj}^M l_{n+r}^+ + e_j^M e^{-u_{n+j}(\xi_{n+j})}, \quad j = 1, 2, \dots, m;
\end{aligned}$$

namely,

$$b_i^M e^{2u_i(\xi_i)} - \big(a_i^l - \sum_{r=1, r\neq i}^{n} c_{ir}^M l_r^+ - \sum_{k=1}^{m} d_{ik}^M l_{n+k}^+ \big) e^{u_i(\xi_i)} + h_i^M > 0, \quad i = 1, 2, \dots, n;$$

$$\beta_j^M e^{2u_{n+j}(\xi_{n+j})} - \big(\alpha_j^l - \sum_{r=1,r\neq j}^{m} \gamma_{rj}^M l_{n+r}^+\big) e^{u_{n+j}(\xi_{n+j})} + e_j^M > 0, \quad j = 1,2,\dots,m,$$

which implies

$$u_i(\xi_i) < \ln A_i^- \quad \text{or} \quad u_i(\xi_i) > \ln A_i^+, \quad i = 1,2,\dots,n; \tag{2.10}$$

$$u_{n+j}(\xi_{n+j}) < \ln A_{n+j}^- \quad \text{or} \quad u_{n+j}(\xi_{n+j}) > \ln A_{n+j}^+, \quad j = 1,2,\dots,m. \tag{2.11}$$

According to (2.5), we analogously have

$$u_i(\eta_i) < \ln A_i^- \quad \text{or} \quad u_i(\eta_i) > \ln A_i^+, \quad i = 1,2,\dots,n; \tag{2.12}$$

$$u_{n+j}(\eta_{n+j}) < \ln A_{n+j}^- \quad \text{or} \quad u_{n+j}(\eta_{n+j}) > \ln A_{n+j}^+, \quad j = 1,2,\dots,m. \tag{2.13}$$

From (2.6)-(2.13) and Lemma 2.3, we have that for any $t \in \mathbb{R}$,

$$\ln l_i^- < u_i(t) < \ln A_i^- \quad \text{or} \quad \ln A_i^+ < u_i(t) < \ln l_i^+, \quad i = 1,2,\dots,n; \tag{2.14}$$

$$\ln l_{n+j}^- < u_{n+j}(t) < \ln A_{n+j}^- \quad \text{or} \quad \ln A_{n+j}^+ < u_{n+j}(t) < \ln l_{n+j}^+, \quad j = 1,2,\dots,m. \tag{2.15}$$

For convenience, we denote

$$G_i = \big(\ln l_i^-, \ln A_i^-\big), \quad H_i = \big(\ln A_i^+, \ln l_i^+\big), \quad i = 1,2,\dots,n+m.$$

Clearly, $l_i^\pm$, $i = 1,2,\dots,n+m$ are independent of λ. For each $i = 1,2,\dots,n+m$, we choose one of interval among the two intervals G_i and H_i and denote it as Δ_i, then define the set

$$\big\{u = (u_1,u_2,\dots,u_{n+m})^T \in X : u_i(t) \in \Delta_i, t \in \mathbb{R}, i = 1,2,\dots,n+m\big\}.$$

Obviously, the number of the above sets is 2^{n+m}. We denote these sets as Ω_k, $k = 1,2,\dots,2^{n+m}$. $\Omega_k, k = 1,2,\dots,2^{n+m}$ are bounded open subsets of $X, \Omega_i \cap \Omega_j = \phi, i \neq j$. Thus Ω_k $(k = 1,2,\dots,2^{n+m})$ satisfies the requirement (a) in Lemma 2.1.

Now we show that (b) of Lemma 2.1 holds; i.e., we prove when $u \in \partial\Omega_k \cap \ker L = \partial\Omega_k \cap R^{n+m}, QN(u,0) \neq (0,0,\dots,0)^T$, $k = 1,2,\dots,2^{n+m}$. If it is not true, then when $u \in \partial\Omega_k \cap \ker L = \partial\Omega_k \cap R^{n+m}$, $i = 1,2,\dots,2^{n+m}$, constant vector $u = (u_1,u_2,\dots,u_{n+m})^T$ with $u \in \partial\Omega_k, k = 1,2,\dots,2^{n+m}$, satisfies

$$\int_0^\omega a_i(t)\,\mathrm{d}t - \int_0^\omega b_i(t)e^{u_i}\,\mathrm{d}t - \int_0^\omega h_i(t)e^{-u_i}\,\mathrm{d}t = 0, \quad i = 1,2,\dots,n;$$
$$\int_0^\omega \alpha_j(t)\,\mathrm{d}t - \int_0^\omega \beta_j(t)e^{u_{n+j}}\,\mathrm{d}t - \int_0^\omega e_j(t)e^{-u_{n+j}}\,\mathrm{d}t = 0, \quad j = 1,2,\dots,m.$$

In view of the mean value theorem of calculous, there exist $n+m$ points t_i $(i = 1,2,\dots,n+m)$ such that

$$a_i(t_i) - b_i(t_i)e^{u_i} - h_i(t_i)e^{-u_i} = 0, \quad i = 1,2,\dots,n; \tag{2.16}$$

$$\alpha_j(t_{n+j}) - \beta_j(t_{n+j})e^{u_{n+j}} - e_j(t_{n+j})e^{-u_{n+j}} = 0, \ j = 1,2,\dots,m. \tag{2.17}$$

By (2.16) and (2.17), we have

$$u_i^\pm = \frac{a_i(t_i) \pm \sqrt{(a_i(t_i))^2 - 4b_i(t_i)h_i(t_i)}}{2b_i(t_i)}, \quad i = 1,2,\dots,n;$$
$$u_{n+j}^\pm = \frac{\alpha_j(t_{n+j}) \pm \sqrt{(\alpha_j(t_{n+j}))^2 - 4\beta_j(t_{n+j})e_j(t_{n+j})}}{2\beta_j(t_{n+j})}, \quad j = 1,2,\dots,m.$$

According to Lemma 2.3, we obtain

$$\ln l_i^- < \ln u_i^- < \ln A_i^- < \ln A_i^+ < \ln u_i^+ < \ln l_i^+, \quad i = 1, 2, \dots, n+m.$$

Then u belongs to one of $\Omega_k \cap R^{n+m}$, $k = 1, 2, \dots, 2^{n+m}$. This contradicts the fact that $u \in \partial\Omega_k \cap R^{n+m}$, $k = 1, 2, \dots, 2^{n+m}$. This proves (b) in Lemma 2.1 holds.

Finally, we show that (c) in Lemma 2.1 holds. Since (H) holds, the system of algebraic equations

$$\begin{gathered} a_i(t_i) - b_i(t_i)e^{x_i} - h_i(t_i)e^{-x_i} = 0, \quad i = 1, 2, \dots, n, \\ \alpha_j(t_{n+j}) - \beta_j(t_{n+j})e^{x_{n+j}} - e_j(t_{n+j})e^{-x_{n+j}} = 0, \quad j = 1, 2, \dots, m \end{gathered}$$

has 2^{n+m} distinct solutions:

$$(x_1^*, x_2^*, \dots, x_{n+m}^*) = (\ln \hat{x}_1, \ln \hat{x}_2, \dots, \ln \hat{x}_{n+m}),$$

where $\hat{x}_i = x_i^-$ or $\hat{x}_i = x_i^+$,

$$x_i^{\pm} = \frac{a_i(t_i) \pm \sqrt{(a_i(t_i))^2 - 4b_i(t_i)h_i(t_i)}}{2b_i(t_i)}, \quad i = 1, 2, \dots, n$$

and $\hat{x}_{n+j} = x_{n+j}^-$ or $\hat{x}_{n+j} = x_{n+j}^+$,

$$x_{n+j}^{\pm} = \frac{\alpha_j(t_{n+j}) \pm \sqrt{(\alpha_j(t_{n+j}))^2 - 4\beta_j(t_{n+j})e_j(t_{n+j})}}{2\beta_j(t_{n+j})}, \quad j = 1, 2, \dots, m.$$

By Lemma 2.2, it is easy to verify that

$$\ln l_i^- < \ln x_i^- < \ln A_i^- < \ln A_i^+ < \ln x_i^+ < \ln l_i^+, \ i = 1, 2, \dots, n+m.$$

Therefore, $(x_1^*, x_2^*, \dots, x_{n+m}^*)$ belongs to the corresponding Ω_k. Since $\ker L = \operatorname{Im} Q$, we can take $J = I$. A direct computation gives, for $k = 1, 2, \dots, 2^{n+m}$,

$$\begin{aligned} &\deg\big\{JQN(u,0), \Omega_k \cap \ker L, (0,0,\dots,0)^T\big\} \\ &= \operatorname{sign}\Big[\prod_{i=1}^{n}\prod_{j=1}^{m}\Big(-b_i(t_i)x_i^* + \frac{h_i(t_i)}{x_i^*}\Big)\Big(-\beta_j(t_{n+j})x_{n+j}^* + \frac{e_j(t_{n+j})}{x_{n+j}^*}\Big)\Big]. \end{aligned}$$

Since

$$\begin{gathered} a_i(t_i) - b_i(t_i)x_i^* - \frac{h_i(t_i)}{x_i^*} = 0, \quad i = 1, 2, \dots, n, \\ \alpha_j(t_{n+j}) - \beta_j(t_{n+j})x_{n+j}^* - \frac{e_j(t_{n+j})}{x_{n+j}^*} = 0, \quad j = 1, 2, \dots, m, \end{gathered}$$

it follows that

$$\begin{aligned} &\deg\big\{JQN(u,0), \Omega_k \cap \ker L, (0,0,\dots,0)^T\big\} \\ &= \operatorname{sign}\Big[\prod_{i=1}^{n}\prod_{j=1}^{m}\big(a_i(t_i) - 2b_i(t_i)x_i^*\big)\big(\alpha_j(t_{n+j}) - 2\beta_j(t_{n+j})x_{n+j}^*\big)\Big] \\ &= \pm 1, \quad k = 1, 2, \dots, 2^{n+m}. \end{aligned}$$

So far, we have proved that Ω_k $(k = 1, 2, \dots, 2^{n+m})$ satisfies all the assumptions in Lemma 2.1. Hence, system (2.2) has at least 2^{n+m} different ω-periodic solutions. Thus by (2.1) system (1.1) has at least 2^{n+m} different positive ω-periodic solutions. This completes the proof of Theorem 2.1. $\square$

For system (1.1), assume that $c_{ir}(t) \geq 0$ $(i = 1, 2, \dots, n; r \neq i)$, $\gamma_{rj}(t) \geq 0$ $(j = 1, 2, \dots, m; r \neq j)$, $d_{ik}(t) \geq 0$ $(i = 1, 2, \dots, n; k = 1, 2, \dots, m)$ and $\delta_{kj}(t) \geq 0$ $(j = 1, 2, \dots, m; k = 1, 2, \dots, n)$ and $a_i(t) > 0$, $b_i(t) > 0$, $h_i(t) > 0$, $\alpha_j(t) > 0$, $\beta_j(t) > 0$, $e_j(t) > 0$ are continuous periodic functions, similar to the proof of Theorem 2.1, one can prove the following result.

Theorem 2.5. *Assume that (H) hold. Then* (1.1) *has at least* 2^{n+m} *positive* ω*-periodic solutions.*

3. Example

Now, let us consider the following network-like predator-prey system with harvesting terms which have one prey species ($n = 1$) and two predator species ($m = 2$):

$$
\begin{aligned}
\dot{x}_1(t) &= x_1(t)\Big(3 + \sin t - \frac{4 + \sin t}{10} x_1(t) - d_{12}(t)x_2(t) - d_{13}(t)x_3(t)\Big) - \frac{9 + \cos t}{20}, \\
\dot{x}_2(t) &= x_2(t)\Big(3 + \cos t - \frac{5 + \cos t}{10} x_2(t) + \delta_{11}(t)x_1(t) - \gamma_{21}(t)x_3(t)\Big) - \frac{2 + \cos t}{5}, \\
\dot{x}_3(t) &= x_3(t)\Big(3 + \sin 2t - \frac{8 + \sin 2t}{10} x_3(t) + \delta_{12}(t)x_1(t) - \gamma_{12}(t)x_2(t)\Big) - \frac{8 + \cos 2t}{10}.
\end{aligned}
\tag{3.1}
$$

In this case, $a_1(t) = 3 + \sin t$, $b_1(t) = \frac{4+\sin t}{10}$, $h_1(t) = \frac{9+\cos t}{20}$, $\alpha_1(t) = 3 + \cos t$, $\beta_1(t) = \frac{5+\cos t}{10}$, $e_1(t) = \frac{2+\cos t}{5}$, $\alpha_2(t) = 3 + \sin 2t$, $\beta_2(t) = \frac{8+\sin 2t}{10}$, $e_2(t) = \frac{8+\cos 2t}{10}$. Since

$$
l_1^{\pm} = \frac{a_1^M \pm \sqrt{(a_1^M)^2 - 4b_1^l h_1^l}}{2b_1^l} = 5 \pm \frac{1}{2}\sqrt{97},
$$

taking $\delta_{11}(t) = \delta_{11}(t+2\pi) > 0$, $\delta_{12}(t) = \delta_{12}(t+2\pi) > 0$ such that $\delta_{11}^M l_1^+ = \delta_{12}^M l_1^+ = 1$, then we have

$$
l_3^{\pm} = \frac{(\alpha_2^M + \delta_{12}^M l_1^+) \pm \sqrt{(\alpha_2^M + \delta_{12} l_1^+)^2 - 4\beta_2^l e_2^l}}{2\beta_2^l} = \frac{25 \pm \sqrt{576}}{7}.
$$

Take $d_{12}(t) = d_{12}(t + 2\pi) > 0, d_{13}(t) = d_{13}(t + 2\pi) > 0, \gamma_{21}(t) = \gamma_{21}(t + 2\pi) > 0, \gamma_{12}(t) = \gamma_{12}(t + 2\pi) > 0$ such that $d_{11}^l l_2^+ = d_{12}^l l_3^+ = \gamma_{21}^M l_3^+ = \gamma_{12}^M l_2^+ = \frac{1}{10}$, then $1.8 = a_1^l - d_{11}^l l_2^+ - d_{12}^l l_3^+ > 2\sqrt{b_1^M h_1^M} = 1$, $1.9 = \alpha_1^l - \gamma_{21}^M l_3^+ > 2\sqrt{\beta_1^M e_1^M} = \frac{6}{5}$, $1.9 = \alpha_2^l - \gamma_{12}^M l_2^+ > 2\sqrt{\beta_2^M e_2^M} = \frac{18}{10}$. Therefore, all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, system (3.1) has at least eight positive 2π-periodic solutions.

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